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A CHANGE OF SCALE FORMULA FOR WIENER INTEGRALS OF UNBOUNDED FUNCTIONS

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ABSTRACT. Cameron and Storvick discovered change of scale formulas for Wiener integrals of bounded functions in a Banach algebra S on classical Wiener space. Yoo and Skoug extended these results to abstract Wiener space for a more generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$ which corresponds to the Banach algebra S on classical Wiener space. In this paper we present a change of scale formula for Wiener integrals of functions on abstract Wiener space which need not be bounded or continuous.

1. Introduction. It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [**3**] and under translations [**2**]. Cameron and Storvick [**5**] expressed the analytic Feynman integral for a rather large class of functionals as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0,1], m_w)$ [**6**]. In [**20, 21, 22**], Yoo, Yoon and Skoug extended these results to classical Yeh-Wiener space and to an abstract Wiener space (H, B, ν). In particular, Yoo and Skoug [**20**] established a change of scale formula for Wiener integrals of functions in the Fresnel class $\mathcal{F}(B)$ on abstract Wiener space, and then they [**21**] developed this formula for a more generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$. But functions in $\mathcal{F}(B)$ and $\mathcal{F}_{A_1,A_2}$ are bounded.

In this paper we establish a change of scale formula for Wiener integrals of functions of the form

$$F(x) = G(x)\Psi((e_1, x)^\sim, \dots, (e_n, x)^\sim)$$

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for $G \in \mathcal{F}(B)$ and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbf{R}^n)$, $1 \le p < \infty$, and ϕ is a Fourier transform of a measure of bounded variation over $\mathbf{R}^n$. Note that $F(x)$ need not be bounded or continuous.

2. Definitions and preliminaries. Let H be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $|||\cdot|||$ be a measurable norm on H with respect to the Gaussian cylinder set measure σ on H. Let B denote the completion of H with respect to $|||\cdot|||$. Let ι denote the natural injection from H to B. The adjoint operator ι^* of ι is one-to-one and maps B^* continuously onto a dense subset of H^* where B^* and H^* are the topological dual of B and H respectively. By identifying H with H^* and B^* with $\iota^* B^*$, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle h, x \rangle = (h, x)$ for all h in H and x in B^* where $(\cdot, \cdot)$ denotes the natural dual pairing between B and B^*. By a well-known result of Gross [**14**] $\sigma \circ \iota^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H, B, ν) is called an abstract Wiener space. For more details, see [**13, 16, 17, 18**].

Let $\mathbf{C}, \mathbf{C}_+$ and $\mathbf{C}_+^\sim$ denote the complex numbers, the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively.

Definition 2.1. Let F be a functional on B such that the integral

$$J_F(\lambda) = \int_B F(\lambda^{-1/2} x)\, d\nu(x) \tag{2.1}$$

exists for all $\lambda > 0$. If there exists an analytic function $J_F^*(z)$ on $\mathbf{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then we call $J_F^*(z)$ the analytic Wiener integral of F over B with parameter z, and for $z \in \mathbf{C}_+$ we write

$$I_a^z[F(\cdot)] = J_F^*(z). \tag{2.2}$$

Let q be a nonzero real number. If the following limit (2.3) exists, we define it to be the analytic Feynman integral of F over B with parameter q and we write

$$I_a^q[F(\cdot)] = \lim_{z \to -iq} I_a^z[F(\cdot)] \tag{2.3}$$

where z approaches $-iq$ through values in $\mathbf{C}_+$.

Let $\{e_n\}$ denote a complete orthonormal (CON) system in H such that the e_n's are in B^*. For each $h \in H$ and $x \in B$, we introduce a stochastic inner product $(\cdot,\cdot)^\sim$ on $H \times B$ defined by

$$(2.4)\qquad (h,x)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $h \in H$, $(h,x)^\sim$ exists for ν-almost everywhere $x \in B$ and it is a Borel measurable function on B having a Gaussian distribution with mean 0 and variance $\|h\|^2$. Also if both h and x are in H, then $(h,x)^\sim = \langle h, x\rangle$.

Let $M(H)$ denote the space of finite complex Borel measures μ on H. Then $M(H)$ is a Banach algebra over the complex numbers under convolution as multiplication with the norm $\|\mu\|$ where $\|\mu\|$ is the total variation of μ.

Given two $\mathbf{C}$-valued functions F and G on B, we say that $F = G$, s-almost everywhere if $F(\alpha x) = G(\alpha x)$ for ν-almost everywhere $x \in B$ for all $\alpha > 0$. For a function F on B we denote by $[F]$ the s equivalence class of functions which equal F s-almost everywhere.

Definition 2.2. The Fresnel class $\mathcal{F}(B)$ is defined as the space of all functions G on B which have the form

$$(2.5)\qquad G(x) = \int_H \exp\{i(h,x)^\sim\}\, d\mu(h)$$

for $\mu \in M(H)$.

In fact, $\mathcal{F}(B)$ is the space of all s-equivalence classes of functions of the form (2.5) since we identify functions which coincide s-almost everywhere on B. It is well known [**10, 16**] that $\mathcal{F}(B)$ is a Banach algebra and the mapping $\mu \to G$ is a Banach algebra isomorphism where μ and G are related by (2.5).

