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THE DIRICHLET PROBLEM FOR QUASIMONOTONE SYSTEMS OF SECOND ORDER EQUATIONS

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ABSTRACT. We prove the existence of a solution of the Dirichlet problem $u'' + f(t,u) = 0$, $u(0) = u(1) = 0$ between upper and lower solutions, where $f : [0,1] \times E \to E$ is quasimonotone increasing in its second variable with respect to a general solid cone.

1. Introduction. Let E be a finite-dimensional real vector space ordered by a cone K. A cone K is a nonempty closed convex subset of E with $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual, $x \leq y :\Longleftrightarrow y - x \in K$. Furthermore we assume that K is solid, that is, $K^0 \neq \varnothing$, and we write $x \ll y$ if $y - x \in K^0$. For $x \leq y$ let $[x,y]$ denote the order interval of all z with $x \leq z \leq y$. Let K^* denote the dual cone of K, that is, the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ $(x \geq 0)$. We fix $p \in K^0$ and consider E to be normed by $\|\cdot\|$, the Minkowski functional of $[-p,p]$. Note that $-\|x\|p \leq x \leq \|x\|p$, $x \in E$.

A function $g : E \to E$ is called quasimonotone increasing (qmi for short), in the sense of Volkmann [**16**], if

$$x, y \in E, \quad x \leq y, \quad \varphi \in K^*, \quad \varphi(x) = \varphi(y) \Longrightarrow \varphi(g(x)) \leq \varphi(g(y)).$$

A function $f : [0,1] \times E \to E$ is called qmi if $x \mapsto f(t,x)$ is qmi for each $t \in [0,1]$.

In the sequel let $f : [0,1] \times E \to E$ be continuous and qmi. We consider the Dirichlet boundary value problem

$$u''(t) = f(t, u(t)), \quad t \in [0,1], \quad u(0) = u(1) = 0. \tag{1}$$

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2. Upper and lower solutions. As usual, functions $v, w \in C^2([0,1], E)$ are called lower and upper solutions for problem (1) in case that

$$v''(t) + f(t, v(t)) \geq 0, \quad t \in [0,1], \quad v(0) \leq 0, \quad v(1) \leq 0, \tag{2}$$

$$w''(t) + f(t, w(t)) \leq 0, \quad t \in [0,1], \quad w(0) \geq 0, \quad w(1) \geq 0, \tag{3}$$

respectively. We will later consider lower and upper solutions such that $v(t) \leq w(t)$, $t \in [0,1]$. If f satisfies certain invariance conditions, (2) and (3) already imply this property, see, for example, [**7, 9, 15**].

The use of lower and upper solutions to obtain existence of solutions of boundary value problems dates back to Perron's method for the Dirichlet problem for elliptic equations, and meanwhile hundreds of papers use lower and upper solutions for all kinds of equations and boundary conditions. For a survey on the history of this subject we refer to [**4**], Chapter 4.3 and the references given there.

If $v(t) \leq w(t)$ ($t \in [0,1]$), the question whether (1) has a solution $u \in C^2([0,1], E)$ between v and w is answered positively only in some special cases. For $E = \mathbf{R}^n$ ordered by the natural cone

$$K_{\text{nat}} = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_1, \dots, x_n \geq 0\}$$

this follows by a result of Lakshmikantham and Vatsala [**8**, Theorem 3.1]. Moreover, many authors proved existence of solutions between lower and upper solutions for several kinds of boundary value problems by the method of monotone iteration, see, for example, [**1, 2, 8, 11, 13**] and the references given there. Roughly speaking, for this method it is always assumed that the function in the differential equation under consideration is such that addition of $\lambda \mathrm{id}$ leads to an increasing function for some $\lambda \geq 0$. As it is to our knowledge there are no answers to the question above for quasimonotone systems which do not satisfy a condition of this type. In an intermediate step of our considerations we will also make use of such a condition:

There exists $\lambda \geq 0$ such that

$$t \in [0,1], \quad v(t) \leq x \leq y \leq w(t) \Rightarrow f(t,x) + \lambda x \leq f(t,y) + \lambda y. \tag{4}$$

The reason why it is interesting to get rid of (4) is that getting a monotone mapping by adding λid to a qmi mapping sometimes works but often fails for two different reasons:

