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RECURRENCES FOR THE PARTITION FUNCTION AND ITS RELATIVES

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ABSTRACT. For each integer $n \geq 0$, (i) $p(n) :=$ the number of unrestricted partitions of n, (ii) $q(n) :=$ the number of partitions of n into distinct parts and (iii) $q_0(n) :=$ the number of partitions of n into distinct odd parts. Conventionally, $p(0) = q(0) = q_0(0) := 1$. Presented here are: two apparently new recurrences for $p(\cdot)$ and three formulas expressing the functions $q_0(\cdot)$ and $q(\cdot)$ in terms of the function $p(\cdot)$.

1. Introduction. We begin our discussion with a definition.

Definition 1.1. As usual, $\mathbf{P} := \{1, 2, 3, \dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$ and $\mathbf{Q} :=$ the set of all rational numbers. Then, for each $n \in \mathbf{N}$, (i) $p(n) :=$ the number of unrestricted partitions of n, (ii) $q(n) :=$ the number of partitions of n into distinct parts and (iii) $q_0(n) :=$ the number of partitions of n into distinct odd parts. Conventionally, $p(0) = q(0) = q_0(0) := 1$. We also adopt the convention that $p(x) = q(x) = q_0(x) := 0$ whenever $x \in \mathbf{Q} - \mathbf{N}$.

Euler's pentagonal number recurrence for the partition function $p(\cdot)$, viz.,

$$p(n) = \sum_{k \in \mathbf{P}} (-1)^{k-1} \{p(n - k(3k-1)/2) + p(n - k(3k+1)/2)\},$$

for each $n \in \mathbf{P}$, where $p(0) = 1$, has been known for more than 250 years.

Doubtless, any new recurrence for $p(\cdot)$ will always be compared with Euler's recurrence. In this paper we present two new recurrences for $p(\cdot)$, Theorems 1.2 and 1.3, below stated. Our concluding remarks

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provide a numerical example to show that the first of these two recurrences is actually more efficient than Euler's recurrence.

We also present three formulas which express the functions $q_0(\cdot)$ and $q(\cdot)$ in terms of the function $p(\cdot)$.

In responding to a question of Ramanujan regarding the parity of the value $p(1000)$, MacMahon [**3**, pp. 281–283] foresaw to a limited degree the possibility of speeding up Euler. Then, modulo 2, he utilized the fifth identity in our proof of Theorem 1.2.

In Section 2, proofs of the following six results are given.

Theorem 1.2. *For each* $n \in \mathbf{N}$,

$$p(n) = \sum_{k\in\mathbf{N}} p\left(\frac{n-k(k+1)/2}{4}\right) + 2\sum_{k\in\mathbf{P}} (-1)^{k-1} p(n-2k^2). \tag{1.1}$$

Theorem 1.3. *For each* $n \in \mathbf{N}$,

$$\begin{aligned} p(n) = &\sum_{k\in\mathbf{N}} p\left(\frac{n-k(k+1)/2}{2}\right) \\ &+ \sum_{k\in\mathbf{P}} (-1)^{k-1}\{p(n-k(3k-1)) + p(n-k(3k+1))\}. \end{aligned} \tag{1.2}$$

Theorem 1.4. *For each* $n \in \mathbf{N}$,

$$\begin{aligned} \frac{1-(-1)^n}{2} q_0(n) + &\sum_{k\in\mathbf{P}} (-1)^k q_0(n-k^2) \\ &- (-1)^n \sum_{k\in\mathbf{P}} (-1)^k q_0(n-2k^2) = 0. \end{aligned} \tag{1.3}$$

Theorem 1.5. *For each* $n \in \mathbf{N}$,

$$q_0(n) = p(n) + 2\sum_{k\in\mathbf{P}} (-1)^k (n-2k^2). \tag{1.4}$$

Theorem 1.6. *For each* $n \in \mathbf{N}$,

$$q_0(n) = (-1)^n p(n) + 2(-1)^n \sum_{k\in\mathbf{P}} (-1)^k p(n-k^2). \tag{1.5}$$