Theorem 2.3 [**16**]. *Let $G \in \mathcal{F}(B)$ be given by* (2.5). *Then the analytic Feynman integral of F over B exists for all real $q \neq 0$ and*

$$(2.6)\qquad I_a^q[G(\cdot)] = \int_H \exp\left\{-\frac{i}{2q}\|h\|^2\right\} d\mu(h).$$

In particular, for each $z \in \mathbf{C}_+$,

$$I_a^z[G(\cdot)] = \int_H \exp\left\{-\frac{1}{2z}\|h\|^2\right\} d\mu(h). \tag{2.7}$$

3. Change of scale formulas. We begin this section by giving some existence theorems of the analytic Wiener integral and the analytic Feynman integral of functions on abstract Wiener space which need not be bounded or continuous.

Theorem 3.1. *Let* $F(x) = G(x)\psi((e_1,x)^\sim,\dots,(e_n,x)^\sim)$ *where* $G \in \mathcal{F}(B)$, $\psi \in L_p(\mathbf{R}^n)$, $1 \le p < \infty$ *and* $\{e_1,\dots,e_n\}$ *is an orthonormal set in* H. *Then for each* $z \in \mathbf{C}_+$, F *is analytic Wiener integrable*; *and if* G *is given by* (2.5), *then*

(3.1)

$$I_a^z[F(\cdot)] = \left(\frac{z}{2\pi}\right)^{n/2} \int_H \int_{\mathbf{R}^n} \exp\left\{\frac{1}{2z}\left[\sum_{k=1}^n (izv_k + \langle e_k,h\rangle)^2 - \|h\|^2\right]\right\} \times \psi(v_1,\dots,v_n)\, dv_1 \dots dv_n\, d\mu(h).$$

Proof. Let λ be a positive real number. We begin by evaluating the Wiener integral

$$\int_B F(\lambda^{-1/2}x)\, d\nu(x) = \int_B \int_H \exp\{i\lambda^{-\frac{1}{2}}(h,x)^\sim\}\psi(\lambda^{-\frac{1}{2}}(e_1,x)^\sim,\dots,\lambda^{-\frac{1}{2}}(e_n,x)^\sim) d\mu(h)\, d\nu(x).$$

Using the Fubini theorem, we change the order of integration in the above equation. In fact, since $\psi \in L_p(\mathbf{R}^n)$ and $\mu \in M(H)$, we have

$$\int_H \int_B \left|\psi(\lambda^{-1/2}(e_1,x)^\sim,\dots,\lambda^{-1/2}(e_n,x)^\sim)\right| d\nu(x)\, d\mu(h) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_H \int_{\mathbf{R}^n} |\psi(v_1,\dots,v_n)| \exp\left\{-\frac{\lambda}{2}\sum_{k=1}^n v_k^2\right\} dv_1 \dots dv_n\, d\mu(h) < \infty.$$

)

For a given $h \in H$, using the Gram-Schmidt process, we obtain $e_{n+1} \in H$ such that $\{e_1, \dots, e_n, e_{n+1}\}$ forms an orthonormal set in H and $h = \sum_{k=1}^{n+1} c_k e_k$, where

$$(3.2) \qquad c_k = \begin{cases} \langle e_k, h\rangle & k = 1, \dots, n, \\ \left(\|h\|^2 - \sum_{j=1}^{n} \langle e_j, h\rangle^2\right)^{1/2} & k = n+1. \end{cases}$$

Hence by the Wiener integration formula, we have

$$\begin{aligned} (3.3) \quad & \int_B F(\lambda^{-1/2}x)\, d\nu(x) \\ &= \int_H \int_B \exp\left\{ i\lambda^{-1/2} \sum_{k=1}^{n+1} c_k (e_k, x)^{\sim} \right\} \\ &\qquad \times \psi(\lambda^{-1/2}(e_1, x)^{\sim}, \dots, \lambda^{-1/2}(e_n, x)^{\sim})\, d\nu(x)\, d\mu(h) \\ &= \left(\frac{\lambda}{2\pi}\right)^{(n+1)/2} \int_H \int_{\mathbf{R}^{n+1}} \exp\left\{ i \sum_{k=1}^{n+1} c_k v_k - \frac{\lambda}{2} \sum_{k=1}^{n+1} v_k^2 \right\} \\ &\qquad \times \psi(v_1, \dots, v_n)\, dv_1 \cdots dv_{n+1}\, d\mu(h) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_H \int_{\mathbf{R}^n} \exp\left\{ \frac{1}{2\lambda} \left[\sum_{k=1}^{n} (i\lambda v_k + \langle e_k, h\rangle)^2 - \|h\|^2 \right] \right\} \\ &\qquad \times \psi(v_1, \dots, v_n)\, dv_1 \cdots dv_n\, d\mu(h). \end{aligned}$$

The third equality in (3.3) is obtained by applying the following integration formula

$$(3.4) \qquad \int_{\mathbf{R}} \exp\{-au^2 + ibu\}\, du = \left(\frac{\pi}{a}\right)^{1/2} \exp\left\{ -\frac{b^2}{4a} \right\},$$

for any $a \in \mathbf{C}_+$ and real number b.

