ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 2004

OSCILLATIONS OF SECOND-ORDER NONLINEAR PARTIAL DIFFERENCE EQUATIONS

SHU TANG LIU AND GUANRONG CHEN

ABSTRACT. Oscillations of the second-order nonlinear partial difference equation

$$T(\triangle_1, \triangle_2)\,[c_{mn}T(\triangle_1, \triangle_2)(y_{mn})] + p_{mn}(y_{m+1,n} + y_{m,n+1})^{\nu} = 0$$

is investigated. Some sufficient conditions for oscillations of solutions of the above equation with $\nu > 1$ and $\nu < 1$ are obtained, where ν is a fraction of odd positive integers, $m, n \in N_i = \{i, i+1, \dots,\}$, i is a nonnegative integer, $T(\triangle_1, \triangle_2) = \triangle_1 + \triangle_2 + I$, $\triangle_1 y_{mn} = y_{m+1,n} - y_{mn}$, $\triangle_2 y_{mn} = y_{m,n+1} - y_{mn}$, $I_{mn}y_{mn} = y_{mn}$.

1. Introduction. Partial difference equations are popular and important in many applications such as those involving population dynamics with spatial migrations, chemical reactions, etc., and also in computation and analysis of finite difference equations [**2, 3, 9, 10**]. In the past several years, the qualitative theory of partial difference equations have been extensively investigated, see [**1, 5–8, 11–17**] and references therein. In particular, oscillations of all solutions of the second order nonlinear partial difference equation

$$T(\triangle_1, \triangle_2)[c_{mn} \triangle_1 (y_{mn})] + \sum_{i=1}^{s} a_i(m,n) f_i(y_{m+1,n}, \triangle_1(y_{mn})) = 0$$

have been studied [**4**], where $T(\triangle_1, \triangle_2) = \triangle_1 + \triangle_2 + I$, $\triangle_1 y_{mn} = y_{m+1,n} - y_{mn}$, $\triangle_2 y_{mn} = y_{m,n+1} - y_{mn}$ and $I(y_{mn}) = y_{mn}$. Let $N_i = \{i, i+1, \dots,\}$, where i is a nonnegative integer, $\{a_i(m,n)\}_{(m,n)\in N_0^2}$ are real double sequences, $i = 1, 2, \dots, s$, and s is a positive integer, the double sequence $\{c_{mn}\}_{(m,n)\in N_0^2}$ is assumed to be positive.

The research was partially supported by the NNSF of China (no. 60372028), the project sponsored by SRF for ROCS, SEM and the Hong Kong CERG City U 1115/03E.
Key words and phrases. Nonlinear partial difference equation, oscillation.
Received by the editors on June 26, 2001, and in revised form on April 22, 2002.

In this paper, we consider the oscillatory behaviors of all solutions of the second-order nonlinear partial difference equation of the form

$$T(\triangle_1, \triangle_2)\left[c_{mn}T(\triangle_1, \triangle_2)(y_{mn})\right] + p(m,n)(y_{m+1,n} + y_{m,n+1})^{\nu} = 0, \tag{1}$$

where ν is a quotient of odd positive integers, $m, n \in N_i$. Some sufficient conditions for oscillation of all solutions of the above equation with $\nu > 1$ and $\nu < 1$ are obtained.

By a solution of Equation (1), we mean a real double sequence $\{y_{mn}\}$ satisfying (1) for $m, n \in N_0$. We consider only such solutions that are nontrivial for all large m, n. A solution $\{y_{mn}\}$ of (1) is called *nonoscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

2. Main results. The following elementary identity for double sequences will be needed later.

Lemma 1 [**13, 17**].

$$\sum_{i=m-k}^{m}\sum_{j=n-l}^{n}(A_{i+1,j} + A_{i,j+1} - A_{ij})$$
$$= \sum_{i=m+1-k}^{m+1}\sum_{j=n+1-l}^{n}A_{ij} + \sum_{i=m-k}^{m}A_{i,n+1} - A_{m-k,n-l} + A_{m+1,n-l}.$$

We consider the case where $c_{ij} > 0$ for all $i \geq 0$, $j \geq 0$ and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}1/c_{ij} < \infty. \tag{2}$$

