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ON THE NORM OF IDEMPOTENTS IN C^*-ALGEBRAS

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ABSTRACT. In this paper we study norms of idempotents in C^*-algebras. Results of Ljance, Vidav, Buckholtz and Wimmer on idempotent operators in Hilbert spaces are considered in the setting of C^*-algebras, and simpler new proofs, based on algebraic and spectral–rather than spatial–arguments, are given. We give an application to projections with respect to a-involutions.

1. Introduction. The paper addresses the twin problem of the existence of an idempotent h in a C^*-algebra $\mathcal{A}$ satisfying $h\mathcal{A} = p\mathcal{A}$ and $(1-h)\mathcal{A} = q\mathcal{A}$, where p, q are given projections (self-adjoint idempotents) in $\mathcal{A}$, and of the exact value of $\|h\|$ if h exists. We denote such an idempotent h by $\pi(p, q)$.

Ljance [**10**] showed in 1959 that, for Hilbert space operators, $\|h\| = (1 - \|pq\|^2)^{-1/2}$. In 1964 Vidav [**15**] found necessary and sufficient conditions for the existence of $\pi(p, q)$, again in the case of Hilbert space operators. Pták [**13**], apparently unaware of the work of Vidav, and originally also of Ljance, gave in 1984 a solution to both problems, and applied it to extremal operators.

Recently the Hilbert space version of the topic was revisited by Buckholtz [**3, 4**], Galántai [**5**], Wimmer [**16, 17**], and the second author [**14**]. The first author [**8**] extended Vidav's results to C^*-algebras.

The purpose of this paper is to consider the existence of $\pi(p, q)$ and Ljance's formula in C^*-algebras, and to give alternative simpler proofs of these theorems. The spectral results on two projections in a C^*-algebra given in Lemma 2.4 hold the key to this simplification. We believe that avoiding spatial arguments in Hilbert spaces in favor of

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simpler algebraic and spectral techniques gives a greater insight into both problems.

2. Preliminaries. We denote by $\mathcal{A}$ a C^*-algebra with unit 1 and by $\mathcal{A}^{-1}$ the set of all invertible elements in $\mathcal{A}$. For an element $a \in \mathcal{A}$ we denote by $\sigma(a)$ the spectrum of a and by $r(a)$ the spectral radius of a.

The term *projection* will be reserved for an element p of a C^*-algebra $\mathcal{A}$ which is self-adjoint and idempotent, that is, $p^* = p = p^2$. If $f, g \in \mathcal{A}$ are idempotents, then $f\mathcal{A} \subset g\mathcal{A} \iff gf = f$; consequently,

$$f\mathcal{A} = g\mathcal{A} \iff gf = f \quad \text{and} \quad fg = g. \tag{2.1}$$

This provides a geometrical motivation for the definition of the range projection. Let $f \in \mathcal{A}$ be an idempotent. Following Koliha [**8**], we say that $p \in \mathcal{A}$ is a *range projection* of f if p is a projection satisfying

$$pf = f \quad \text{and} \quad fp = p. \tag{2.2}$$

If $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H, then (2.2) holds if and only if p is the (orthogonal) projection onto the range of f. Let us recall [**8**, Theorem 1.3] that, for every idempotent $f \in \mathcal{A}$, there exists a unique range projection of f denoted by $f^\perp$ given explicitly by the Kerzman-Stein formula [**7**]

$$f^\perp = f(f + f^* - 1)^{-1}. \tag{2.3}$$

If p is a projection, then $p^\perp = p$. Recall that [**8**, Proposition 1.4]

$$1 - f^\perp = (1 - f^*)^\perp \quad \text{and} \quad 1 - (f^*)^\perp = (1 - f)^\perp. \tag{2.4}$$

Definition 2.1. Let $e, f \in \mathcal{A}$ be idempotents. By $\pi(e, f)$ we denote an idempotent $h \in \mathcal{A}$, if it exists, satisfying the conditions

$$h^\perp = e^\perp, \quad (1 - h)^\perp = f^\perp. \tag{2.5}$$

Motivated by results obtained for bounded linear operators on Hilbert spaces by Labrousse [**9**], Vidav [**15**], Pták [**13**] and Buckholtz [**3, 4**], the first author [**8**] considered the problem of finding $\pi(p,q)$ in the case when p, q are projections in a C^*-algebra $\mathcal{A}$. In this paper we give a new proof of the existence of $\pi(p,q)$ for projections p, q in Theorem 4.1, and discuss a more general case in Theorem 5.2. In comparison with the proofs in [**3, 4, 9, 10, 12, 13, 15**–**17**], which depend on spatial arguments, our proofs use algebraic and spectral techniques in C^*-algebras.