Now we will show that the righthand side of the third equality in (3.3) is an analytic function of $\lambda \in \mathbf{C}_+$. Let $\lambda_l \to \lambda$ in $\mathbf{C}_+$. Then there exists $\alpha > 0$ such that $\operatorname{Re} \lambda_l \ge \alpha$ for sufficiently large l and, by using the Bessel inequality, we have

$$\begin{aligned}&\Big|\exp\Big\{\frac{1}{2\lambda_l}\Big[\sum_{k=1}^{n}(i\lambda_l v_k+\langle e_k,h\rangle)^2-\|h\|^2\Big]\Big\}\psi(v_1,\dots,v_n)\Big|\\&=\exp\Big\{-\frac{\operatorname{Re}\lambda_l}{2|\lambda_l|^2}\Big[\|h\|^2-\sum_{k=1}^{n}\langle e_k,h\rangle^2\Big]-\frac{\operatorname{Re}\lambda_l}{2}\sum_{k=1}^{n}v_k^2\Big\}|\psi(v_1,\dots,v_n)|\\&\le\exp\Big\{-\frac{\alpha}{2}\sum_{k=1}^{n}v_k^2\Big\}|\psi(v_1,\dots,v_n)|.\end{aligned}$$

Since $\psi\in L_p(\mathbf{R}^n)$ and $\mu\in M(H)$, the righthand side of the above inequality is integrable on $H\times\mathbf{R}^n$. Hence, by the dominated convergence theorem, the last expression in (3.3) is a continuous function of $\lambda\in\mathbf{C}_+$. Moreover, by using the Morera theorem, we can easily show that it is an analytic function of λ throughout $\mathbf{C}_+$, and this completes the proof. □

If we restrict our attention to the case $p=1$, we obtain the following existence theorem of the analytic Feynman integral. But, if $p>1$, we are not able to justify the application of the dominated convergence theorem in the proof of Corollary 3.2 below. Thus, in this case we could not claim the existence of the analytic Feynman integral.

Corollary 3.2. *Let* $F(x)=G(x)\psi((e_1,x)^\sim,\dots,(e_n,x)^\sim)$ *where* $G\in\Gamma(B)$, $\psi\in L_1(\mathbf{R}^n)$ *and* $\{e_1,\dots,e_n\}$ *is an orthonormal set in* H. *Then for each real* $q\neq 0$, F *is analytic Feynman integrable*; *and if* G *is given by* (2.5), *then*

(3.5)

$$I_a^q[F(\cdot)]=\Big(-\frac{iq}{2\pi}\Big)^{n/2}\int_H\int_{\mathbf{R}^n}\exp\Big\{\frac{i}{2q}\Big[\sum_{k=1}^{n}(qv_k+\langle e_k,h\rangle)^2-\|h\|^2\Big]\Big\}\times\psi(v_1,\dots,v_n)\,dv_1\cdots dv_n\,d\mu(h).$$

Let $\hat{M}(\mathbf{R}^n)$ be the set of functions ϕ defined on $\mathbf{R}^n$ by

$$\phi(r_1,\dots,r_n)=\int_{\mathbf{R}^n}\exp\Big\{i\sum_{k=1}^{n}r_kt_k\Big\}\,d\rho(t_1,\dots,t_n) \tag{3.6}$$

where ρ is a complex Borel measure of bounded variation on $\mathbf{R}^n$.

Theorem 3.3. *Let* $F(x) = G(x)\phi((e_1, x)^\sim, \dots, (e_n, x)^\sim)$ *where* $G \in \mathcal{F}(B)$, $\phi \in \hat{M}(\mathbf{R}^n)$ *and* $\{e_1, \dots, e_n\}$ *is an orthonormal set in* H. *Then for each* $z \in \mathbf{C}_+$, F *is analytic Wiener integrable*; *and if* G *and* ϕ *are given by* (2.5) *and* (3.6), *respectively, then*

(3.7)
$$I_a^z[F(\cdot)] = \int_H \int_{\mathbf{R}^n} \exp\left\{-\frac{1}{2z}\left[\|h\|^2 + \sum_{k=1}^n 2t_k\langle e_k, h\rangle + \sum_{k=1}^n t_k^2\right]\right\} \times d\rho(t_1, \dots, t_n)\, d\mu(h).$$

Moreover, the righthand side of (3.7) *is a continuous function of* z *on* $\mathbf{C}_+^\sim$.

Proof. By the same method as in the proof of Theorem 3.1, we have for a positive real number λ,

$$\begin{aligned}
&\int_B F(\lambda^{-1/2}x)\, d\nu(x) \\
&= \int_H \int_{\mathbf{R}^n} \int_B \exp\left\{i\lambda^{-1/2} \sum_{k=1}^{n+1} c_k (e_k, x)^\sim + i\lambda^{-1/2} \sum_{k=1}^n t_k (e_k, x)^\sim\right\} \\
&\qquad \times d\nu(x)\, d\rho(t_1, \dots, t_n)\, d\mu(h) \\
&= (2\pi)^{-\frac{n+1}{2}} \int_H \int_{\mathbf{R}^n} \int_{\mathbf{R}^{n+1}} \exp\left\{i\lambda^{-\frac{1}{2}} \sum_{k=1}^{n+1} c_k u_k + i\lambda^{-\frac{1}{2}} \sum_{k=1}^n t_k u_k \right. \\
&\qquad \left. - \frac{1}{2}\sum_{k=1}^{n+1} u_k^2\right\} du_1 \cdots du_{n+1}\, d\rho(t_1, \dots, t_n)\, d\mu(h) \\
&= \int_H \int_{\mathbf{R}^n} \exp\left\{-\frac{1}{2\lambda}\left[c_{n+1}^2 + \sum_{k=1}^n (c_k + t_k)^2\right]\right\} d\rho(t_1, \dots, t_n) d\mu(h) \\
&= \int_H \int_{\mathbf{R}^n} \exp\left\{-\frac{1}{2\lambda}\left[\|h\|^2 + \sum_{k=1}^n 2t_k\langle e_k, h\rangle + \sum_{k=1}^n t_k^2\right]\right\} d\rho(t_1, \dots, t_n) d\mu(h).
\end{aligned}$$