Theorem 1. *Assume that $c_{ij} > 0$ for all $i \geq 0$, $j \geq 0$ and* (2) *holds. Further, assume that $\nu > 1$ and $p_{ij} > 0$ for all $i \geq 0$, $j \geq 0$, and*

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty}\left(\frac{1}{2}\right)^{i+j}p_{ij}\rho_{i+1,j}^{\nu} = \infty, \tag{3}$$

where

$$\rho_{mn} = \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} 1/c_{ij}. \tag{4}$$

Then all solutions of Equation (1) *are oscillatory.*

Proof. Assume the contrary, namely, that there exists a nonoscillatory solution $\{y_{mn}\}$. Without loss of generality, assume that $y_{mn} > 0$ for $m \geq M, n \geq N$. Then

$$T(\triangle_1, \triangle_2)[c_{mn}T(\triangle_1, \triangle_2)(y_{mn})] \leq 0 \quad \text{for } m \geq M, n \geq N. \tag{5}$$

In view of (5), we have

$$c_{m+1,n}T(\triangle_1, \triangle_2)(y_{m+1,n}) \leq c_{mn}T(\triangle_1, \triangle_2)(y_{mn}),$$
$$c_{m,n+1}T(\triangle_1, \triangle_2)(y_{m,n+1}) \leq c_{mn}T(\triangle_1, \triangle_2)(y_{mn}),$$

that is, $\{c_{mn}T(\triangle_1, \triangle_2)(y_{mn})\}$ is nonincreasing, thus

(6)

$$c_{mn}T(\triangle_1, \triangle_2)(y_{mn}) \leq c_{MN}T(\triangle_1, \triangle_2)(y_{MN}) \quad \text{for } m \geq M,\ n \geq N,$$

or

$$T(\triangle_1, \triangle_2)(y_{mn}) \leq c_{MN}T(\triangle_1, \triangle_2)(y_{MN})/c_{mn} \ \text{ for } m \geq M, n \geq N.$$

Applying Lemma 1 and summing the above inequality from M, N to m, n, we obtain

$$\begin{aligned} y_{mn} - y_{MN} &\leq y_{mn} + \left(\sum_{i=M+1}^{m-1} \sum_{j=N+1}^{n} y_{ij} + \sum_{j=N+1}^{n-1} y_{mj} + \ . \ . \ . \right) - y_{MN} \\ &= \sum_{i=M+1}^{m+1} \sum_{j=N+1}^{n} y_{ij} + \sum_{i=M}^{m} y_{i,n+1} + y_{m+1,N} - y_{MN} \\ &\leq [c_{MN}T(\triangle_1, \triangle_2)(y_{MN})] \sum_{i=M}^{m} \sum_{j=N}^{n} 1/c_{ij}, \end{aligned}$$

that is,

$$y_{mn} - y_{MN} \le [c_{MN}T(\triangle_1, \triangle_2)(y_{MN})] \sum_{i=M}^{m} \sum_{j=N}^{n} 1/c_{ij} \tag{7}$$
$$\text{for } m \ge M, n \ge N.$$

Hence, y_{mn} is bounded above. From (7), we have

(8)

$$y_{MN} \ge -[c_{MN}T(\triangle_1, \triangle_2)(y_{MN})] \sum_{i=M}^{m} \sum_{j=N}^{n} 1/c_{ij} \quad \text{for } m \ge M, n \ge N.$$

Letting $m, n \to \infty$ gives

$$y_{MN} \ge -[c_{MN}T(\triangle_1, \triangle_2)(y_{MN})]\rho_{MN}, \tag{9}$$

where ρ_{MN} is defined by (4) and M, N are two sufficiently large numbers.