Basic auxiliary results are summarized in the following three lemmas. The first is the well-known Akhiezer-Glazman equality. See [**1**] for the Hilbert space setting and [**11**, Lemma 1 (i)] for a C^*-algebra formulation.

Lemma 2.2. *If p, q are projections in a C^*-algebra $\mathcal{A}$, then*

$$\|p-q\| = \max\{\|p(1-q)\|, \|q(1-p)\|\}. \tag{2.6}$$

The following result was obtained for bounded linear operators on Hilbert spaces by Del Pasqua [**12**], see also [**6, 8, 10, 14**]. We give a proof based on matrix representations.

Lemma 2.3. *If $h \in \mathcal{A}$ is a nontrivial idempotent, then*

$$\|h\| = \|1-h\| = \|h+h^*-1\|. \tag{2.7}$$

Proof. Let $p = h^{\perp}$. The C^*-algebra $\mathcal{A}$ has a matrix representation which preserves the involution in $\mathcal{A}$, namely

$$x = \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p). \end{bmatrix}$$

Recall that since p is a projection, $p\mathcal{A}p$ and $(1-p)\mathcal{A}(1-p)$ are C^*-algebras with units p and $1-p$, respectively.

Let $u = h-p$. Then $(h+h^*-1)^2 = 1+uu^*+u^*u$, and

$$(h+h^*-1)^2 = \begin{bmatrix} 1+uu^* & 0 \\ 0 & 1+u^*u \end{bmatrix}.$$

Similarly,

$$h^*h = \begin{bmatrix} 1+uu^* & 0 \\ 0 & 0 \end{bmatrix}, \quad (1-h)^*(1-h) = \begin{bmatrix} 0 & 0 \\ 0 & 1+u^*u \end{bmatrix}.$$

As $\sigma(1+uu^*) = \sigma(1+u^*u)$, we have

$$\sigma((h+h^*-1)^2) \cup \{0\} = \sigma(h^*h) = \sigma((1-h)^*(1-h)),$$

from which (2.7) follows via the formula $\|x\| = \|x^*x\|^{1/2} = r(x^*x)^{1/2}$. □

The next result summarizes pertinent spectral properties of a pair of projections. This lemma, in particular part (v), is the key to the proof of Theorem 3.1.

Lemma 2.4. *Let p, $q \in \mathcal{A}$ be nontrivial projections. Then the following are true.*

(i) $\sigma(pq) = \sigma(pqp) \subset [0, r(pq)] \subset [0, 1]$.

(ii) $r(pq) = r(pqp) = \|pqp\| = \|pq\|^2$.

(iii) $1 - pq \in \mathcal{A}^{-1}$ *if and only if* $\|pq\| < 1$.

(iv) $\sigma(p-q) \subset [-1, 1]$.

(v) *If* $\lambda \in \mathbf{C} \setminus \{0, 1, -1\}$, *then* $\lambda \in \sigma(p-q)$ *if and only if* $1 - \lambda^2 \in \sigma(pq)$.

Proof. (i) For any $\lambda \in \mathbf{C}$,

$$\lambda - pq = \begin{bmatrix} p(\lambda - pqp)p & -pq(1-p) \\ 0 & \lambda(1-p), \end{bmatrix},$$

which implies that $\sigma(pq) = \sigma'(pqp) \cup \{0\}$, where $\sigma'(x)$ stands for the spectrum of $x \in p\mathcal{A}p$ in the algebra $p\mathcal{A}p$. From the equation $\lambda - pqp = p(\lambda - pqp)p + \lambda(1-p)$ we conclude that $\sigma(pqp) = \sigma'(pqp) \cup \{0\}$, and $\sigma(pq) = \sigma(pqp)$ follows. The rest follows from the positivity of $pqp = (pq)(pq)^*$ and the inequality $r(pq) \le \|pq\| \le \|p\|\|q\| = 1$.