Using the Bessel inequality in the last expression above, we know that the exponential in the expression is bounded in absolute value by unity for $\lambda \in \mathbf{C}_+^{\sim}$. Since ρ is a complex Borel measure of bounded variation on $\mathbf{R}^n$, it follows that the righthand side of the last equality above is analytic in λ for $\lambda \in \mathbf{C}_+$ and is continuous in λ for $\lambda \in \mathbf{C}_+^{\sim}$, and hence this completes the proof. □

The following corollary follows immediately from Theorem 3.3.

Corollary 3.4. *Let* $F(x) = G(x)\phi((e_1, x)^{\sim}, \dots, (e_n, x)^{\sim})$ *where* $G \in \mathcal{F}(B)$, $\phi \in \hat{M}(\mathbf{R}^n)$ *and* $\{e_1, \dots, e_n\}$ *is an orthonormal set in* H. *Then for each real* $q \neq 0$, F *is analytic Feynman integrable*; *and if* G *and* ϕ *are given by* (2.5) *and* (3.6), *respectively, then*

$$(3.8)\quad I_a^q[F(\cdot)] = \int_H \int_{\mathbf{R}^n} \exp\left\{-\frac{i}{2q}\left[\|h\|^2 + \sum_{k=1}^n 2t_k\langle e_k, h\rangle + \sum_{k=1}^n t_k^2\right]\right\} d\rho(t_1, \dots, t_n)\, d\mu(h).$$

From the above results and the linearity of the analytic Wiener integral and the analytic Feynman integral on abstract Wiener space, we have the following corollary.

Corollary 3.5. *Let* $F(x) = G(x)\Psi((e_1, x)^{\sim}, \dots, (e_n, x)^{\sim})$ *where* $G \in \mathcal{F}(B)$, $\Psi = \psi + \phi \in L_p(\mathbf{R}^n) + \hat{M}(\mathbf{R}^n)$, $1 \leq p < \infty$, *and* $\{e_1, \dots, e_n\}$ *is an orthonormal set in* H. *Then for each* $z \in \mathbf{C}_+$, F *is analytic Wiener integrable. Moreover, if* G *and* ϕ *were given by* (2.5) *and* (3.6), *respectively, and* $\psi \in L_p(\mathbf{R}^n)$, *then*

$$(3.9)\quad \begin{aligned} I_a^z[F(\cdot)] = \left(\frac{z}{2\pi}\right)^{n/2} \int_H \int_{\mathbf{R}^n} \exp\left\{\frac{1}{2z}\left[\sum_{k=1}^n (izv_k + \langle e_k, h\rangle)^2 - \|h\|^2\right]\right\} \\ \times \psi(v_1, \dots, v_n)\, dv_1 \cdots dv_n\, d\mu(h) \\ + \int_H \int_{\mathbf{R}^n} \exp\left\{-\frac{1}{2z}\left[\|h\|^2 + \sum_{k=1}^n 2t_k\langle e_k, h\rangle + \sum_{k=1}^n t_k^2\right]\right\} \\ \times d\rho(t_1, \dots, t_n)\, d\mu(h). \end{aligned}$$

In case $p=1$ *for each real* $q\neq 0$, F *is analytic Feynman integrable and*

(3.10)
$$I_a^q[F(\cdot)] = \left(-\frac{iq}{2\pi}\right)^{n/2}\int_H\int_{\mathbf{R}^n}\exp\left\{\frac{i}{2q}\left[\sum_{k=1}^n (qv_k+\langle e_k,h\rangle)^2-\|h\|^2\right]\right\}$$
$$\times\,\psi(v_1,\dots,v_n)\,dv_1\cdots dv_n\,d\mu(h)$$
$$+\int_H\int_{\mathbf{R}^n}\exp\left\{-\frac{i}{2q}\left[\|h\|^2+\sum_{k=1}^n 2t_k\langle e_k,h\rangle+\sum_{k=1}^n t_k^2\right]\right\}$$
$$\times\,d\rho(t_1,\dots,t_n)\,d\mu(h).$$

Next we introduce two lemmas which play a key role in the rest of this section.

Lemma 3.6. *Let* $\psi\in L_p(\mathbf{R}^r)$, $1\le p<\infty$, *and* $z\in\mathbf{C}_+$ *and let* $\{e_1,\dots,e_n\}$ *be an orthonormal set in* H *with* $n>r$. *Let* $h\in H$ *and let*

$$K\equiv\int_B\exp\left\{\frac{1-z}{2}\,[(e_k,x)^\sim]^2+i(h,x)^\sim\right\}$$
$$\times\,\psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)\,d\nu(x).$$

Then

$$K=\left(\frac{z}{2\pi}\right)^{r/2}z^{-n/2}\exp\left\{\frac{z-1}{2z}\sum_{k=1}^n\langle e_k,h\rangle^2-\frac{1}{2}\,\|h\|^2\right\}$$
$$\times\int_{\mathbf{R}^r}\exp\left\{\frac{1}{2z}\sum_{k=1}^r(izu_k+\langle e_k,h\rangle)^2\right\}\psi(u_1,\dots,u_r)\,du_1\cdots du_r.$$