Theorem 1.7. *For each* $n \in \mathbf{N}$,

$$q(n) = p(n) + \sum_{k\in\mathbf{P}} (-1)^k \{p(n-k(3k-1)) + p(n-k(3k+1))\}. \tag{1.6}$$

2. Proofs. Our proofs depend on the following identities, each of which is valid for all complex numbers x such that $|x| < 1$.

$$\prod_1^\infty (1+x^n)(1-x^{2n-1}) = 1, \tag{2.1}$$

$$\prod_1^\infty (1-x^n) = 1 + \sum_{k\in\mathbf{P}} (-1)^k \{x^{1/2k(3k-1)} + x^{1/2k(3k+1)}\}, \tag{2.2}$$

$$\prod_1^\infty (1-x^n)(1-x^{2n-1}) = 1 + 2\sum_{k\in\mathbf{P}} (-1)^k x^{k^2}, \tag{2.3}$$

$$\prod_1^\infty \frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{k\in\mathbf{N}} x^{k(k+1)/2}. \tag{2.4}$$

For a proof of (2.1), see [**2**, p. 277]; and for proofs of (2.2), (2.3) and (2.4), see [**2**, pp. 282–284].

Proof of Theorem 1.2. First of all, we let $x \to x^2$ in (2.3), and observe that the lefthand side of the resulting identity is

$$\begin{aligned}\prod_1^\infty (1-x^{2n})(1-x^{4n-2}) &= \prod_1^\infty (1-x^n)(1+x^n)(1-x^{2n-1})(1+x^{2n-1}) \\ &= \prod_1^\infty (1-x^n)(1+x^{2n-1}).\end{aligned}$$

The last step is justified by (2.1). Hence,

$$\prod_1^\infty (1+x^{2n-1}) = \prod_1^\infty (1-x^n)^{-1}\Big\{1+2\sum_1^\infty (-1)^k x^{2k^2}\Big\}. \tag{2.5}$$

Next, with the help of (2.1), we write the lefthand side of (2.4) as

$$\begin{aligned}\prod_1^\infty (1-x^{2n})(1+x^n) &= \prod_1^\infty (1-x^{2n})(1+x^{2n})(1+x^{2n-1})\\ &= \prod_1^\infty (1-x^{4n})(1+x^{2n-1}).\end{aligned}$$

Therefore, (2.4) becomes

$$\prod_1 (1+x^{2n-1}) = \prod_1^\infty (1-x^{4n})^{-1}\sum_0^\infty x^{k(k+1)/2}.$$

Between (2.5) and the foregoing identity we eliminate the product $\prod_{n=1}^\infty (1+x^{2n-1})$ to get

$$\prod_1^\infty (1-x^n)^{-1}\Big\{1+2\sum_1^\infty (-1)^k x^{2k^2}\Big\} = \prod_1^\infty (1-x^{4n})^{-1}\sum_0^\infty x^{k(k+1)/2},$$

or, equivalently,

$$\sum_{j=0}^\infty p(j)x^j\Big\{1+2\sum_{k=1}^\infty (-1)^k x^{2k^2}\Big\} = \sum_{j=0}^\infty p(j)x^{4j}\sum_{k=0}^\infty x^{k(k+1)/2}.$$

Expanding both sides of the foregoing identity, and subsequently equating coefficients of like powers of x, we prove Theorem 1.2. □

Proof of Theorem 1.3. On the one hand, because of (2.2), where $x \to x^2$, we get

$$\begin{aligned}\prod_1^\infty (1+x^n) &= \prod_1^\infty (1-x^n)^{-1}\left\{1+\sum_1^\infty (-1)^k (x^{k(3k-1)}+x^{k(3k+1)})\right\}\\ &= \sum_{j=0}^\infty p(j)x^j\left\{1+\sum_1^\infty (-1)^k (x^{k(3k-1)}+x^{k(3k+1)})\right\}\\ &= \sum_{n=0}^\infty p(n)x^n + \sum_{n=2}^\infty x^n \sum_{k\ge 1}(-1)^k p(n-k(3k-1))\\ &\quad + \sum_{n=4}^\infty x^n \sum_{k\ge 1}(-1)^k p(n-k(3k+1)).\end{aligned}$$