To prove (ii) we only need to observe that $\|pq\|^2 = \|(pq)(pq)^*\| = \|pqp\|$.

Property (iii) is a consequence of (i) and (ii), and the inclusion (iv) follows from the Akhiezer-Glazman equality (2.6).

For (v) it is enough to note that, for any $\lambda \in \mathbf{C}$,

$$\begin{aligned}(\lambda - 1 + p)[\lambda - (p-q)](\lambda + 1 - q) \\ &= [(\lambda-1)(\lambda+q) + pq](\lambda+1-q) \\ &= (\lambda-1)(\lambda+1)(\lambda+q) + (\lambda+1)pq - (\lambda-1)(\lambda+1)q - pq \\ &= \lambda(\lambda^2 - 1 + pq). \qquad \square\end{aligned}$$

3. The norm of $h = \pi(p,q)$. In this section we give a formula for the norm of an idempotent h in $\mathcal{A}$ in terms of the range projections of h and $1-h$, a C^*-algebra version of the result obtained for bounded linear operators on Hilbert spaces by Ljance [**10**]. The result was proved also in [**3, 13, 14, 16**] in the setting of Hilbert spaces. Our approach is different in eschewing spatial arguments, and using algebra—and a little analysis.

Theorem 3.1. *Let $h \in \mathcal{A}$ be a nontrivial idempotent. Then*

$$\|h\| = \frac{1}{\sqrt{1 - \|h^\perp(1-h)^\perp\|^2}}. \tag{3.1}$$

Proof. Write $p = h^\perp$ and $q = (1-h)^\perp$. Using equations $ph = h$, $hp = p$, $qh = h + q - 1$, $hq = 0$, we verify that

$$(1-pq)(1+hh^*-h) = 1 = (1+hh^*-h)(1-pq).$$

Hence $1 - pq \in \mathcal{A}^{-1}$, and $\|pq\| < 1$ by Lemma 2.4 (iii).

By the Kerzman-Stein formula (2.3),

$$p = h(h+h^*-1)^{-1}, \qquad q = (h-1)(h+h^*-1)^{-1}.$$

Therefore $p - q = (h+h^*-1)^{-1}$, that is

$$p - q \in \mathcal{A}^{-1} \quad \text{with} \quad (p-q)^{-1} = h + h^* - 1.$$

Since $\|h\| = \|h + h^* - 1\|$ by (2.7), we have

$$\|h\| = \|(p-q)^{-1}\|. \tag{3.2}$$

By Lemma 2.4 (v) we obtain

(3.3)

$$\|(p-q)^{-1}\| = r((p-q)^{-1}) = \frac{1}{\inf\{|\lambda| : \lambda \in \sigma(p-q)\}}$$
$$= \frac{1}{\inf\{|\lambda| : \lambda^2 = 1-t, t \in [0, \|pq\|^2]\}} = \frac{1}{\sqrt{1-\|pq\|^2}}.$$

From (3.2) and (3.3) we get (3.1). □

The theorem has the following useful corollary.

Corollary 3.2. *Let $h \in \mathcal{A}$ be a nontrivial idempotent. Then*

$$\|h^\perp (1-h)^\perp\| = \frac{\sqrt{\|h\|^2 - 1}}{\|h\|}. \tag{3.4}$$

Proof. Clearly (3.1) implies (3.4). □

For P_R and P_K, the projections onto the range R and the null space K of a bounded idempotent operator M in a Hilbert space H, Vidav [**15**, Proof of Theorem 1] proved the inequality

$$\|P_R P_K\| \le \frac{\|M\|}{\sqrt{1+\|M\|^2}}. \tag{3.5}$$

Note that $P_R = M^\perp$ and $P_K = (I - M)^\perp$. Then (3.5) follows from our sharper estimate (3.4).