Proof. Using (3.2) and the integration formula (3.4), we have

$$\begin{aligned} K &= \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^{n}[(e_k,x)^\sim]^2 + i\sum_{k=1}^{n+1} c_k(e_k,x)^\sim\right\} \\ &\qquad\qquad \times \psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)\, d\nu(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathbf{R}^{n+1}} \exp\left\{\frac{1-z}{2}\sum_{k=1}^{n} u_k^2 + i\sum_{k=1}^{n+1} c_k u_k - \frac{1}{2}\sum_{k=1}^{n+1} u_k^2\right\} \\ &\qquad\qquad \times \psi(u_1,\dots,u_r)\, du_1\cdots du_{n+1} \\ &= \left(\frac{z}{2\pi}\right)^{r/2} z^{-n/2} \exp\left\{-\frac{1}{2z}\sum_{k=r+1}^{n} c_k^2 - \frac{1}{2}c_{n+1}^2\right\} \\ &\quad \times \int_{\mathbf{R}^r} \exp\left\{-\frac{z}{2}\sum_{k=1}^{r} u_k^2 + i\sum_{k=1}^{r} c_k u_k\right\}\psi(u_1,\dots,u_r)\, du_1\cdots du_r. \end{aligned}$$

By (3.2) we have the desired result. □

By the same method as in the proof of Lemma 3.6, we have the following lemma.

Lemma 3.7. *Let* $z, h, \{e_1,\dots,e_n\}$ *be given as in Lemma* 3.6. *Let*

$$K \equiv \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^{n}[(e_k,x)^\sim]^2 + i(h,x)^\sim + i\sum_{k=1}^{r} t_k(e_k,x)^\sim\right\} d\nu(x).$$

Then

$$K = z^{-n/2}\exp\left\{\frac{z-1}{2z}\sum_{k=1}^{n}\langle e_k,h\rangle^2 - \frac{1}{z}\sum_{k=1}^{r} t_k\langle e_k,h\rangle - \frac{1}{2z}\sum_{k=1}^{r} t_k^2 - \frac{1}{2}\|h\|^2\right\}.$$

Now we give a relationship between Wiener integral and analytic Wiener integral on abstract Wiener space.

Theorem 3.8. *Let* $\{e_n\}$ *be a complete orthonormal set in* H. *Let* $F(x) = G(x)\psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\psi \in$

$L_p(\mathbf{R}^r)$, $1 \leq p < \infty$. *Then for each* $z \in \mathbf{C}_+$, *we have*

$$\text{(3.11)}\quad I_a^z[F(\cdot)] = \lim_{n\to\infty} z^{n/2} \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^n [(e_k,x)^\sim]^2\right\} F(x)\,d\nu(x).$$

Proof. Let n be a natural number with $n > r$, and let

$$\Gamma(n) = \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^n [(e_k,x)^\sim]^2\right\} F(x)\,d\nu(x).$$

By the Fubini theorem, (3.2), (3.4) and Lemma 3.6,

$$\begin{aligned}\Gamma(n) &= \int_H \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^n [(e_k,x)^\sim]^2 + i(h,x)^\sim\right\} \\ &\qquad\qquad \times \psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)\,d\nu(x)\,d\mu(h) \\ &= \left(\frac{z}{2\pi}\right)^{r/2}\left(\frac{1}{z}\right)^{n/2} \int_H \int_{\mathbf{R}^r} \exp\left\{\frac{z-1}{2z}\sum_{k=1}^n \langle e_k,h\rangle^2 - \frac{1}{2}\|h\|^2\right\} \\ &\quad \times \exp\left\{\frac{1}{2z}\sum_{k=1}^r (izv_k + \langle e_k,h\rangle)^2\right\}\psi(v_1,\dots,v_r)\,dv_1\cdots dv_r\,d\mu(h).\end{aligned}$$

Note that, by the Bessel inequality, we have

$$\begin{aligned}&\left|\exp\left\{\frac{z-1}{2z}\sum_{k=1}^n \langle e_k,h\rangle^2 - \frac{1}{2}\|h\|^2 + \frac{1}{2z}\sum_{k=1}^r (izv_k + \langle e_k,h\rangle)^2\right\}\psi(v_1,\dots,v_r)\right| \\ &\qquad\qquad \leq \exp\left\{-\frac{\operatorname{Re} z}{2}\sum_{k=1}^r v_k^2\right\}|\psi(v_1,\dots,v_r)|\end{aligned}$$

and the righthand side of the inequality above is integrable on $H \times \mathbf{R}^r$ since $\psi \in L_p(\mathbf{R}^r)$ and $\mu \in M(H)$. Hence, by the dominated convergence theorem and Parseval's relation, we obtain

$$\begin{aligned}\lim_{n\to\infty} z^{\frac{n}{2}}\Gamma(n) &= \left(\frac{z}{2\pi}\right)^{r/2} \int_H \int_{\mathbf{R}^r} \exp\left\{\frac{1}{2z}\left[\sum_{k=1}^r (izv_k + \langle e_k,h\rangle)^2 - \|h\|^2\right]\right\} \\ &\qquad\qquad \times \psi(v_1,\dots,v_r)\,dv_1\cdots dv_r\,d\mu(h).\end{aligned}$$

By equation (3.1) in Theorem 3.1, the proof is completed. □

Moreover, if $p = 1$, we obtain the following relationship between the Wiener integral and the analytic Feynman integral on abstract Wiener space.