For example, if $E = \mathbf{R}$ is ordered by the cone $[0, \infty)$ each mapping $g : \mathbf{R} \to \mathbf{R}$ is qmi, but for example $t \mapsto \sqrt{|t|} + \lambda t$ is never monotone, thus f may fail to be sufficiently smooth in its second variable. The other reason is that the cone may be such that even for linear mappings adding λid never leads to a monotone mapping. The first example of this type was given in [**16**], Beispiel 5, for $\mathbf{R}^3$ ordered by the ice-cream cone $K_{\text{ice}} = \{x \in \mathbf{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$. For a characterization of the linear qmi mappings with respect to this cone, see [**14**].

For lower and upper solutions v, w we consider the following additional properties which are often satisfied in applications:

$$w''(t) + f(t, w(t)) \ll v''(t) + f(t, v(t)), \quad t \in [0, 1], \tag{5}$$

$$v(t) \ll w(t), \quad t \in [0, 1]. \tag{6}$$

We will handle the general case under the assumption that one of these conditions is satisfied. In particular (5) is satisfied in the case that one of the differential inequalities in (2) or (3) is strict.

Theorem 1. *Let $f : [0, 1] \times E \to E$ be continuous and qmi, and let v, w be lower and upper solutions for problem* (1) *with $v(t) \leq w(t)$ ($t \in [0, 1]$), satisfying in addition* (5) *or* (6). *Then problem* (1) *has a solution $u \in C^2([0, 1], E)$ such that*

$$v(t) \leq u(t) \leq w(t), \quad t \in [0, 1].$$

Remark. It would be interesting to know whether Theorem 1 is valid without the conditions (5) and (6). For the natural cone this is true by the result of Lakshmikantham and Vatsala mentioned above, and it is obtained by Nagumo's cutting off method. This method does not work for general cones in an obvious manner. Our method to prove Theorem 1 is an approximation method and needs strict inequalities since perturbations are used.

For applications it is of interest to have conditions on f such that lower or upper solutions exist. The following result is just an example for such a condition in our case.

Theorem 2. *Let $f : [0,1] \times E \to E$ be continuous and qmi, and let there exist p_1 or $p_2 \in K^0$ and real numbers $\mu_0 > 0$, $c < \pi^2$ such that*

$$\frac{f(t, -\mu p_1)}{\mu} \geq -cp_1 \quad or \quad \frac{f(t, \mu p_2)}{\mu} \leq cp_2, \quad \mu \geq \mu_0, t \in [0,1].$$

Then problem (1) *has a lower or upper solution v or w with $v(t) \ll 0$ or $w(t) \gg 0$ ($t \in [0,1]$), respectively.*

Theorem 2 follows by considering v or w defined by

$$v(t) = -\frac{\mu_0 \cos((\pi - \varepsilon)(t - 1/2))}{\cos((\pi - \varepsilon)/2)}\, p_1, \quad w(t) = \frac{\mu_0 \cos((\pi - \varepsilon)(t - 1/2))}{\cos((\pi - \varepsilon)/2)}\, p_2$$

($t \in [0,1]$), with $\varepsilon \in (0, \pi)$ such that $(\pi - \varepsilon)^2 > c$. In particular, (6) is satisfied in case that p_1 and p_2 exist. Then (1) is solvable according to Theorem 1.

3. Preliminaries. For the proof of Theorem 1 we will make use of the following theorem.

Theorem 3. *Let $f : [0,1] \times E \to E$ be continuous and qmi, let v, w be lower and upper solutions for problem* (1) *with $v(t) \leq w(t)$ ($t \in [0,1]$) and for some $\lambda \geq 0$, let* (4) *be valid. Then problem* (1) *has a solution $u \in C^2([0,1], E)$ satisfying $v(t) \leq u(t) \leq w(t)$ ($t \in [0,1]$).*

This result follows by the method of monotone iteration, as mentioned above, and we give a sketch of the proof for the convenience of the reader.