It follows from (5) that there are two possible cases of $T(\triangle_1, \triangle_2)(y_{mn})$. First, we consider the case where $T(\triangle_1, \triangle_2)(y_{mn}) \ge 0$ for $m \ge M$, $n \ge N$. Summing (1) from M, N to m, n, we obtain

$$\sum_{i=M}^{m} \sum_{j=N}^{n} T(\triangle_1, \triangle_2)\left[c_{ij}T(\triangle_1, \triangle_2)(y_{ij})\right] + \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij}(y_{m+1,n} + y_{m,n+1})^{\nu} = 0. \tag{10}$$

Applying Lemma 1 again, we obtain

$$c_{mn}T(\triangle_1, \triangle_2)(\ y_{mn}) - c_{MN}T(\triangle_1, \triangle_2)(y_{MN}) + \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij}(y_{m+1,n} + y_{m,n+1})^{\nu} \le 0,$$

or

$$\sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij}(y_{m+1,n} + y_{m,n+1})^{\nu} \le c_{MN}T(\triangle_1, \triangle_2)(y_{MN}).$$

Letting $m, n \to \infty$ yields

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} p_{ij}(y_{m+1,n} + y_{m,n+1})^{\nu} < \infty. \tag{11}$$

Since $c_{mn}T(\triangle_1, \triangle_2)(y_{mn}) \geq 0$ for $m \geq M$, $n \geq N$, that is,

$$y_{m+1,n} + y_{m,n+1} \geq y_{mn},$$

there exists a positive number, c, such that $y_{mn} > c > 0$ for $m > M, n \geq N$. Thus, there exist $M_1 \geq M$, $N_1 \geq N$, such that

$$y_{m+1,n} + y_{m,n+1} \geq y_{m+1,n} \geq \rho_{m+1,n} \quad \text{for } m \geq M_1,\ n \geq N_1, \tag{12}$$

since $\rho_{mn} \to 0$ as $m, n \to \infty$. Combining (11) and (12), we have

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} p_{ij}\rho_{i+1,j}^{\nu} < \infty, \tag{13}$$

that is,

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} \left(\frac{1}{2}\right)^{i+j} p_{ij}\rho_{i+1,j}^{\nu} < \infty, \tag{14}$$

which contradicts (3).

Next, we consider the other case where

$$c_{mn}T(\triangle_1, \triangle_2)(y_{mn}) < 0, \quad \text{for } m \geq M, n \geq N.$$

We have

$$\begin{aligned}
&T(\triangle_1,\triangle_2)\Big[\Big(\frac{1}{2}\Big)^{m+n-1}(c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))^{-\nu+1}\Big]\\
&=\Big(\frac{1}{2}\Big)^{m+n}(c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n}))^{-\nu+1}\\
&\quad+\Big(\frac{1}{2}\Big)^{m+n}(c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1}))^{-\nu+1}\\
&\quad-\Big(\frac{1}{2}\Big)^{m+n-1}(c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))^{-\nu+1}\\
&=\Big(\frac{1}{2}\Big)^{m+n}\left\{\begin{smallmatrix}(c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n}))^{-\nu+1}+(c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1}))^{-\nu+1}\\ -2(c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))^{-\nu+1}\end{smallmatrix}\right\}\\
&=\Big(\frac{1}{2}\Big)^{m+n}\left\{\begin{smallmatrix}(c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n}))^{-\nu+1}-(c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))^{-\nu+1}\\ +(c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1}))^{-\nu+1}-(c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))^{-\nu+1}\end{smallmatrix}\right\}\\
&=(-\nu+1)\Big(\frac{1}{2}\Big)^{m+n}\left\{\begin{smallmatrix}\xi^{-\nu}(c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n})-c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))\\ +\eta^{-\nu}(c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1})-c_{mn}T(\triangle_1,\triangle_2)(y_{mn}))\end{smallmatrix}\right\}\\
&\le(-\nu+1)\xi^{-\nu}\Big(\frac{1}{2}\Big)^{m+n}\left(\begin{smallmatrix}c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n})+c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1})\\ -2c_{mn}T(\triangle_1,\triangle_2)(y_{mn})\end{smallmatrix}\right)\\
&\le(-\nu+1)\xi^{-\nu}\Big(\frac{1}{2}\Big)^{m+n}\left(\begin{smallmatrix}c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n})+c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1})\\ -c_{mn}T(\triangle_1,\triangle_2)(y_{mn})\end{smallmatrix}\right)\\
&=(-\nu+1)\xi^{-\nu}\Big(\frac{1}{2}\Big)^{m+n}\Big\{T(\triangle_1,\triangle_2)\Big[c_{mn}T(\triangle_1,\triangle_2)(y_{mn})\Big]\Big\}\\
&=(-\nu+1)\xi^{-\nu}\Big(\frac{1}{2}\Big)^{m+n}\Big[-p_{mn}(y_{m+1,n}+y_{m,n+1})^{\nu}\Big]
\end{aligned}$$