Theorem 3.9. *Let $\{e_n\}$ be a complete orthonormal set in H. Let $F(x) = G(x)\psi((e_1, x)^\sim, \dots, (e_r, x)^\sim)$ where $G \in \mathcal{F}(B)$ and $\psi \in L_1(\mathbf{R}^r)$. Let $\{z_n\}$ be a sequence of complex numbers in $\mathbf{C}_+$ such that $z_n \to -iq$. Then*

$$I_a^q[F(\cdot)] = \lim_{n\to\infty} z_n^{n/2}\int_B \exp\left\{\frac{1-z_n}{2}\sum_{k=1}^{n}[(e_k, x)^\sim]^2\right\}F(x)\,d\nu(x). \tag{3.12}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.8. Let n be a natural number with $n > r$, and let

$$\Gamma(n, z_n) = \int_B \exp\left\{\frac{1-z_n}{2}\sum_{k=1}^{n}[(e_k, x)^\sim]^2\right\}F(x)\,d\nu(x).$$

By the same method as in the proof of Theorem 3.8, we have

$$\Gamma(n, z_n) = \left(\frac{z_n}{2\pi}\right)^{r/2}\left(\frac{1}{z_n}\right)^{n/2}\int_H\int_{\mathbf{R}^r}\exp\left\{\frac{z_n - 1}{2z_n}\sum_{k=1}^{n}\langle e_k, h\rangle^2 - \frac{1}{2}\|h\|^2\right\}$$
$$\times \exp\left\{\frac{1}{2z_n}\sum_{k=1}^{r}(iz_n v_k + \langle e_k, h\rangle)^2\right\}\psi(v_1, \dots, v_r)\,dv_1\cdots dv_r\,d\mu(h).$$

Using the Bessel inequality in the first exponent above, we have that the absolute value of the exponentials above is bounded by unity. And also $|\psi(v_1, \dots, v_r)|$ is integrable on $H \times \mathbf{R}^r$ since $\psi \in L_1(\mathbf{R}^r)$ and $\mu \in M(H)$. Hence by the dominated convergence theorem and Parseval's relation, we obtain

$$\lim_{n\to\infty} z_n^{\frac{n}{2}}\Gamma(n, z_n) = \left(-\frac{iq}{2\pi}\right)^{\frac{r}{2}}\int_H\int_{\mathbf{R}^r}\exp\left\{\frac{i}{2q}\left[\sum_{k=1}^{r}(qv_k + \langle e_k, h\rangle)^2 - \|h\|^2\right]\right\}$$
$$\times\, \psi(v_1, \dots, v_r)\,dv_1\cdots dv_r\,d\mu(h).$$

By equation (3.5) in Theorem 3.2, the proof is completed. □

Theorem 3.10. *Let* $\{e_n\}$ *be a complete orthonormal set in* H. *Let* $F(x) = G(x)\phi((e_1,x)^\sim,\dots,(e_r,x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\phi \in \hat{M}(\mathbf{R}^r)$. *Then equation* (3.11) *holds.*

Proof. Let n be a natural number with $n > r$ and let

$$\Gamma(n) = \int_B \exp\left\{\frac{1-z}{2}\sum_{k=1}^{n}[(e_k,x)^\sim]^2\right\}F(x)\,d\nu(x).$$

By the Fubini theorem, (3.2), (3.4) and Lemma 3.7, we have

$$\begin{aligned}\Gamma(n) = \int_H\int_{\mathbf{R}^r}\int_B \exp\bigg\{&\frac{1-z}{2}\sum_{k=1}^{n}[(e_k,x)^\sim]^2 + i(h,x)^\sim \\ &+ i\sum_{k=1}^{r}t_k(e_k,x)^\sim\bigg\}\,d\nu(x)\,d\rho(t_1,\cdots,t_r)\,d\mu(h) \\ = \left(\frac{1}{z}\right)^{n/2}\int_H\int_{\mathbf{R}^r}\exp\bigg\{&\frac{z-1}{2z}\sum_{k=1}^{n}\langle e_k,h\rangle^2 - \frac{1}{z}\sum_{k=1}^{r}t_k\langle e_k,h\rangle \\ &- \frac{1}{2z}\sum_{k=1}^{r}t_k^2 - \frac{1}{2}\|h\|^2\bigg\}\,d\rho(t_1,\cdots,t_r)\,d\mu(h).\end{aligned}$$

Using the Bessel inequality, we have that the exponential of the last expression above is bounded in absolute value by unity. Hence by the dominated convergence theorem and Parseval's relation, we obtain

$$\begin{aligned}\lim_{n\to\infty} z^{n/2}\Gamma(n) = \int_H\int_{\mathbf{R}^r}\exp\left\{-\frac{1}{2z}\left[\|h\|^2 + \sum_{k=1}^{r}2t_k\langle e_k,h\rangle + \sum_{k=1}^{r}t_k^2\right]\right\} \\ \times\, d\rho(t_1\dots,t_r)\,d\mu(h).\end{aligned}$$

By equation (3.7) in Theorem 3.3, the proof is completed. □

Modifying the proof of Theorem 3.10, by replacing "z" by "z_n" whenever it occurs, we have the following corollary.

