ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 2003

INFLECTION POINTS AND NONSINGULAR EMBEDDINGS OF SURFACES IN $\mathbf{R}^5$

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ABSTRACT. We define asymptotic direction fields on surfaces embedded in $\mathbf{R}^5$ and characterize their critical points both as umbilics of height functions and as singular points of order 2 of the embedding in Feldman's sense. We show that there are at least one and at most five of these fields defined locally at each point of a generically embedded closed surface. We use this viewpoint in order to consider the existence of singular points of order 2 on a given surface.

1. Introduction. The osculating space of order k at a point p of a m-dimensional manifold M in $\mathbf{R}^n$ is the linear subspace $T_p^k M$ spanned by the osculating k-spaces of all the curves contained in M passing through p. A smooth map $f : M \to N$ between smooth manifolds M and N is said to be **nondegenerate** or **non singular of order k** if it induces an injective linear map $T_p^k f : T_p^k M \to T_{f(p)}^k N$, $\forall\, p \in M$. These maps were studied by E.A. Feldman ([**5**]–[**7**]), who determined the dimensions m, n of the manifolds M and N for which the set of non degenerate embeddings of order k is open and dense in the set of all the embeddings of M in N with the Whitney C^∞-topology and developed several geometrical applications of these methods.

The existence of nondegenerate embeddings of order k from M to N appears to be related to the global geometry of these manifolds. An interesting question arising in this context is that of which surfaces admit nondegenerate embeddings of order 2 in $\mathbf{R}^n$. For this question to make sense we must consider $n = 5, 6$, for when $n < 5$ there are no such maps, and for $n > 6$, Feldman proved that they form a dense set in $Emb(M, \mathbf{R}^n)$. We consider here the case $n = 5$. To approach this problem we use the family of height functions induced by an embedding

Work of the first author partially supported by FAPESP grant no. 1997/13335-6.
Work of the second author partially supported by DGCYT grant no. BFM2000-1110.
Work of the third author partially supported by FAPESP grant no. 97/10735-3 and CNPq grant no. 300066/88-03.
Received by the editors on October 6, 2000, and in revised form on February 15, 2001.

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of a surface M in $\mathbf{R}^5$ to define the concept of asymptotic direction on M. The study of the singularities of this family leads to the characterization of the singular points of order 2 of the embedding, on the one hand as the umbilic singularities of height functions on M, and on the other as the singular points of the fields of asymptotic directions on M. So the question of analyzing the existence of nonsingular embeddings of order 2 of surfaces in $\mathbf{R}^5$ is reduced to the problem of studying the behavior of the asymptotic direction fields on these surfaces.

We observe that the singular points under consideration may be of different types according to the type of contact the surface has with its osculating hyperplanes at the point. We present a geometrical characterization of this fact by analyzing the projection of the surface into convenient 4-spaces. In any case, we prove that all these points lie, generically, along regular closed curves.

A further analysis of the asymptotic lines tells us that a surface generically embedded in $\mathbf{R}^5$ admits at least one and at most five locally defined fields of asymptotic directions. When some of these fields are globally defined on a surface M with nonvanishing Euler number, it is possible to deduce the existence of singular points of order 2 on it.

On the other hand, it can be seen that stereographic projection takes inflection points of surfaces in S^4 (considered as embedded in $\mathbf{R}^5$) to semi-umbilic points (i.e., points where the curvature ellipse degenerates into a segment, see [**10**],[**12**]) of their images in $\mathbf{R}^4$. It then follows that the semi-umbilic points of a generic surface in 4-space also form regular closed curves, which provides an alternative proof to the one given by Montaldi in [**12**] for this fact.

There are still many results to be obtained in this direction, and we expect that further analysis of the global behavior of the singularities of corank 2 of height functions, as well as a deeper study of the generic configurations of the associated fields of asymptotic directions, will contribute to this purpose.