where

$$\begin{aligned}
&c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n})<\xi<c_{mn}T(\triangle_1,\triangle_2)(y_{mn}),\\
&c_{m,n+1}T(\triangle_1,\triangle_2)(y_{m,n+1})\le\eta\le c_{mn}T(\triangle_1,\triangle_2)(y_{mn}),
\end{aligned}$$

and, without loss of generality, let $\xi^{-\nu}=\min(\xi^{-\nu},\eta^{-\nu})$.

We note that $y_{mn}\ge-[c_{mn}T(\triangle_1,\triangle_2)(y_{mn})]\rho_{mn}$, for $m\ge M, n\ge N$ by (9). Hence, we also have

$$y_{m+1,n}\ge-[c_{m+1,n}T(\triangle_1,\triangle_2)(y_{m+1,n})]\rho_{m+1,n},$$

so that

(15)

$$y_{m,n+1} + y_{m+1,n} \geq y_{m+1,n} \geq -[c_{m+1,n}T(\triangle_1, \triangle_2)(y_{m+1,n})]\rho_{m+1,n}.$$

Equation (15) implies that

$$\begin{aligned}
&T(\triangle_1, \triangle_2)\left[\left(\frac{1}{2}\right)^{m+n-1}(c_{mn}T(\triangle_1, \triangle_2)(y_{mn}))^{-\nu+1}\right] \\
&\leq \left(\frac{1}{2}\right)^{m+n}(-\nu+1)\xi^{-\nu}\left[-p_{mn}(y_{m+1,n}+y_{m,n+1})^{\nu}\right] \\
&\leq \left(\frac{1}{2}\right)^{m+n}(-\nu+1)\xi^{-\nu}\left\{p_{mn}\left[(c_{m+1,n}T(\triangle_1, \triangle_2)(y_{m+1,n}))\,\rho_{m+1,n}\right]^{\nu}\right\} \\
&\leq \left(\frac{1}{2}\right)^{m+n}(-\nu+1)\left\{\,p_{mn}\left[(c_{m+1,n}T(\triangle_1, \triangle_2)(y_{m+1,n}))\,\rho_{m+1,n}\right]^{\nu}\right\} \\
&\cdot\left[c_{m+1,n}T(\triangle_1, \triangle_2)(y_{m+1,n})\right]^{-\nu}. \\
&= -(\nu-1)\left(\frac{1}{2}\right)^{m+n}p_{mn}\rho^{\nu}_{m+1,n}.
\end{aligned}$$

Hence,

$$T(\triangle_1, \triangle_2)\left[\left(\frac{1}{2}\right)^{m+n-1}(c_{mn}T(\triangle_1, \triangle_2)(y_{mn}))^{-\nu+1}\right] \leq -(\nu-1)\left(\frac{1}{2}\right)^{m+n}p_{mn}\rho^{\nu}_{m+1,n}. \tag{16}$$