Corollary 3.11. *Let* $\{e_n\}$ *and* $\{z_n\}$ *be given as in Theorem* 3.9 *and let* F *be given as in Theorem* 3.10. *Then equation* (3.12) *holds.*

From Theorem 3.8 and Theorem 3.10 and the linearity of the analytic Wiener integral on abstract Wiener space, we obtain

Corollary 3.12. *Let* $\{e_n\}$ *be given as in Theorem* 3.9. *Let* $F(x) = G(x)\ \Psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\Psi = \psi + \phi \in L_p(\mathbf{R}^r) + \hat{M}(\mathbf{R}^r)$, $1 \le p < \infty$. *Then equation* (3.11) *holds.*

Similarly, from Theorem 3.9, Corollary 3.11 and the linearity of the analytic Feynman integral on abstract Wiener space, we have

Corollary 3.13. *Let* $\{e_n\}$ *and* $\{z_n\}$ *be given as in Theorem* 3.9. *Let* $F(x) = G(x)\Psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\Psi = \psi + \phi \in L_1(\mathbf{R}^r) + \hat{M}(\mathbf{R}^r)$. *Then equation* (3.12) *holds.*

Our main result, namely a change of scale formula for Wiener integrals on abstract Wiener space, now follows from Corollary 3.12.

Theorem 3.14. *Let* $\{e_n\}$ *be given as in Theorem* 3.9. *Let* $F(x) = G(x)\Psi((e_1,x)^\sim,\dots,(e_r,x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\Psi = \psi + \phi \in L_p(\mathbf{R}^r) + \hat{M}(\mathbf{R}^r)$, $1 \le p < \infty$. *Then, for any* $\rho > 0$,

(3.13)
$$\int_B F(\rho x)\,d\nu(x) = \lim_{n\to\infty} \rho^{-n} \int_B \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}[(e_k,x)^\sim]^2\right\} F(x)\,d\nu(x).$$

Proof. By letting $z = \rho^{-2}$ in (3.11), we have equation (3.13). □

Obviously the constant function $\phi \equiv 1$ is a member of $\hat{M}(\mathbf{R}^r)$. Hence we have the following corollary which is a change of scale formula for Wiener integrals on an abstract Wiener space given in [**20**].

Corollary 3.15. *Let* $\{e_n\}$ *be given as in Theorem* 3.9. *Let* $F \in \mathcal{F}(B)$. *Then, for any* $\rho > 0$, *equation* (3.13) *holds.*

4. Corollaries. In this section we apply our results to the classical Wiener space.

Let $H_0 = H_0[a,b]$ be the space of real-valued functions f on $[a,b]$ which are absolutely continuous and whose derivative Df is in $L_2[a,b]$. The inner product on H_0 is given by

$$\langle f, g\rangle = \int_a^b (Df)(s)(Dg)(s)\,ds.$$

Then H_0 is a real separable infinite dimensional Hilbert space. Let $B_0 = B_0[a,b]$ be the space $C_0[a,b]$ of all continuous functions x on $[a,b]$ with $x(0) = 0$ and equip B_0 with the sup norm. Let ν_0 be classical Wiener measure. Then (H_0, B_0, ν_0) is an example of an abstract Wiener space. Note that if $\{e_n\}$ is a complete orthonormal set in H_0, then $\{De_n\}$ is also a complete orthonormal set in $L_2[a,b]$ and $(e_n, x)^\sim$ equals the Paley-Wiener-Zygmund stochastic integral $\int_a^b (De_n)(s)\tilde{d}x(s)$ for s-almost everywhere $x \in B_0$.

In [**5**], Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of functionals on $C_0[a,b]$ which are expressible in the form

$$F(x) = \int_{L_2[a,b]} \exp\left\{ i\int_a^b v(s)\,\tilde{d}x(s)\right\} d\sigma(v) \tag{4.1}$$

for s-almost everywhere $x \in C_0[a,b]$, where $\sigma \in M(L_2[a,b])$. Then we know that $F \in \mathcal{F}(B_0)$ if and only if $F \in \mathcal{S}$.

Corollary 4.1 ([**7**, Theorem 1]). *Let $F(x) = G(x)\psi(x(b))$ where $G \in \mathcal{S}$, $\psi \in L_p(\mathbf{R})$, $1 \le p < \infty$. Then, for each $z \in \mathbf{C}^+$, F is analytic Wiener integrable, and if G is given by* (4.1), *then*

(4.2)

$$I_a^z[F(\cdot)] = \left(\frac{z}{2\pi(b-a)}\right)^{1/2} \int_{L_2[a,b]} \int_{\mathbf{R}} \exp\left\{ \frac{1}{2z(b-a)} \left[\left(iz\xi + \int_a^b v(s)\,ds\right)^2 - (b-a)\int_a^b (v(s))^2\,ds\right]\right\} \psi(\xi)\,d\xi\,d\sigma(v).$$

In case $p=1$, for each real $q\neq 0$, F is analytic Feynman integrable and

$$I_a^q[F(\cdot)]=\Big(-\frac{iq}{2\pi(b-a)}\Big)^{1/2}\int_{L_2[a,b]}\int_{\mathbf{R}}\exp\bigg\{\frac{i}{2q(b-a)}\bigg[\Big(q\xi+\int_a^b v(s)\,ds\Big)^2 - (b-a)\int_a^b (v(s))^2\,ds\bigg]\bigg\}\psi(\xi)\,d\xi\,d\sigma(v). \tag{4.3}$$