On the other hand, by (2.1) and (2.4), we get

$$\begin{aligned}\prod_1^\infty (1+x^n) &= \prod_1^\infty (1-x^{2n})^{-1}\sum_0^\infty x^{k(k+1)/2}\\ &= \sum_{j=0}^\infty p(j)x^{2j}\sum_{k=0}^\infty x^{k(k+1)/2}\\ &= \sum_{n=0}^\infty x^n \sum_{k\ge 0} p\left(\frac{n-k(k+1)/2}{2}\right).\end{aligned}$$

Between these two identities we now eliminate the product $\prod(1+x^n)$, and subsequently equate coefficients of like powers of x to obtain the desired conclusion. □

Proof of Theorem 1.4. In (2.3) we let $x \to -x$, to get

$$\prod_1^\infty (1+x^{2n-1}) = \prod_1^\infty (1-(-x)^n)^{-1}\left\{1+2\sum_1^\infty x^{k^2}\right\}. \tag{2.6}$$

By (2.5) and (2.6),

$$\prod_1^\infty (1-x^n)\left\{1+2\sum_1^\infty x^{k^2}\right\} = \prod_1^\infty (1-x^{2n})(1+x^{2n-1})\left\{1+2\sum_1^\infty (-1)^k x^{2k^2}\right\},$$

whence

$$1+2\sum_1^{\infty} x^{k^2} = \prod_1^{\infty}(1+x^n)(1+x^{2n-1})\left\{1+2\sum_1^{\infty}(-1)^k x^{2k^2}\right\},$$

whence, due to (2.1),

$$\prod_1^{\infty}(1-x^{2n-1})\left\{1+2\sum_1^{\infty} x^{k^2}\right\} = \prod_1^{\infty}(1+x^{2n-1})\left\{1+2\sum_1^{\infty}(-1)^k x^{2k^2}\right\},$$

whence

$$\sum_{j=0}^{\infty}(-1)^j q_0(j)x^j\left\{1+2\sum_1^{\infty} x^{k^2}\right\} = \sum_{j=0}^{\infty} q_0(j)x^j\left\{1+2\sum_1^{\infty}(-1)^k x^{2k^2}\right\}.$$

Now, expanding both sides of the foregoing identity, and subsequently equating coefficients of like powers of x, we prove our theorem. □

Proof of Theorem 1.5. Since

$$\prod_1^{\infty}(1+x^{2n-1}) = \sum_0^{\infty} q_0(n)x^n,$$

(2.5) yields

$$\begin{aligned}\sum_{n=0}^{\infty} q_0(n)x^n &= \sum_{j=0}^{\infty} p(j)x^j\left\{1+2\sum_{k=1}^{\infty}(-1)^k x^{2k^2}\right\}\\ &= \sum_{n=0}^{\infty} p(n)x^n + 2\sum_{n=2}^{\infty} x^n \sum_{k\geq 1}(-1)^k p(n-2k^2).\end{aligned}$$

The conclusion follows upon equating coefficients of like powers of x. □

Proof of Theorem 1.6. Beginning with (2.6) and arguing as in the foregoing proof, we get

$$\sum_{n=0}^{\infty} q_0(n)x^n = \sum_{n=0}^{\infty}(-1)^n p(n)x^n + 2\sum_{n=1}^{\infty} x^n \sum_{k\geq 1}(-1)^{n-k} p(n-k^2).$$