4. The existence of $h = \pi(p, q)$. The results of the preceding section lead to a simple algebraic proof of the following theorem which extends [**8**, Theorem 2.2] and [**8**, Corollary 2.2]. In the setting of Hilbert spaces, the equivalence of (ii) and (v) is Vidav's result [**15**, Theorem 1], and the equivalence of (i), (ii), (vii) and (viii) was derived by Buckholtz [**3, 4**]. Recall that, for projections $p, q \in \mathcal{A}$, $\pi(p,q)$ denotes an idempotent $h \in \mathcal{A}$ satisfying $p = h^\perp$ and $q = (1-h)^\perp$.

Theorem 4.1. *Let $p, q \in \mathcal{A}$ be nontrivial projections. Then the following conditions are equivalent*:

(i) $\mathcal{A} = p\mathcal{A} \oplus q\mathcal{A}$.

(ii) *The idempotent* $\pi(p,q)$ *exists.*

(iii) $\|pq\| < 1$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(iv) $1 - pq \in \mathcal{A}^{-1}$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(v) $\|pqp\| < 1$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(vi) $1 - pqp \in \mathcal{A}^{-1}$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(vii) $\|p + q - 1\| < 1$.

(viii) $p - q \in \mathcal{A}^{-1}$.

The idempotent $\pi(p,q)$ *is given by the formulae*

$$\pi(p,q) = (1-pqp)^{-1}(p-pq) = (p-q)^{-1}(1-q). \tag{4.1}$$

Proof. (i) $\Longleftrightarrow$ (ii). First assume that $\mathcal{A} = p\mathcal{A} \oplus q\mathcal{A}$. The unit 1 is uniquely decomposed as $1 = h + g$, where $h = pu$ and $g = qv$ for some $u, v \in \mathcal{A}$. From this decomposition we obtain $h = h^2 + hg$ and $g = hg + g^2$, which implies $h - h^2 = g - g^2 = 0$ in view of $p\mathcal{A} \cap q\mathcal{A} = \{0\}$. Hence h, g are idempotents, and $g = 1 - h$. Expressing $p - hp$ in two ways as $p - hp = p(1-up)$ and $p - hp = (1-h)p = qvp$ we conclude that $p - hp = 0$, that is, $hp = p$. On the other hand, $ph = p^2u = pu = h$. This proves that $p = h^\perp$. By symmetry, $q = (1-h)^\perp$.

Conversely, if $h = \pi(p,q)$, then $\mathcal{A} = h\mathcal{A} \oplus (1-h)\mathcal{A} = p\mathcal{A} \oplus q\mathcal{A}$ by (2.1).

(ii) $\Longrightarrow$ (iii). Write $h = \pi(p,q)$. Since $h\mathcal{A} = p\mathcal{A}$ and $(1-h)\mathcal{A} = q\mathcal{A}$, we have $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$. By Corollary 3.2, $\|pq\| = (\|h\|^2 - 1)^{1/2}\|h\|^{-1} < 1$.

The equivalence of (iii)–(vi) follows from Lemma 2.4.

The implication (vi) $\Longrightarrow$ (ii) is established in the proof of [**8**, Theorem 2.1] by verifying that $h = (1-pqp)^{-1}(p-pq) = \pi(p,q)$. Note that the condition $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$ is used to show that $(1-h)q = q$: Indeed, $1-h = pa+qb$ for some a, $b \in \mathcal{A}$. Then $0 = h(1-h) = hpa+hqb = pa$, so that $1-h = qb$ and $q(1-h) = qqb = qb = 1-h$.

(ii) $\Longrightarrow$ (vii). By hypothesis, $h = \pi(p,q)$ exists. We show that

$$\|p+q-1\| = \|pq\| = \|(1-q)(1-p)\|. \tag{4.2}$$

By (2.4), $(h^*)^\perp = 1-(1-h)^\perp = 1-q$ and $(1-h^*)^\perp = 1-h^\perp = 1-p$. Hence $\|pq\| = \|(1-q)(1-p)\|$. Equation (4.2) follows from the Akhiezer-Glazman equality (2.6).

(vii) $\Longrightarrow$ (viii) follows from the equation $(p-q)^2 = 1-(p+q-1)^2$.