Using Lemma 1 and summing (16) from M, N to m, n, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{M+n}(c_{M,n+1}T(\triangle_1, \triangle_2)(y_{M,n+1}))^{-\nu+1} \\
&\quad - \left(\frac{1}{2}\right)^{M+N-1}(c_{MN}T(\triangle_1, \triangle_2)(y_{MN}))^{-\nu+1} \\
&\quad \leq \sum_{i=M+1}^{m}\sum_{j=N}^{n}\left[\left(\frac{1}{2}\right)^{i+j}(c_{i,j+1}T(\triangle_1, \triangle_2)(y_{i,j+1}))^{-\nu+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=N}^{n}\left[\left(\frac{1}{2}\right)^{m+j}(c_{m+1,j}T(\triangle_1,\triangle_2)(y_{m+1,j}))^{-\nu+1}\right]\\
&+\left(\frac{1}{2}\right)^{M+n}(c_{M,n+1}T(\triangle_1,\triangle_2)(y_{M,n+1}))^{-\nu+1}\\
&-\left(\frac{1}{2}\right)^{M+N-1}(c_{MN}T(\triangle_1,\triangle_2)(y_{MN}))^{-\nu+1}\\
=&\sum_{i=M}^{m}\sum_{j=N}^{n}T(\triangle_1,\triangle_2)\left[\left(\frac{1}{2}\right)^{i+j-1}(c_{ij}T(\triangle_1,\triangle_2)(y_{ij}))^{-\nu+1}\right]\\
\le& -(\nu-1)\sum_{i=M}^{m}\sum_{j=N}^{n}\left(\frac{1}{2}\right)^{i+j}p_{ij}\rho_{i+1,j}^{\nu},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&-\left(\frac{1}{2}\right)^{M+N-1}(c_{MN}T(\triangle_1,\triangle_2)(y_{MN}))^{-\nu+1}\\
\ge& (\nu-1)\sum_{i=M}^{m}\sum_{j=N}^{n}\left(\frac{1}{2}\right)^{i+j}p_{ij}\rho_{i+1,j}^{\nu}\\
&-\left(\frac{1}{2}\right)^{M+N-1}(c_{MN}T(\triangle_1,\triangle_2)(y_{MN}))^{-\nu+1}.
\end{aligned}
$$

So, letting $m,n\to\infty$, we have

$$
\sum_{i=M}^{m}\sum_{j=N}^{n}\left(\frac{1}{2}\right)^{i+j}p_{ij}\rho_{i+1,j}^{\nu}<\infty,
$$

which contradicts (3). This completes the proof of the theorem. □

Example 1. Consider the partial difference equation

$$
(\mathrm{E}_1)\quad T(\Delta_1,\Delta_2)\left(2^{m+n}T(\Delta_1,\Delta_2)(y_{mn})\right) + 3\times 2^{3m+n+1}(y_{m+1,n}+y_{m,n+1})^3=0,
$$

where $c_{mn}=2^{m+n}$, $p_{mn}=3\times 2^{3m+n+1}$, and $\nu=3$. It is easy to see that all assumptions of Theorem 1 hold. So, Equation (E_1) has an oscillatory solution $\{y_{mn}\}$. In fact, $\{y_{mn}\} = \{(-1)^n/2^m\}$ is such a solution.

Now, we consider the sublinear case, i.e., $0 < \nu < 1$.

Theorem 2. *Assume that* $c_{ij} > 0$ *for all* $i \ge 0$, $j \ge 0$, *and* (2) *holds. Further, assume that* $0 < \nu < 1$ *and* $p_{ij} > 0$ *for all* $i \ge 0$, $j \ge 0$, *and*

$$\sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \left(\frac{1}{2}\right)^{i+j} p_{ij}\rho_{i+1,j} = \infty. \tag{17}$$

Then, all solutions of Equation (1) *oscillate.*

Proof. Assume the contrary, namely, that there exists a nonoscillatory solution, $\{y_{mn}\}$. Without loss of generality, assume that $y_{mn} > 0$ for $m \ge M$, $n \ge N$. Then

$$T(\triangle_1, \triangle_2)[c_{mn}T(\triangle_1, \triangle_2)(y_{mn})] \le 0 \quad \text{for } m \ge M, n \ge N.$$

If $c_{mn}T(\triangle_1, \triangle_2)(y_{mn}) \ge 0$ for $m \ge M$, $n \ge N$, we have (11) and (13). For large m, n, we have $\rho_{mn} \le 1$ and $\rho_{mn}^{\nu} \ge \rho_{mn}$. Therefore, from (13), we have

$$\sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \left(\frac{1}{2}\right)^{i+j} p_{ij}\rho_{i+1,j} < \infty, \tag{18}$$

which contradicts (17).