2. Degenerate directions and binormals. Consider an embedding of a surface

$$f : M \longrightarrow \mathbf{R}^5.$$

The family of height functions on M associated to this embedding is given by

$$\lambda(f) : M \times S^4 \longrightarrow \mathbf{R}^5$$
$$(p, v) \longmapsto f_v(p) = \langle f(p), v \rangle.$$

Clearly, f_v has a singularity at $p \in M$ if and only if v is normal to M at p. It follows from Looijenga's genericity theorem ([**11**]) that there is a residual set of embeddings in $C^\infty(M, \mathbf{R}^5)$ with the Whitney C^∞-topology, for which the family $\lambda(f)$ is locally stable and thus, for any $v \in N_pM$, the height function f_v has a singularity of one of the following types at p: Morse (A_1), fold (A_2), cusp (A_3), swallowtail (A_4), butterfly (A_5), elliptic umbilic (D_4^+), hyperbolic umbilic (D_4^-) or parabolic umbilic (D_5). The series $\{A_k\}_{k \geq 1}$ is known as the cuspoids family. They represent singularities of corank 1 (corank(f_v)=corank($Hess\,(f_v)$))and $\mathcal{A}$-codimension $k-1$. The $\{D_k^\pm\}_{k \geq 4}$ series is known as the umbilics family. These singularities have corank 2, and have $\mathcal{A}$-codimension $k-1$ (see [**1**]).

A vector $v \in N_pM$ shall be called **degenerate direction** for M provided that p is a non Morse singularity of f_v, that is, a singularity of $\mathcal{A}$-codimension at least 1.

Given a generic embedding $f : M \to \mathbf{R}^5$, we shall characterize the global distribution of its degenerate directions over the surface M in terms of the coefficients of the second fundamental form of f. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be an orthonormal frame in a neighborhood of a point p in M, such that $\{e_1, e_2\}$ is a tangent frame and $\{e_3, e_4, e_5\}$ is a normal frame in this neighborhood. The matrix of the second fundamental form of f with respect to this frame is given by

$$\alpha_f(p) = \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \\ c_{20} & c_{11} & c_{02} \end{bmatrix}$$

where $a_{20} = \langle f_{xx}, e_3 \rangle$, $a_{11} = \langle f_{xy}, e_3 \rangle$, $a_{02} = \langle f_{yy}, e_3 \rangle$, $b_{20} = \langle f_{xx}, e_4 \rangle$, $b_{11} = \langle f_{xy}, e_4 \rangle$, $b_{02} = \langle f_{yy}, e_4 \rangle$, $c_{20} = \langle f_{xx}, e_5 \rangle$, $c_{11} = \langle f_{xy}, e_5 \rangle$ and $c_{02} = \langle f_{yy}, e_5 \rangle$.

This vector valued quadratic form induces, for each $p \in M$, a linear map A_p from the normal space, N_pM, of M at p to the space Q of quadratic forms in the variables x and y. If we represent a vector $v \in N_pM$ by its coordinates (v_3, v_4, v_5) with respect to the basis

$\{e_3, e_4, e_5\}$, we have

$$A_p(v_3, v_4, v_5) = v_3(d^2 f \cdot e_3) + v_4(d^2 f \cdot e_4) + v_5(d^2 f \cdot e_5).$$

Now, by using the natural identifications (through the basis induced by the above frame) of N_pM and Q with $\mathbf{R}^3$, we can view this as the linear map $A_p : \mathbf{R}^3 \to \mathbf{R}^3$, whose matrix is $\alpha_f(p)$.

Denote by Σ_p the projectivized normal space of $f(M)$ at $f(p)$. That is, Σ_p represents the set of all (nonoriented) normal directions at $f(p)$. Then we can show the following.