(viii) $\Longrightarrow$ (ii). Set $h = (p-q)^{-1}(1-q)$. Since $(p-q)p = (1-q)p = (1-q)(p-q)$, we have also $h = p(p-q)^{-1}$. We show that $h = \pi(p,q)$. First,

$$h^2 = (p-q)^{-1}(1-q)p(p-q)^{-1} = (p-q)^{-1}(1-q)(p-q)(p-q)^{-1} = h,$$

and h is idempotent. Clearly, $ph = h$ and $(1-h)q = q$. From $(1-q)p = (p-q)p$ we obtain $hp = p$. Finally, from $1-h = 1-p(p-q)^{-1} = -q(p-q)^{-1}$ we get $q(1-h) = 1-h$. □

From the proof of Theorem 4.1 we distill the following result.

Theorem 4.2. *Let p, q be nontrivial projections in A satisfying one of the equivalent conditions of Theorem* 4.1. *Then* (4.2) *holds.*

Example 4.3. Equation (4.2) does not hold for general projections p, q. Consider the C^*-algebra $\mathbf{C}^{3,3}$ of all 3×3 complex matrices with the spectral norm, and let

$$p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 1-p-q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then p and q are projections in $\mathbf{C}^{3,3}$, and $pq = 0$. Hence $\|pq\| = 0 \neq 1 = \|p+q-1\|$.

5. Applications. Our aim in this section is to further extend the problem considered in Theorem 4.1, and to give simpler algebraic proofs for recent Wimmer's results [**17**].

But first the following generalization of Theorem 4.1.

Theorem 5.1. *Let a be a positive invertible element of $\mathcal{A}$. If p, $q \in \mathcal{A}$ are nontrivial idempotents satisfying $ap = p^*a$ and $aq = q^*a$, then the following conditions are equivalent*:

(i) $\mathcal{A} = p\mathcal{A} \oplus q\mathcal{A}$.

(ii) *There exists an idempotent $f \in \mathcal{A}$ such that*

$$p = a^{-1/2} f^{\perp} a^{1/2} \quad \textit{and} \quad q = a^{-1/2}(1-f)^{\perp} a^{1/2}.$$

(iii) $\|a^{-1/2}pqa^{-1/2}\| < 1$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(iv) $1 - pq \in \mathcal{A}^{-1}$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(v) $\|a^{1/2}pqa^{-1/2} < 1$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(vi) $1 - pqp \in \mathcal{A}^{-1}$ *and* $\mathcal{A} = p\mathcal{A} + q\mathcal{A}$.

(vii) $\|a^{1/2}(p + q - 1)a^{-1/2}\| < 1$.

(viii) $p - q \in \mathcal{A}^{-1}$.

The idempotent f is given by the formula

$$f = a^{1/2}(p - q)^{-1}(1 - q)a^{-1/2}.$$

Proof. It is known (see, for instance, [**2**]) that $x^{*a} = a^{-1}x^*a$ is an involution on $\mathcal{A}$ and that $\mathcal{A}$ becomes a C^*-algebra with the involution $x \mapsto x^{*a}$ and the norm $\|x\|_a = \|a^{1/2}xa^{-1/2}\|$. We denote this C^*-algebra by $\mathcal{A}_a$. The condition $ax = x^*a$ means that x is self-adjoint in $\mathcal{A}_a$; hence, the hypotheses of the theorem imply that p, q are projections in $\mathcal{A}_a$. We then apply Theorem 4.1 to $\mathcal{A}_a$: There exists an idempotent $h \in \mathcal{A}_a$ such that $h^{\perp a} = p$ and $(1-h)^{\perp a} = q$, where ${}^{\perp a}$ denotes the range projection in $\mathcal{A}_a$.