For the case where $c_{mn}T(\triangle_1, \triangle_2)(y_{mn}) < 0$ for $m \ge M$, $n \ge N$, using Lemma 1 and summing (1) from M, N to m, n, we obtain

$$\sum_{i=M}^{m} \sum_{j=N}^{n} T(\triangle_1, \triangle_2)\left[c_{ij}T(\triangle_1, \triangle_2)(y_{ij})\right] + \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij}(y_{i+1,j} + y_{i,j+1})^{\nu} = 0,$$

that is,

$$\begin{aligned}\sum_{i=M+1}^{m+1} \sum_{j=N+1}^{n} [c_{ij}T(\triangle_1, \triangle_2)(y_{ij})] + \sum_{i=M}^{m} [c_{i,n+1}T(\triangle_1, \triangle_2)(y_{i,n+1})] \\ + [c_{m+1,N}T(\triangle_1, \triangle_2)(y_{m+1,N})] - [c_{MN}T(\triangle_1, \triangle_2)(y_{MN})] \\ + \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij}(y_{i+1,j} + y_{i,j+1})^{\nu} = 0.\end{aligned}$$

Hence,

$$
\begin{aligned}
&- [c_{m+1,n} T(\triangle_1, \triangle_2)(y_{m+1,n})] \\
&\qquad\qquad \geq - \begin{cases} \sum_{i=M+1}^{m+1} \sum_{j=N+1}^{n} [c_{ij} T(\triangle_1, \triangle_2)(y_{ij})] \\ + \sum_{i=M}^{m} [c_{i,n+1} T(\triangle_1, \triangle_2)(y_{i,n+1})] \\ + [c_{m+1,N} T(\triangle_1, \triangle_2)(y_{m+1,N})] \end{cases} \\
&\qquad\qquad \geq \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij} (y_{i+1,j} + y_{i,j+1})^{\nu} \quad \text{for } m \geq M, \ n \geq N,
\end{aligned}
$$

that is,

$$
\begin{aligned}
(19) \qquad - [T(\triangle_1, \triangle_2)(y_{m+1,n})] \geq \frac{1}{c_{m+1,n}} \sum_{i=M}^{m} \sum_{j=N}^{n} p_{ij} (y_{i+1,j} + y_{i,j+1})^{\nu} \\
\text{for } m \geq M, \ n \geq N.
\end{aligned}
$$

We consider the partial difference $T(\triangle_1, \triangle_2) \left[(1/2)^{m+n-1} y_{m+1,n}^{2\varepsilon} \right]$, where $\varepsilon > 0$, such that $2\varepsilon < 1 - \nu$. Note that y_{mn} is nonincreasing. Thus,

$$
\begin{aligned}
(20) \quad & - T(\triangle_1, \triangle_2) \left[\left(\frac{1}{2} \right)^{m+n-1} y_{m+1,n}^{2\varepsilon} \right] \\
&= - \left(\frac{1}{2} \right)^{m+n} \left[y_{m+2,n}^{2\varepsilon} + y_{m+1,n+1}^{2\varepsilon} - 2 y_{m+1,n}^{2\varepsilon} \right] \\
&= - \left(\frac{1}{2} \right)^{m+n} \left[\left(y_{m+2,n}^{2\varepsilon} - y_{m+1,n}^{2\varepsilon} \right) + \left(y_{m+1,n+1}^{2\varepsilon} - y_{m+1,n}^{2\varepsilon} \right) \right] \\
&= -2\,\varepsilon \left(\frac{1}{2} \right)^{m+n} \Bigg[\lambda^{2\varepsilon - 1} \left(y_{m+2,n} - y_{m+1,n} \right) \\
&\qquad + \mu^{2\varepsilon - 1} \left(y_{m+1,n+1} - y_{m+1,n} \right) \Bigg] \\
&\geq -2\,\varepsilon \left(\frac{1}{2} \right)^{m+n} \lambda^{2\varepsilon - 1} \left(y_{m+2,n} + y_{m+1,n+1} - 2 y_{m+1,n} \right) \\
&\geq -2\,\varepsilon \left(\frac{1}{2} \right)^{m+n} \lambda^{2\varepsilon - 1} \left(y_{m+2,n} + y_{m+1,n+1} - y_{m+1,n} \right) \\
&= 2\,\varepsilon \left(\frac{1}{2} \right)^{m+n} \lambda^{2\varepsilon - 1} \left[-T(\triangle_1, \triangle_2)(y_{m+1,n}) \right],
\end{aligned}
$$