Lemma 1. *An embedding of a surface M in $\mathbf{R}^5$ induces a decomposition of M into subsets $M_i = \{p \in M : rank(\alpha_f(p)) = i\},\ i = 0, 1, 2, 3.$ Moreover,*

i) *If $p \in M_3$ then there is a closed curve in Σ_p of degenerate directions at p.*

ii) *The subset M_2 is subdivided, in turn, into the following*:

$M_2(2) = \{p \in M_2 :$ there are two lines of degenerate directions in $\Sigma_p\}$
$M_2(1) = \{p \in M_2 :$ there is a unique line of degenerate directions in $\Sigma_p\}$
$M_2(0) = \{p \in M_2 :$ there is a unique degenerate direction in $\Sigma_p\}$

iii) *If $p \in M_1$ then there is either a unique line of degenerate directions in Σ_p or the whole projective plane Σ_p is made of degenerate directions.*

iv) *If $p \in M_0$ then all the directions in Σ_p are degenerate.*

Proof. Let C represent the cone of degenerate quadratic forms in Q.

i) If $rank(\alpha_f(p)) = 3$, then $Im(A_p)$ fills the whole Q and hence $A_p^{-1}(C)$ is a closed curve in Σ_p.

ii) If $rank(\alpha_f(p)) = 2$, then $Im(A_p)$ is a 2-plane and we may have one of the 3 following possibilities according to the relative position of this plane with respect to the cone C.

a) $ImA_p \cap C$ is a pair of intersecting lines. Then $A_p^{-1}(C)$ gives a couple of intersecting planes in $\mathbf{R}^3$, defining a pair of projective lines in Σ_p.

b) $ImA_p \cap C = \{0\}$. In this case $A_p^{-1}(C) = Ker(A_p)$, which defines a unique point in Σ_p.

c) $ImA_p \cap C$ is tangent to C along one of its generatrices. Then $A_p^{-1}(C)$ is a plane in $\mathbf{R}^3$, defining a projective line in Σ_p.

iii) If $rank(\alpha_f(p)) = 1$ we can also have 3 possibilities according to the relative position of the line $Im(A_p)$ with respect to C: lying inside, outside or on the cone. We have that in the two first cases $A_p^{-1}(C) = Ker(A_p)$ which defines a projective line in Σ_p. On the other hand, it is not difficult to see that under the third assumption $A_p^{-1}(C) = N_pM$ and thus all the directions are degenerate.

iv) If $rank(\alpha_f(p)) = 0$, we clearly have that $A_p^{-1}(C) = A_p^{-1}(0) = N_pM$ and the result follows. □

Proposition 2. *Let f be a generic embedding of a closed surface M in $\mathbf{R}^5$. Then $M = M_3 \cup M_2$.*

Proof. We must prove that generically the rank of the second fundamental form at any point is at least 2. In fact, if it were lower at some point p we would have that the three normal vectors $a_{20}e_3 + b_{20}e_4 + c_{20}e_5$, $a_{11}e_3 + b_{11}e_4 + c_{11}e_5$ and $a_{02}e_3 + b_{02}e_4 + c_{02}e_5$ would be mutually linearly dependent at this point. But this is equivalent to the vanishing of at least 4 quadratic equations in the variables a_{ij}, b_{ij} and c_{ij}, which can be taken as coordinates in the jet space $J^2(M, \mathbf{R}^5)$. The zeroes of these equations determine a stratified subset V of codimension at least 4 in $J^2(M, \mathbf{R}^5)$ and the Thom transversality theorem ([**9**]) ensures us that the image of the 2-jet, $j^2 f : M \to J^2(M, \mathbf{R}^5)$ of any generic embedding $f : M \to \mathbf{R}^5$ must avoid such a subset. □

Let $\Delta(p) = det(\alpha_f(p))$. It is clear that $\Delta^{-1}(0) = M_2 \cup M_1 \cup M_0$. In the following proposition we analyze the properties of the set $\Delta^{-1}(0)$ for a generic embedding f.

Proposition 3. *Let f be a generic embedding of a closed surface M in $\mathbf{R}^5$. Then,*

a) *M_3 is an open subset of M.*

b) $\Delta^{-1}(0) = M_2$ *is a regularly embedded curve.*

Proof. $M - M_3 = \Delta^{-1}(0)$ and since Δ is a continuous function on M, M_3 must be an open region in M.