The conclusion follows. $\square$

Proof of Theorem 1.7. Since

$$\prod_{1}^{\infty}(1+x^n) = \sum_{0}^{\infty} q(n)x^n,$$

the identity at the beginning of the proof of Theorem 1.3 yields

$$\sum_{n=0}^{\infty} q(n)x^n = \sum_{n=0}^{\infty} p(n)x^n + \sum_{n=2}^{\infty} x^n \sum_{k\geq 1}(-1)^k(n-k(3k-1)) + \sum_{n=4}^{\infty} x^n \sum_{k\geq 1}(-1)^k p(n-k(3k+1)).$$

Upon equating coefficients of like powers of x, we prove our theorem. $\square$

Theorems 1.5, 1.6 and 1.7 present formulas for the functions $q_0(\cdot)$ and $q(\cdot)$ expressed in terms of the function $p(\cdot)$. The first such expressions seem to be due to Watson [**4**, p. 551]. There he stated two identities, essentially variants of (2.4), whose developments lead to

$$q_0(n) = \sum_{k\geq 0} p\left(\frac{n-k(k+1)/2}{4}\right), \quad n \in \mathbf{N}$$

and

$$q(n) = \sum_{k\geq 0} p\left(\frac{n-k(k+1)/2}{2}\right), \quad n \in \mathbf{N}.$$

Although the foregoing formulas are valid, they do not provide effective determinations of the functions $q_0(\cdot)$ and $q(\cdot)$. The trouble turns on using large values of $p(\cdot)$ to compute the relatively smaller values of $q_0(\cdot)$ and $q(\cdot)$. Effective recurrences for $q_0(\cdot)$ and $q(\cdot)$ are presented by the author [**1**, pp. 1–2], viz., for each $n \in \mathbf{N}$,

$$\sum_{k\in\mathbf{N}} (-1)^{k(k+1)/2} q_0(n-k(k+1)/2) = \begin{cases} (-1)^m & \text{if } n = m(3m\pm 1) \text{ for some } m \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$q(n) + 2\sum_{k\in\mathbf{P}}(-1)^k q(n-k^2) = \begin{cases} (-1)^m & \text{if } n = m(3m\pm 1)/2 \text{ for some } m\in\mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Concluding remarks. As observed in [**2**, p. 286], for given $n \in \mathbf{P}$, computation of $p(n)$ by Euler's recurrence requires about $\sqrt{(8n)/3}$ of the values $p(m)$, $0 \le m < n$. Note that, for given $n \in \mathbf{P}$, the second sum on the righthand side of (1.1) requires about $\sqrt{n/2}$ of the values $p(m)$, $0 \le m < n$, and the first sum requires less than $\sqrt{2n}/2 = \sqrt{n/2}$ of these earlier values. Hence, computation of $p(n)$ by (1.1) requires less than $2\sqrt{n/2} = \sqrt{2n}$ of the values $p(m)$, $0 \le m < n$. In a word, the recurrence of Theorem 1.2 is more efficient than Euler's recurrence. We illustrate this with the numerical example $n = 15$, where $\sqrt{(8n)/3} = \sqrt{40}$ so that $[\sqrt{40}] = 6$; and $\sqrt{2n} = \sqrt{30}$, with $[\sqrt{30}] = 5$.

By Euler's recurrence,

$$\begin{aligned} p(15) &= p(15-1) + p(15-2) - p(15-5) - p(15-7) \\ &\quad + p(15-12) + p(15-15) \\ &= p(14) + p(13) - p(10) - p(8) + p(3) + p(0) \\ &= 135 + 101 - 42 - 22 + 3 + 1 = 176. \end{aligned}$$

By recurrence (1.1),

$$\begin{aligned} p(15) &= p\left(\frac{15-3}{4}\right) + p\left(\frac{15-15}{4}\right) + 2\{p(15-2) - p(15-8)\} \\ &= p(3) + p(0) + 2\{p(13) - p(7)\} \\ &= 3 + 1 + 2\{101 - 15\} = 4 + 172 = 176. \end{aligned}$$

Comparison of recurrence (1.2) with Euler's recurrence is not as striking as the foregoing comparison. Details are left for the reader.

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