where $y_{m+2,n} \le \lambda \le y_{m+1,n}$, $y_{m+1,n+1} \le \eta \le y_{m+1,n}$, and without loss of generality, $\lambda^{2\varepsilon-1} = \min(\lambda^{2\varepsilon-1}, \mu^{2\varepsilon-1})$. Substituting (19) into (20) gives

$$\begin{aligned}
&-T(\triangle_1,\triangle_2)\left[\left(\frac{1}{2}\right)^{m+n-1} y_{m+1,n}^{2\varepsilon}\right]\\
&\ge 2\,\varepsilon\left(\frac{1}{2}\right)^{m+n} \lambda^{2\varepsilon-1}\frac{1}{c_{m+1,n}}\sum_{i=M}^{m}\sum_{j=N}^{n} p_{ij}(y_{i+1,j}+y_{i,j+1})^{\nu}\\
&\ge 2\,\varepsilon\left(\frac{1}{2}\right)^{m+n} y_{m+1,n}^{2\varepsilon-1}\frac{1}{c_{m+1,n}}\sum_{i=M}^{m}\sum_{j=N}^{n} p_{ij}(y_{i+1,j}+y_{i,j+1})^{\nu}\\
&\ge 2\,\varepsilon\left(\frac{1}{2}\right)^{m+n} \frac{(y_{m+1,n}+y_{m,n+1})^{2\varepsilon-1}}{c_{m+1,n}}\sum_{i=M}^{m}\sum_{j=N}^{n} p_{ij}(y_{i+1,j}+y_{i,j+1})^{\nu}\\
&\ge \left(\frac{1}{2}\right)^{m+n} \frac{2\,\varepsilon}{c_{m+1,n}}\sum_{i=M}^{m}\sum_{j=N}^{n} p_{ij}(y_{i+1,j}+y_{i,j+1})^{\nu+2\varepsilon-1}.
\end{aligned}$$

Since $H \ge y_{mn} > 0$ for $m \ge M$, $n \ge N$, where $H > 0$ is a constant, there exists a positive number K such that

$$-T(\triangle_1,\triangle_2)\left[\left(\frac{1}{2}\right)^{m+n-1} y_{m+1,n}^{2\varepsilon}\right] \ge \frac{K}{c_{m+1,n}}\left(\frac{1}{2}\right)^{m+n}\sum_{i=M}^{m}\sum_{j=N}^{n} p_{ij},$$

Summing both sides of this inequality gives

$$\begin{aligned}
-\Bigg(\sum_{i=M+1}^{m}\sum_{j=N}^{n}\left(\frac{1}{2}\right)^{i+j} y_{i,j+1}^{2\varepsilon} &+ \sum_{j=N}^{n}\left(\frac{1}{2}\right)^{m+j} y_{m+1,j}^{2\varepsilon}\\
&+\left(\frac{1}{2}\right)^{M+n} y_{M,n+1}^{2\varepsilon} - \left(\frac{1}{2}\right)^{M+N-1} y_{MN}^{2\varepsilon}\Bigg)\\
&= -\sum_{i=M}^{m}\sum_{j=N}^{n} T(\triangle_1,\triangle_2)\left[\left(\frac{1}{2}\right)^{i+j-1} y_{i+1,j}^{2\varepsilon}\right]\\
&\ge K\sum_{i=M}^{m}\sum_{j=N}^{n}\frac{1}{c_{i+1,j}}\left(\frac{1}{2}\right)^{i+j}\sum_{u=M}^{i}\sum_{v=N}^{j} p_{uv}.
\end{aligned}$$

Hence,

$$\left(\frac{1}{2}\right)^{M+N-1} y_{MN}^{2\varepsilon} - \left(\frac{1}{2}\right)^{m+n} y_{m+1,n}^{2\varepsilon} \geq K \sum_{i=M}^{m} \sum_{j=N}^{n} \frac{1}{c_{i+1,j}} \left(\frac{1}{2}\right)^{i+j} \sum_{u=M}^{i} \sum_{v=N}^{j} p_{uv}.$$

By rearranging the double sum, we have

$$\left(\frac{1}{2}\right)^{M+N-1} y_{MN}^{2\varepsilon} - \left(\frac{1}{2}\right)^{m+n} y_{m+1,n}^{2\varepsilon} \geq K \sum_{u=M}^{m} \sum_{v=N}^{n} \left(\frac{1}{2}\right)^{u+v} p_{uv} \sum_{i=u}^{m} \sum_{j=v}^{n} \frac{1}{c_{i+1,j}},$$

and so letting $m, n \to \infty$ yields

$$\sum_{u=M}^{\infty} \sum_{v=N}^{\infty} \left(\frac{1}{2}\right)^{u+v} p_{uv} \rho_{u+1,v} < \infty,$$

which is a contradiction.