Let

$$\begin{aligned} f : \mathbf{R}^2, 0 &\longrightarrow \mathbf{R}^5 \\ (x, y) &\longmapsto (x, y, f_1(x, y), f_2(x, y), f_3(x, y)) \end{aligned}$$

be the local representation of M in the Monge form at a point $p \in M$. In these coordinates $\Delta(p) = f_{1xx}f_{2xy}f_{3yy} - f_{1xy}f_{2xx}f_{3yy} - f_{1xx}f_{2yy}f_{3xy} + f_{1yy}f_{2xx}f_{3xy} + f_{1xy}f_{2yy}f_{3xx} - f_{1yy}f_{2xy}f_{3xx}$. It follows from this expression that, under appropriate transversality conditions on the 3-jet of f, the set $\Delta = 0$ represents a curve possibly with isolated singular points determined by the vanishing of the derivatives of the function Δ. We observe that the orthogonality property of the frame $\{e_1, e_2, e_3, e_4, e_5\}$ is irrelevant for our study. In fact, a change of basis in N_pM preserves the relative position of $Im(A_p)$ with the cone C in Q, and thus the sets M_3, $M_2(j)$ and M_1. So we can take $\{e_3, e_4, e_5\}$ such that e_5 generates $Ker(A_p)$.

If $p \in M_2(2)$, we choose $\{e_3, e_4\}$ as the two degenerate directions in N_pM. Furthermore, we can also make a change of coordinates in the source, such that the two degenerate directions correspond to the quadratic forms x^2 and y^2 in C. Thus f can be locally written as

$$f(x, y) = (x, y, x^2 + R_1(x, y), y^2 + R_2(x, y), R_3(x, y)),$$

where $R_i \in m^3$, i.e., all the derivatives of the R_i vanish up to order 3, $i = 1, 2, 3$.

If $p \in M_2(0)$, then $Im(A_p) \cap C = \{(0, 0, 0)\}$ is tangent to C and we take e_5 as the generator of $A_p^{-1}(Im(A_p) \cap C)$. With additional change of coordinates in the source, f can be written as

$$f(x, y) = (x, y, x^2 - y^2 + R_1(x, y), xy + R_2(x, y), R_3(x, y)).$$

If $p \in M_2(1)$ then analogously, f can be written as

$$f(x, y) = (x, y, x^2 + R_1(x, y), xy + R_2(x, y), R_3(x, y)).$$

In each of the above cases it is a simple (but tedious) calculation to verify that under generic conditions on the 3-jet of f at $(0,0)$, the point p is a regular point of $\Delta^{-1}(0)$. $\square$

Remark. It follows from its definition that the set $M_2(1)$ is the union of the isolated regular points of $\Delta^{-1}(0)$.

A unit vector $v \in \Sigma_p$ is called **binormal direction** for M if and only if f_v has a singularity of cusp type or worse (i.e., the $\mathcal{A}$-codimension of f_v is at least 2) at p.

It follows from above that these singularities are generically of one of the following types

1) A_3 (cusp) for points p in an open region of M.

2) A_4 (swallowtail) for points p lying along curves in M.

3) A_5 (butterfly) at isolated points of M.

4) D_4 (elliptic and hyperbolic umbilic) for the point p varying along curves in M.

5) D_5 (parabolic umbilic) at isolated points of M.

We call these directions binormal by analogy to the case of curves in $\mathbf{R}^3$. In this case the tangent hyperplane orthogonal to the binormal direction has higher order of contact with M (see [**12**]).

The genericity conditions for a locally stable family of height functions on a closed surface ensure that at each point of the open and dense subset M_3, the number of binormals must be finite. We shall see in the next section that this number is different from zero at every point of M.

Following Feldman ([**5**], [**7**]), we say that a point p of M is **2-singular** or an **inflection point** whenever the linear map $T_p^2 f : T_p^2 M \to T_{f(p)}^2 \mathbf{R}^5$ is not injective. By choosing local coordinates $\{x, y\}$ at p in M, we have that the linear subspace $T_p^2 M$ is generated by the vectors $\left\{ \frac{\partial f}{\partial x}|_p, \frac{\partial f}{\partial y}|_p, \frac{\partial^2 f}{\partial x^2}|_p, \frac{\partial^2 f}{\partial x \partial y}|_p, \frac{\partial^2 f}{\partial y^2}|_p \right\}$. Thus the definition of 2-singular point amounts to asking that the vectors $\left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2} \right\}$ be linearly dependent at p. An embedding $f : M \to \mathbf{R}^5$ is said to be

regular of order 2 if there are no 2-singular points in M.