Proof. Let $\phi(t)=1/\sqrt{b-a}$ and $e(t)=\int_a^{a+t}\phi(s)\,ds$. Then $\{e(t)\}$ is an orthonormal set in H_0 and

$$x(b)=\sqrt{b-a}\int_a^b \phi(t)\,\tilde{d}x(t)=\sqrt{b-a}(e,x)^{\sim}.$$

Thus by (3.1) in Theorem 3.1, we have

$$\begin{aligned}I_a^z[F(\cdot)]&=\Big(\frac{z}{2\pi}\Big)^{1/2}\int_{L_2[a,b]}\int_{\mathbf{R}}\exp\bigg\{\frac{1}{2z}\bigg[\Big(iz\xi+\frac{1}{\sqrt{b-a}}\int_a^b v(s)\,ds\Big)^2 \\ &\qquad - \int_a^b (v(s))^2\,ds\bigg]\bigg\}\psi(\sqrt{b-a}\xi)\,d\xi\,d\sigma(v) \\ &=\Big(\frac{z}{2\pi(b-a)}\Big)^{1/2}\int_{L_2[a,b]}\int_{\mathbf{R}}\exp\bigg\{\frac{1}{2z(b-a)}\bigg[\Big(iz\xi+\int_a^b v(s)\,ds\Big)^2 \\ &\qquad -(b-a)\int_a^b (v(s))^2\,ds\bigg]\bigg\}\psi(\xi)\,d\xi\,d\sigma(v),\end{aligned}$$

as desired. Equation (4.3) can be proved similarly. □

The following corollary is obtained easily from Theorem 3.3 and Corollary 3.4.

Corollary 4.2 ([**7**, Theorem 2]). *Let $F(x)=G(x)\phi(x(b))$ where $G\in\mathcal{S}$, $\phi\in\hat{M}(\mathbf{R})$. Then for each $z\in\mathbf{C}_+$, F is analytic Wiener integrable. Moreover if G is given by* (4.1) *and ϕ is given by*

$$\phi(r)=\int_{\mathbf{R}}\exp\{irt\}\,d\rho(t) \tag{4.4}$$

then,

$$(4.5)\quad I_a^z[F(\cdot)] = \int_{L_2[a,b]}\int_{\mathbf{R}} \exp\left\{-\frac{1}{2z}\int_a^b (v(s)+t)^2\,ds\right\} d\rho(t)\,d\sigma(v).$$

Moreover, for each real $q \neq 0$, F is analytic Feynman integrable and

$$(4.6)\quad I_a^q[F(\cdot)] = \int_{L_2[a,b]}\int_{\mathbf{R}} \exp\left\{-\frac{i}{2q}\int_a^b (v(s)+t)^2\,ds\right\} d\rho(t)\,d\sigma(v).$$

From the above results and the linearity of the analytic Wiener integral, we can obtain the formula for the analytic Wiener integral of functions of the form $F(x) = G(x)\Psi(x(b))$ where $G \in \mathcal{S}$, $\Psi = \psi + \phi \in L_p(\mathbf{R}) + \hat{M}(\mathbf{R})$, $1 \leq p < \infty$. Similarly we can also have the formula for the analytic Feynman integral of functions of the form $F(x) = G(x)\Psi(x(b))$ where $G \in \mathcal{S}$, $\Psi = \psi + \phi \in L_1(\mathbf{R}) + \hat{M}(\mathbf{R})$.

The following corollary is a relationship between the Wiener integral and the analytic Wiener (Feynman) integral on classical Wiener space.

Corollary 4.3. *Let $F(x) = G(x)\Psi(x(b))$ where $G \in \mathcal{S}$, $\Psi = \psi + \phi \in L_p(\mathbf{R}) + \hat{M}(\mathbf{R})$, $1 \leq p < \infty$. Let $\{\phi_n\}$ be a complete orthonormal set in $L_2[a,b]$ with $\phi_1(t) = 1/\sqrt{b-a}$. Then, for each $z \in \mathbf{C}_+$, we have*

$$(4.7)\quad I_a^z[F(\cdot)] = \lim_{n\to\infty} z^{n/2} \int_{C_0[a,b]} \exp\left\{\frac{1-z}{2}\sum_{k=1}^n \left(\int_a^b \phi_k(t)\,\tilde{d}x(t)\right)^2\right\} F(x)\,d\nu_0(x).$$

In case $p = 1$, if we let $\{z_n\}$ be a sequence of complex numbers in $\mathbf{C}_+$ such that $z_n \to -iq$, then

$$(4.8)\quad I_a^q[F(\cdot)] = \lim_{n\to\infty} z_n^{n/2} \int_{C_0[a,b]} \exp\left\{\frac{1-z_n}{2}\sum_{k=1}^n \left(\int_a^b \phi_k(t)\,\tilde{d}x(t)\right)^2\right\} F(x)\,d\nu_0(x).$$

From the above equation we obtain a change of scale formula for the Wiener integral on classical Wiener space.

Corollary 4.4. *Let* F *and* $\{\phi_n\}$ *be given as in Corollary* 4.3. *Then for any* $\rho > 0$,

(4.9)

$$\int_{C_0[a,b]} F(\rho x)\, d\nu_0(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_0[a,b]} \exp\left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \left(\int_a^b \phi_k(t)\, \tilde{d}x(t) \right)^2 \right\} F(x)\, d\nu_0(x).$$

Corollary 4.5 ([**6**, Theorem 2]). *Let* ρ *and* $\{\phi_n\}$ *be given as in Corollary* 4.3. *Then if* $F \in \mathcal{S}$, *equation* (4.9) *holds.*

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