Write $f = a^{1/2}ha^{-1/2}$. Then f is an idempotent, and

$$\begin{aligned} h^{\perp a} &= h(h + a^{-1}h^*a - 1)^{-1} \\ &= h[a^{-1/2}(a^{1/2}ha^{-1/2} + a^{-1/2}h^*a^{1/2} - 1)a^{1/2}]^{-1} \\ &= ha^{-1/2}(f + f^* - 1)^{-1}a^{1/2} \\ &= a^{-1/2}f(f + f^* - 1)^{-1}a^{-1/2} \\ &= a^{-1/2}f^{\perp}a^{1/2}. \end{aligned}$$

Similarly, $1-f = a^{1/2}(1-h)a^{-1/2}$, and $(1-h)^{\perp a} = a^{-1/2}(1-f)^{\perp}a^{1/2}$. The rest follows from Theorem 4.1. □

The following theorem is motivated by Wimmer's result [**17**, Theorem 2.1], proved for finite dimensional Hilbert spaces. Recall that, for idempotents $u, v \in \mathcal{A}$, $\pi(u,v) = \pi(u^{\perp}, v^{\perp})$.

Theorem 5.2. *Let $h \in \mathcal{A}$ be a nontrivial idempotent and $f \in \mathcal{A}$ a nontrivial projection such that*

$$\|h\|\|f - (1-h)^{\perp}\| < 1. \tag{5.1}$$

Then $g := \pi(h,f)$ exists and

$$\|g - h\| \le \frac{\|h\|^2\|f - (1-h)^{\perp}\|}{1 - \|h\|\|f - (1-h)^{\perp}\|}. \tag{5.2}$$

Proof. From the proof of Theorem 3.1 we recall that $h^{\perp} - (1-h)^{\perp} = (h + h^* - 1)^{-1}$. In view of (5.1) and Lemma 2.3,

$$\|f - (1-h)^{\perp}\| < \frac{1}{\|h\|} = \frac{1}{\|(h^{\perp} - (1-h)^{\perp})^{-1}\|}.$$

Hence $\|f - (1-h)^{\perp}\|\|(h^{\perp} - (1-h)^{\perp})^{-1}\| < 1$, and

$$h^{\perp} - f = (h^{\perp} - (1-h)^{\perp}) - (f - (1-h)^{\perp}) \in \mathcal{A}^{-1}.$$

By Theorem 4.1 (viii) there exists $g = \pi(h, f)$, that is, an idempotent $g \in \mathcal{A}$ such that $g^{\perp} = h^{\perp}$ and $(1-g)^{\perp} = f$. Hence

$$gh^{\perp} = h^{\perp}, \quad h^{\perp} g = g, \quad (1-g)f = f, \quad f(1-g) = 1-g. \tag{5.3}$$

From these equations and properties of range projection we deduce

$$h(1-h)^{\perp} = 0, \quad gh = h, \quad hg = g. \tag{5.4}$$

In the following calculations we will use (5.3) and (5.4) freely.

Consider $s = (1-g)(1-h)^{\perp} = (1-h)^{\perp} - g(1-h)^{\perp}$. We have

$$-g(1-h)^{\perp} = h(1-g)(1-h)^{\perp} = hf(1-g)(1-h)^{\perp} = hfs,$$

and $s = (1-h)^{\perp} + hfs$. Hence $\|s\| \le 1 + \|hf\|\|s\|$, and

$$\|s\| \le \frac{1}{1 - \|hf\|},$$

since $\|hf\| = \|h(f - (1-h)^{\perp})\| \le \|h\|\|f - (1-h)^{\perp}\| < 1$. Therefore

$$g - h = g(1-h) = g(1-h)^{\perp}(1-h) = -hfs(1-h).$$

Applying the norm, we get

$$\|g - h\| \le \|hf\|\|s\|\|1-h\| \le \frac{\|hf\|}{1 - \|hf\|}\,\|h\|,$$

and (5.2) follows. □

From the preceding theorem and its proof we obtain the following result.

Corollary 5.3. *Let $h, g \in \mathcal{A}$ be nontrivial idempotents and $f \in \mathcal{A}$ a nontrivial projection such that $\|hf\| < 1$ and $g = \pi(h, f)$. Then*

$$\|g - h\| \le \frac{\|hf\|}{1 - \|hf\|}\,\|h\|. \tag{5.5}$$

Remark 5.4. From Theorem 5.2 we recover Wimmer's result [**17**, Theorem 2.1 (ii)]. Corollary 5.3 is a C^*-algebra version of [**17**, Theorem 2.1 (i)] with an additional hypothesis that $\pi(h, f)$ exists which compensates for the finite dimensionality assumption of [**17**].

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