Theorem 4. *For an embedding $f : M \to \mathbf{R}^5$, the following conditions are equivalent,*

a) *A point $p \in M$ is 2-singular.*

b) $\Delta(p) = 0$ *(i.e., $p \in M_2(0) \cup M_2(1) \cup M_2(2) \cup M_1 \cup M_0$).*

c) *The point $p \in M$ is a singularity of corank 2 for some height function on M.*

Proof. Let $f : M \to \mathbf{R}^5$ be given in the Monge form in a neighborhood of the point p as in the proof of Proposition 3 and let $\alpha_f(p)$ be the matrix of the second fundamental form of f with respect to this normal form. We observe that a point $p \in M$ is regular of order 2 if and only if the following matrix has maximal rank

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_{20} & a_{11} & a_{02} \\ 0 & 0 & b_{20} & b_{11} & b_{02} \\ 0 & 0 & c_{20} & c_{11} & c_{02} \end{bmatrix}.$$

This is equivalent to asking that $rank(\alpha_f(p)) = 3$ and this proves that a) and b) are equivalent. Suppose on the other hand that p is a corank 2 singularity for some height function, say $f_v = f_{e_5}$ on M. This implies that the quadratic form $d^2 f \cdot e_5$ (i.e., the second fundamental form in the direction e_5) vanishes at p. But this means that the matrix $\alpha_f(p)$ has a null row and therefore we have $\Delta(p) = 0$. Hence c) implies b). Conversely, if $\Delta(p) = 0$, then it is not hard to show that by a convenient change of coordinates we can assume that $d^2 f \cdot e_5 = 0$ and thus p is a corank 2 singularity for the height function in the normal direction e_5. □

An immediate consequence of this and Proposition 2 is the following:

Corollary 5. *The inflection points of a generic embedding $f : M \to \mathbf{R}^5$ form regular closed curves.*

Remark. For a generic embedding, the singularities of corank 2 of height functions on M can only be of umbilic type D_k, with $k = 4, 5$.

Corollary 6. *Given an embedding $f : M \to \mathbf{R}^5$, we have the following:*

a) *If $p \in M_2$ then there is a unique direction $v \in \Sigma_p$ such that the height function f_v has a corank 2 singularity at p. This direction is given by the intersection of the 2 projective lines when $p \in M_2(2)$, and by the unique degenerate direction when $p \in M_2(0)$.*

b) *If $p \in M_1$ then it is a corank 2 singularity for the height functions corresponding to all the degenerate directions of M at p.*

c) *If $p \in M_0$ then it is a corank 2 singularity for the height functions corresponding to all the normal directions to M at p.*

Proof. As before, $p \in M_2$ if and only if $rank(\alpha_f(p)) = 2$. But this means that $Ker(\alpha_f(p))$ is 1-dimensional and this provides the unique direction for which the associated quadratic form vanishes. But this implies that p is a corank 2 singularity for the corresponding height function.

On the other hand, if $p \in M_1$, then $rank(\alpha_f(p)) = 1$ and $Ker(\alpha_f(p))$ has dimension 2. Therefore there is a whole projective line of degenerate directions for which the corresponding height functions have a corank 2 singularity. These are, generically, of type $D_4^{\pm}$.

Finally, if $p \in M_0$, we have that $Ker(\alpha_f(p)) = N_pM$, so all the quadratic parts of all the height functions vanish and thus all these functions have a singularity of corank 2 at p. $\square$

Remark. Observe that when $p \in M_2(0) \cup M_1 \cup M_0$ the degenerate directions coincide with the binormals, whereas when $p \in M_2(2) \cup M_2(1)$ there are degenerate directions which are not binormals for M at p.