Proof. Set $v_0 = v$ and define $(v_k)_{k=0}^{\infty}$ recursively by

$$(7) \qquad \begin{cases} v_{k+1}''(t) - \lambda v_{k+1}(t) = -f(t, v_k(t)) - \lambda v_k(t), & t \in [0,1], \\ v_{k+1}(0) = v_{k+1}(1) = 0. \end{cases}$$

By means of (4) it is easy to check that $v_k(t) \leq v_{k+1}(t) \leq w(t)$ ($t \in [0,1]$), $k \in \mathbf{N}_0$. For the details see for example [**2**]. In particular $(v_k)_{k=0}^{\infty}$ and $(v_k'')_{k=0}^{\infty}$ are bounded, hence $(v_k')_{k=0}^{\infty}$ is bounded, and by means of (7) and Ascoli-Arzelà's theorem $\{v_k : k \in \mathbf{N}_0\}$ is a compact subset of $C^2([0,1],E)$. Since $(v_k)_{k=0}^{\infty}$ is in addition increasing, it is convergent in $C^2([0,1],E)$. By (7) its limit is a solution of (1). $\square$

4. Proof of Theorem 1. Let f, v, w be as in Theorem 1. We first consider the case that (5) is satisfied. We fix $0 < \varepsilon < 1/2$. By choosing

$$q_\varepsilon(t) = -\varepsilon(v''(t) + f(t,v(t)) + w''(t) + f(t,w(t))), \quad t \in [0,1]$$

we obtain

$$v''(t) + f(t,v(t)) + q_\varepsilon(t) \gg 0 \gg w''(t) + f(t,w(t)) + q_\varepsilon(t), \quad t \in [0,1].$$

In particular there exists $\gamma > 0$ such that

$$\text{(8)} \quad v''(t) + f(t,v(t)) + q_\varepsilon(t) \geq \gamma p \gg -\gamma p \geq w''(t) + f(t,w(t)) + q_\varepsilon(t)$$

($t \in [0,1]$), and obviously we can choose $\gamma < \varepsilon$. Set

$$D := \{(t,x) \in [0,1] \times E : v(t) \leq x \leq w(t)\}.$$

We first use Friedrichs' mollifiers to approximate f by a function which is smooth in its second variable. Let $r > 0$ be such that

$$D \subseteq [0,1] \times [-rp, rp].$$

Since f is uniformly continuous on compact sets there exists $0 < \delta < 1$ such that

$$\|f(t,x) - f(t,y)\| \leq \frac{\gamma}{2}$$

for $(t,x), (t,y) \in [0,1] \times [-(r+1)p, (r+1)p]$ with $\|x - y\| \leq \delta$. Let $h \in C^\infty(E, \mathbf{R})$ be such that

$$h(x) \geq 0, \quad x \in E, \quad \operatorname{supp} h \subseteq [-\delta p, \delta p], \quad \int_E h(x)\,dx = 1,$$

and let $f_\gamma : [0,1] \times E \to E$ be defined as

$$f_\gamma(t,x) = \int_E h(\xi - x) f(t,\xi)\,d\xi = \int_E h(\xi) f(t, \xi + x)\,d\xi.$$

By standard reasoning (see for example [**3**]) the function f_γ is C^∞ in its second variable, $D_x f_\gamma : [0,1] \times E \to L(E)$ is continuous ($L(E)$ the algebra of linear mappings on E), and

$$(9) \quad \|f(t,x) - f_\gamma(t,x)\| \le \frac{\gamma}{2} \quad \Big(\Leftrightarrow -\frac{\gamma}{2} p \le f(t,x) - f_\gamma(t,x) \le \frac{\gamma}{2} p \Big)$$

for $(t,x) \in [0,1] \times [-rp, rp]$. Moreover f_γ is qmi:

Let $(t,x),(t,y) \in [0,1] \times E$ and $\varphi \in K^*$ such that $x \le y$ and $\varphi(x) = \varphi(y)$. Then $\xi + x \le \xi + y$ and $\varphi(\xi + x) = \varphi(\xi + y)$ for each $\xi \in E$. Since $h \ge 0$ we get $\varphi(f_\gamma(t,x)) \le \varphi(f_\gamma(t,y))$.

Next, (8) and (9) prove

$$(10) \quad v''(t) + f_\gamma(t, v(t)) + q_\varepsilon(t) \ge \frac{\gamma}{2} p \gg -\frac{\gamma}{2} p \ge w''(t) + f_\gamma(t, w(t)) + q_\varepsilon(t)$$

for $t \in [0,1]$.