Example 2. Consider the partial difference equation

$$(\mathrm{E}_2) \quad T(\Delta_1, \Delta_2)\left(e^{m+n} T(\Delta_1, \Delta_2)(y_{mn})\right) + \left(\frac{3e}{4} - 1\right) 4^{-m+1/3} e^{m+n} (y_{m+1,n} + y_{m,n+1})^{1/3} = 0,$$

where $c_{mn} = e^{m+n}$, $p_{mn} = ((3e/4) - 1)\, 4^{-(m+1/3)} e^{m+n}$ and $\nu = 1/3$. It is easy to see that all assumptions of Theorem 2 hold. So, Equation (E_2) has a oscillatory solution $\{y_{mn}\}$. In fact, $\{y_{mn}\} = \{(-1)^m / 2^m\}$ is such a solution.

REFERENCES

1. I. Gyori and G. Ladas, *Oscillation theorem of delay differential equations with applications*, Clarendon Press, Oxford, 1991.

2. W.G. Kelley and A.C. Peterson, *Difference equations*, Academic Press, New York, 1991.

3. X.-P. Li, *Partial difference equations used in the study of molecular orbits*, Acta Chimica SINICA **40** (1982), 688–698 (in Chinese).

4. S.T. Liu, *Oscillation theorems for second-order nonlinear partial difference equations*, J. Comp. Appl. Math., 2001.

5. S.T. Liu, F.Q. Deng and B.G. Zhang, *Oscillation of delay partial difference equations with positive and negative coefficients*, Southeast Asian Bull. Math., 2001.

6. S.T. Liu, X.P. Guan and J. Yang, *Nonexistence of positive solution of a class of nonlinear delay partial difference equations*, J. Math. Anal. Appl. **234** (1999), 361–371.

7. S.T. Liu and H. Wang, *Necessary and sufficient conditions for oscillations of a class of delay partial difference equations*, Dynam. Syst. Appl. **7** (1998), 495–500.

8. S.T. Liu and B.G. Zhang, *Oscillation of a class of partial difference equations*, Panamer. Math. J. **8** (1998), 93–100.

9. H. Lvy and F. Lessman, *Finite difference equations*, Dover Publications, New York, 1992.

10. J.C. Strikwerda, *Finite difference schemes and partial differential equations*, Wadsworth and Brooks, California, 1989.

11. P.J.Y. Wong and R.P. Agarwal, *On the oscillation of partial difference equations generated by deviating arguments*, Acta Math. Hungar. **79** (1998), 1–29.

12. ———, *Oscillation criteria for nonlinear partial difference equations with delays*, Comput. Math. Appl. **32** (1996), 57–86.

13. B.G. Zhang and S.T. Liu, *On the oscillation of two partial difference equations*, J. Math. Anal. Appl. **206** (1997), 480–492.

14. ———, *Oscillation of partial difference equations*, Panamer. Math. J. **5** (1995), 61–71.

15. ———, *Necessary and sufficient conditions for oscillations of delay partial difference equations*, Discussions on Mathematics–Differential Indusions **15** (1995), 213–219.

16. ———, *Oscillation of partial difference equations with variable coefficients*, Comput. Math. Appl. **36–38** (1998), 235–242.

17. B.G. Zhang, S.T. Liu and S.S. Cheng, *Oscillation of a class of delay partial difference equation*, J. Difference Equations Appl. **1** (1995), 215–226.

College of Control Science and Engineering, Shandong University, Jinan, 250041, P.R. China
Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, P.R. China
E-mail address: `stliu@sdu.edu.cn`

Dept. of Electronic Engineering, City Univ. of Hong Kong, P.R. China
Dept. of Electrical and Computer Engineering, University of Houston, Houston, TX 77204, U.S.A.