We characterize below the different types of inflection points of a surface M in 5-space in terms of the local geometry of the projections of M into convenient hyperplanes. We recall from [**14**] that a point on a surface embedded in 4-space can be hyperbolic, parabolic or elliptic according to the existence of two, one or zero binormal directions for

M at the point. Inflection points for surfaces in 4-space are corank 2 singularities of height functions on the surface. A more geometrical characterization of these points in terms of the curvature ellipses can be found in [**10**].

Theorem 7. *Let $p \in M$ be an inflection point, so there is some $v \in N_pM$ such that p is a singularity of corank 2 of the height function f_v. Let H_v be the orthogonal hyperplane to v passing through p and $\pi_v : \mathbf{R}^5 \to H_v$ the orthogonal projection in the direction v. Then, $g_v = \pi_v \circ f : M \to H_v \equiv \mathbf{R}^4$ provides a local embedding of M in $\mathbf{R}^4$. Moreover,*

i) *$p \in M_2(2)$ if and only if p is a hyperbolic point of the surface $g_v(M)$ in $\mathbf{R}^4$.*

ii) *$p \in M_2(0)$ if and only if p is an elliptic point of $g_v(M)$.*

iii) *$p \in M_2(1)$ if and only if p is a parabolic point of $g_v(M)$.*

iv) *$p \in M_1 \cup M_0$ if and only if p is an inflection point of $g_v(M)$.*

Proof. Since p is an inflection point we have $rank(\alpha_f(p)) < 3$. Let $v \in N_pM$ such that p is an umbilic singularity for the height function f_v. Then $v \in Ker(\alpha_f(p))$. Observe now that $Ker(\pi_v) =< v >\subset Ker(\alpha_f(p)) = Ker(A_p)$. Hence $A_p : N_pM \to Q$ induces a linear map $\tilde{A}_p : N_pM \cap H_v \to Q$. The subset $\tilde{A}_p^{-1}(C)$ gives the binormal directions of the surface $g_v(M)$ at p. It is not difficult to verify that

i) If $p \in M_2(2)$ then $\tilde{A}_p^{-1}(C) = A_p^{-1}(C) \cap H_v$ consists of 2 distinct binormal directions. So p is a hyperbolic point of $g_v(M)$. ii) If $p \in M_2(1)$ then $\tilde{A}_p^{-1}(C) = A_p^{-1}(C) \cap H_v$ is the unique binormal direction. So p is a parabolic point of $g_v(M)$. iii) If $p \in M_2(0)$ then $\tilde{A}_p^{-1}(C) = Ker(A_p) \cap H_v = \{0\}$, which means that there are no binormal directions at p. So p is an elliptic point of $g_v(M)$. iv) If $p \in M_1 \cup M_0$ then $\tilde{A}_p^{-1}(C) = Ker(A_p) \cap H_v$ gives a direction corresponding to a height function of corank 2. So p is an inflection point of $g_v(M)$. $\square$

3. Asymptotic directions. Let $p \in M$ and $v \in \Sigma_p$ a binormal direction at p. Then f_v has a degenerate singularity at p and hence $rank(Hess\,(f_v)) < 2$. Therefore $\dim\{Ker(Hess\,(f_v))\} \geq 1$. An

asymptotic direction at p, associated to the binormal direction v is any unit vector in T_pM that lies in the kernel of the quadratic form given by the hessian of f_v. Therefore, any binormal direction at p defines some asymptotic direction at p. In fact, since p is a singularity of type $A_k,\ k > 2$ of the height function f_v, then $rank(Hess\, f_v) = 1$, and thus there is a unique asymptotic direction associated to the binormal v at p. Whereas if p is a singularity of umbilic type for some height function (i.e., an inflection point), then there is a whole circle of asymptotic directions associated to v at p.

Remark. If b is a binormal direction at p, the hyperplane H_b orthogonal to b passing through p has higher order contact with M at p. This contact occurs along the elements $\theta \in Ker(Hess\,(f_b))$, which means that the straight line through p in the direction θ must have higher order of contact with M at p than most tangent lines to M at p. This is why θ is called an **asymptotic** direction. This clearly generalizes the definition of asymptotic directions for surfaces in $\mathbf{R}^4$ (see [**14**],[**15**]). It is worth pointing out that in the last case the concepts of degenerate and binormal directions coincide.