Since f_γ is qmi each linear mapping $y \mapsto (D_x f_\gamma)(t,x) y$ is qmi ($(t,x) \in [0,1] \times E$). This is an immediate consequence of the Mean Value Theorem and the definition of qmi mappings, see for example, [**5**]. Since D is compact, $(D_x f_\gamma)(D)$ is a compact subset of $L(E)$. Hence there exists $\lambda_0 \in \mathbf{R}$ such that $\lambda_0 > r(A)$ ($r(A)$ the spectral radius of A) for all $A \in (D_x f_\gamma)(D)$. Let $\widetilde{K}$ denote the cone of all monotone linear mappings in $L(E)$, and let $L(E)$ be ordered by this cone. For $\lambda \ge \lambda_0$ and $A \in (D_x f_\gamma)(D)$ we consider the resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$. Then it is known, see for example [**6**], that $R(\lambda, A) \in \widetilde{K}$. In particular we find

$$(11) \qquad \lambda A R(\lambda, A) + \lambda I = \lambda^2 R(\lambda, A) \ge 0, \quad A \in (D_x f_\gamma)(D),$$

and

$$(12) \qquad \lambda A R(\lambda, A) \longrightarrow A, \quad \lambda \to \infty$$

in $L(E)$ uniformly on $(D_x f_\gamma)(D)$. Since K is a solid cone $\widetilde{K}$ is a solid cone too, see [**12**, Lemma 5]. Fix $P \in \widetilde{K}^0$, and let $L(E)$ be normed by $|||\cdot|||$, denoting the Minkowski functional of the order interval $[-P, P]$. We have

$$-|||A|||P \le A \le |||A|||P, \quad A \in L(E).$$

Let $\sigma \in (0,\varepsilon)$ be such that

$$\sigma \max\{\|Pv(t)\|, \|Pw(t)\|\} \leq \frac{\gamma}{4} \quad (t \in [0,1]). \tag{13}$$

Choosing $\lambda \geq \lambda_0$ sufficiently big, we obtain by means of (12) that

$$|||\lambda(D_x f_\gamma)(t,x)R(\lambda,(D_x f_\gamma)(t,x)) - (D_x f_\gamma)(t,x)||| \leq \sigma, \quad (t,x) \in D,$$

and therefore

$$\lambda(D_x f_\gamma)(t,x)R(\lambda,(D_x f_\gamma)(t,x)) \leq \sigma P + (D_x f_\gamma)(t,x), \quad (t,x) \in D.$$

According to (11) we obtain

$$0 \leq \sigma P + (D_x f_\gamma)(t,x) + \lambda I, \quad (t,x) \in D. \tag{14}$$

We define $F_{\varepsilon,\gamma,\sigma} : [0,1] \times E \to E$ by

$$F_{\varepsilon,\gamma,\sigma}(t,x) = f_\gamma(t,x) + \sigma P x + q_\varepsilon(t).$$

Note that $F_{\varepsilon,\gamma,\sigma}$ is continuous and qmi. Now by means of (14) we find

$$F_{\varepsilon,\gamma,\sigma}(t,x) + \lambda x \leq F_{\varepsilon,\gamma,\sigma}(t,y) + \lambda y \quad (t \in [0,1], v(t) \leq x \leq y \leq w(t)),$$

which is (4) for $F_{\varepsilon,\gamma,\sigma}$ instead of f. Moreover, by means of (10) and (13) we have

$$v''(t) + F_{\varepsilon,\gamma,\sigma}(t,v(t)) \geq \frac{\gamma}{4}\, p \gg -\frac{\gamma}{4}\, p \geq w''(t) + F_{\varepsilon,\gamma,\sigma}(t,w(t))$$

for $t \in [0,1]$.