Let v be a degenerate direction at a point p of M_3, so $rank(Hess\,(f_v)) = 1$, and let θ be a tangent vector in the kernel of the quadratic form $Hess\,(f_v)(p)$. We denote by γ_θ the normal section of the surface M in the tangent direction θ. That is, γ_θ is a curve in the 4-space $V_\theta = \langle\theta\rangle \oplus N_pM$, obtained as the intersection of this 4-space with M. The restriction of the family of height functions $\lambda(f)$ to (some parametrization of) the curve γ_θ gives the family of height functions on this curve. Now, if we take into account that the binormal (or 3rd normal vector) n_3 of a curve in 4-space can be characterized by the fact that the height function over the curve, corresponding to the direction n_3, has a singularity of type $A_k,\ k \geq 3$, we obtain the following geometrical characterization of the asymptotic directions at regular points of surfaces in 5-space. (A proof of a similar assertion for curves in 3-space can be found in [**2**], the case of curves in 4-space is proven in [**3**].)

Proposition 8. *Let $p \in M_3$ and $v \in N_pM$ a degenerate direction for M at p. Let θ be a tangent direction in $Ker(Hess\,(f_v)(p))$. Then θ is an asymptotic direction corresponding to the binormal v if and only if v is the binormal direction at p for the curve γ_θ in the 4-space V_θ.*

Theorem 9. *There are at least one and at most* 5 *asymptotic directions at each point of* M_3.

Proof. Let M be given in the Monge form

$$f(x,y) = (x, y, f_1(x,y), f_2(x,y), f_3(x,y))$$

in a neighborhood of $p = (0,0) \in \mathbf{R}^2$, with $f_i(x,y) = Q_i(x,y) + K_i(x,y) + R_i(x,y)$, where Q_i are quadratic forms, K_i are cubic forms and $R_i \in m^4$, $i = 1,2,3$. Since $p \in M_3$, the 3 quadratic forms $Q_1(x,y)$, $Q_2(x,y)$ and $Q_3(x,y)$ must be linearly independent (in Q) and without loss of generality we can take local coordinates at p in such a way that $Q_3(x,y) = -(x^2+y^2)$ and $K_3(x,y) = 0$. To simplify the notation we denote as $Q_i(u,v)$, $i = 1,2,3$ the bilinear form $d^2 f_i(x,y)(u,v)$, where $u, v \in T_qM$ and $q = (x,y)$ varies in a small enough neighborhood of p in $\mathbf{R}^2$. Analogously $K_i(u^3)$, $i = 1,2,3$ denotes the cubic form associated to f at the point q acting on a vector $u \in T_qM$.

Let $v \in N_qM$ be a solution of the equation $A_q(v) = 0$. Then $Hess\,(f_v(q))$ is a degenerate quadratic form and so there is $u \in T_qM$ such that $Hess\,(f_v(q))(u,w) = 0$, $\forall\, w \in T_qM$. By writing $v = v_3e_3 + v_4e_4 + v_5e_5 \in N_qM$ in terms of the normal frame $\{e_3, e_4, e_5\}$, we have $v_3Q_1(u,w) + v_4Q_2(u,w) - v_5\langle u, w\rangle = 0$. This expression must be true in particular for the vector u and a vector $w \in T_qM$ orthogonal to u, so we have the equations,

$$v_3Q_1(u,u) + v_4Q_2(u,u) - v_5(u_1^2 + u_2^2) = 0 \tag{1}$$

$$v_3Q_1(u,w) + v_4Q_2(u,w) = 0. \tag{2}$$

On the other hand, q is a singular point of cusp type or worse if the vector u satisfies $v_3K_1(u^3) + v_4K_2(u^3) + v_5K_3(u^3) = 0$ (see [**12**]). And since in the chosen local coordinates $K_3(x,y) = 0$, this gives

$$v_3K_1(u^3) + v_4K_2(u^3) = 0. \tag{3}$$

Once given v_3 and v_4, we can obtain v_5 from (1). On the other hand, eliminating v_3 and v_4 in (2) and (3) gives

$$K_1(u^3)Q_2(u,w) - K_2(u^3)Q_1(u,w) = 0.$$

Since the coordinates of the vector w (orthogonal to u in T_qM) can be given as a linear combination of those of u, we obtain that for each q in a neighborhood of $p = (0, 0)$, the above equation is defined by a quintic form in two variables. This gives the differential equation for the asymptotic lines in M. We observe that this equation cannot be identically zero on regular points of order 2 of a generic surface. $\square$

Corollary 10. *For a generically embedded surface M in $\mathbf{R}^5$ there are at least 1 and at most 5 locally defined fields of asymptotic directions on M, whose singularities occur on the curve $\Delta^{-1}(0)$.*

It is a well known consequence of the work of H. Hopf that a compact connected manifold M admits a global nowhere zero vector field if and only if its Euler number is zero. Therefore, we can state the following

Corollary 11. *Let M be a compact connected surface with non-vanishing Euler number generically immersed in $\mathbf{R}^5$. If M admits a globally defined field of asymptotic directions, then it has necessarily singular points of order 2.*

An example of a 2-regular embedding of the 2-sphere in 5-space is given by the restriction to S^2 of the Veronese map of order 2 $V : \mathbf{R}^3 \to \mathbf{R}^6$, given by $V(x, y, z) = (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz)$ (see [**4**]). This can be seen as an embedding of the projective plane in 4-sphere. Studying the singularities of the family of height functions on this surface leads to the conclusion that all the binormals give rise to singularities of infinite codimension. This is an extremely degenerate example from the viewpoint of contact of the surface with its set of tangent lines (and hence with the tangent hyperplanes containing these lines).

4. Stereographic projection and surfaces in $\mathbf{R}^4$. Given a surface M in $\mathbf{R}^4$ we define an osculating hypersphere of M at a point p as a hypersphere whose order of contact with M at p is at least 3. This order of contact can be measured through the distance squared function from the center of the given hypersphere (see [**13**]). Having order of contact k means that this function has a singularity of codimension

$k-1$, and at the contact point these are generically of type A_k or D_{k-1}. Semi-umbilic points of M are points where some distance squared function has a singularity of corank 2. These are generically of type $D_4^{\pm}$ along curves in M and D_5 at isolated points of these curves ([**12**]). In fact, Montaldi proved that the semi-umbilic points of a generically embedded surface in $\mathbf{R}^4$ lie along curves on the surface. He saw that the possible singularities of these curves correspond to the vanishing of the second fundamental form of the embedding (i.e., all the height functions at the point have zero quadratic part). Since this cannot happen for a generic embedding of a surface in $\mathbf{R}^4$, these curves are smoothly embedded on the surface.

Composing an embedding $g : M \to \mathbf{R}^4$, of a surface in 4-space with the inverse of the stereographic projection, leads to another embedding of the surface M in 5-space,

$$f : M \longrightarrow \mathbf{R}^4 \longrightarrow S^4 \longrightarrow \mathbf{R}^5$$

where $\psi : \mathbf{R}^4 \to S^4$ denotes the inverse of the stereographic projection, and the map in the right-hand side is the natural inclusion of S^4 in $\mathbf{R}^5$. Now, since the stereographic projection transforms hyperspheres of $\mathbf{R}^4$ into hypercircles of S^4 preserving their respective contacts with $g(M)$ and $f(M)$, it can be seen that any osculating hypersphere having a given contact with $g(M)$ is in correspondence, through ψ, with an osculating hyperplane having the same contact with $f(M)$ in $\mathbf{R}^5$ ([**16**]). Consequently we can state,

Proposition 12. *The inverse ψ of the stereographic projection takes the semi-umbilic points of a surface M immersed in $\mathbf{R}^4$ into the inflection points of the surface $\psi(M) \in S^4 \in \mathbf{R}^5$.*

Now, since we know that for a generic embedding the inflection points form regular closed curves (Corollary 5), we have that Montaldi's result can be considered as a particular case, corresponding to the hyperspherical surfaces of $\mathbf{R}^5$, of ours.

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