According to Theorem 3 there is a solution $z_{\varepsilon,\gamma,\sigma} \in C^2([0,1],E)$ between v and w of the boundary value problem

$$z''(t) = F_{\varepsilon,\gamma,\sigma}(t,z(t)) \quad (t \in [0,1]), \quad z(0) = z(1) = 0. \tag{15}$$

By choosing a sequence $(\varepsilon_n)_{n=1}^\infty$ in $(0,1/2)$ with limit 0 and suitable sequences $\lambda_n, \sigma_n \in (0,\varepsilon_n)$, $n \in \mathbf{N}$, we obtain a sequence of solutions $(z_n)_{n=1}^\infty$ of the corresponding boundary value problems in (15), all between v and w. Moreover

$$F_{\varepsilon_n,\gamma_n,\sigma_n}(t,x) \longrightarrow f(t,x), \quad n \to \infty \tag{16}$$

uniformly on D. By means of (15), (16) and Ascoli-Arzelà's theorem $(z_n)_{n=1}^{\infty}$ has a convergent subsequence in $C^2([0,1],E)$ with limit u, say. We have $v(t) \leq u(t) \leq w(t)$ $(t \in [0,1])$, and u is a solution of problem (1) by means of (16).

Finally we consider f, v, w under condition (6). Then, for $n \in \mathbf{N}$, we have

$$v''(t)+f(t,v(t))+\frac{1}{n}(w(t)-v(t)) \gg 0 \geq w''(t)+f(t,w(t)),$$

$t \in [0,1]$, which means that the case above applies to $f_n : [0,1]\times E \to E$,

$$f_n(t,x) = f(t,x) + \frac{1}{n}(w(t) - x).$$

Thus, each boundary value problem

$$z''(t) = f_n(t, z(t)), \quad t \in [0,1], \quad z(0) = z(1) = 0$$

has a solution z_n between v and w. With the same arguments as above, the sequence $(z_n)_{n=1}^{\infty}$ has a subsequence converging in $C^2([0,1],E)$ to a solution u of problem (1), with u between v and w. $\square$

5. Examples. Consider $\mathbf{R}^3$ ordered by the cone K_{ice} in Section 2. Then

$$x \longmapsto g(x) := (2x_1x_2, -x_1^2 + x_2^2 + x_3^2, 2x_2x_3)$$

is qmi, see [**5**]. Let $q : [0,1] \to K_{\text{ice}}$ be defined as $q(t) = (-t/2, 0, t)$, and consider $f : [0,1] \times \mathbf{R}^3 \to \mathbf{R}^3$, $f(t,x) = g(x) + q(t)$. Then Theorem 1 applies to f, v, w with

$$v(t) = 0, \quad w(t) = (0, 0, t(1-t)), \quad t \in [0,1],$$

since

$$w''(t) + f(t, w(t)) = (-t/2, (t(1-t))^2, -2+t) \ll 0, \quad t \in [0,1].$$

Hence (1) has a solution $u : [0,1] \to K_{\text{ice}}$.

Next, consider the space $E = S_n$ of all real symmetric $n \times n$ matrices ordered by the cone K of all positive semidefinite matrices. Note that $I \in K^0$. For $A, D \in S_n$, $B, C \in \mathbf{R}^{n\times n}$ it is known [**10**] that

$$X \longmapsto XAX + B^TX + XB + C^TXC + D$$

is qmi. Hence, for $A, D \in C([0,1], S_n)$, $B, C \in C([0,1], \mathbf{R}^{n\times n})$, we obtain a continuous and qmi function $f : [0,1] \times S_n \to S_n$,

$$f(t,X) = XA(t)X + B^T(t)X + XB(t) + C^T(t)XC(t) + D(t).$$

Assume that $A(t)$ is negative semidefinite and $D(t)$ is positive semi-definite, $t \in [0,1]$. Then $V(t) = 0$, $t \in [0,1]$, is a lower solution for (1). In case that

$$B^T(t) + B(t) + C^T(t)C(t) \leq \gamma I \quad (t \in [0,1]) \tag{17}$$

for some $\gamma < \pi^2$, we can choose $c \in (\gamma, \pi^2)$ and find $\mu_0 > 0$ such that

$$\frac{f(t,\mu I)}{\mu} \leq \gamma I + \frac{1}{\mu} D(t) \ll cI, \quad \mu \geq \mu_0, t \in [0,1].$$

Hence, according to Theorem 2, there is an upper solution W for problem (1) with $W(t) \gg 0$ $(t \in [0,1])$.

Then according to Theorem 1, there is a solution $U \in C^2([0,1], S_n)$ of problem (1), which is positive semi-definite on $[0,1]$ since $U(t) \geq V(t) = 0$, $t \in [0,1]$.

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