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THE RATIONALITY OF THE MODULI SPACES OF BIELLIPTIC CURVES OF GENUS FIVE WITH MORE BIELLIPTIC STRUCTURES

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0. Introduction and notations. Let C be an irreducible, smooth, projective curve of genus $g \geq 2$, defined over the complex field $\mathbf{C}$. The curve C is called *bielliptic* if it admits a degree 2 morphism $\pi: C \to E$ onto an elliptic curve E: such a morphism is called a *bielliptic structure*.

If $g \geq 6$ then the bielliptic structure is unique. If $g = 3, 4, 5$ this holds true generically, but there exist curves C carrying more than one bielliptic structure.

We denote by $\mathfrak{M}_g^{be,n}$ the locus of points representing curves with at least n bielliptic structures inside the coarse moduli space $\mathfrak{M}_g$ of smooth curves of genus g. There are the following sharp bounds: $n \leq 21, 10, 5$ if $g = 3, 4, 5$ respectively (see Corollary 5.8 of [**3**]).

We focus our interest on the case $g = 5$. It is already known that $\mathfrak{M}_5^{be,1}$ is rational (see [**6**]). The aim of this paper is to prove the following

Main Theorem. *The loci* $\mathfrak{M}_5^{be,2}$, $\mathfrak{M}_5^{be,3}$ *and* $\mathfrak{M}_5^{be,4} = \mathfrak{M}_5^{be,5}$ *are irreducible and rational of respective dimensions* 5, 4 *and* 2. $\square$

The loci $\mathfrak{M}_5^{be,n}$ play a helpful role in the description of the structure of the Chow ring $A(\mathfrak{M}_5)$ (see Section 4 of [**8**] where $\mathfrak{M}_5^{be,n} =: B_n$).

For the proof of the main theorem above we proceed imitating the method used in [**6**] for proving the rationality of $\mathfrak{M}_5^{be,1}$. Let $[C] \in \mathfrak{M}_5$ be the isomorphism class of a curve C. The canonical model $\widetilde{C}$ of C is the base locus of a net of quadric hypersurfaces $\mathcal{N}$ in $\mathbf{P}^4_{\mathbf{C}}$. Let N be a projective plane parametrizing the quadrics in $\mathcal{N}$. The discriminant curve $D \subseteq N$ of $\mathcal{N}$ is a stable plane quintic.

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If $[C] \in \mathfrak{M}_5^{be,n}$, then D is the union of a n distict lines $L_0, \dots, L_{n-1}$ and an integral curve F of degree $5-n$. Moreover F is endowed (in a natural way) with a non-effective theta-characteristic η (i.e. an invertible sheaf η on the normalization $\widehat{F}$ of F such that $\eta^{\otimes 2} \cong \omega_{\widehat{F}}$ and $h^0\big(\widehat{F}, \eta\big) = 0$). One can associate to C the triple $(F, \eta, \cup_{i=0}^{n-1} L_i)$ and the existence of a birational equivalence

$$\mathfrak{M}_5^{be,n} \approx \{(F, \eta, \cup_{i=0}^{n-1} L_i)\} / \operatorname{PGL}_3,$$

can be shown (see Section 1) so that the rationality of $\mathfrak{M}_5^{be,n}$ follows by proving the rationality of the quotient on the right (see Section 2).

Notations. As usual we denote by $\mathcal{O}_X$ and ω_X the structure sheaf and the canonical sheaf of the irreducible, smooth, projective variety X. For each invertible sheaf $\mathcal{L}$ on X we denote by $|\mathcal{L}|$ the projectivization of $H^0\big(X, \mathcal{L}\big)$.

GL_n is the general linear group of order n, PGL_n is the general projective linear group of order n.

If $g_1, \dots, g_h$ are in a certain group (respectively, vector space) G then $\langle g_1, \dots, g_h \rangle$ denotes the subgroup (respectively, the subspace) of G generated by $g_1, \dots, g_h$.

We denote by $\cong$ isomorphisms and by $\approx$ birational equivalences.

1. Bielliptic curves of genus 5. In this section, following [**5**], we will construct a birational model of $\mathfrak{M}_5^{be,n}$. Such a construction is analogous to the one used in [**6**] for proving the rationality of $\mathfrak{M}_5^{be,1}$.

If C is a bielliptic curve of genus $g \geq 5$, then it is neither hyperelliptic nor trigonal by the Castelnuovo-Severi inequality (see [**1**]). Assume now $g = 5$. Then the canonical model $\widetilde{C} \subseteq \mathbf{P}^4_{\mathbf{C}}$ of C is the complete intersection of three quadric hypersurfaces, say Q_0, Q_1, Q_2. Let N denote the projective plane, with homogeneous coordinates ν_0, ν_1, ν_2, parametrizing the quadrics of the net $\mathcal{N} = \{\nu_0 Q_0 + \nu_1 Q_1 + \nu_2 Q_2\}$: if $P \in N$ we denote by Q_P the corresponding quadric.

In N there is defined the discriminant D of the net $\mathcal{N}$, i.e. the locus of points P such that Q_P is singular.

Lemma 1.1. *$D \subseteq N \cong \mathbf{P}^2_{\mathbf{C}}$ is a curve of degree 5. It has at most ordinary double points as singularities. More precisely*

(i) *$P \in N \setminus D$ if and only $\mathrm{rk}(Q_P) = 5$;*

(ii) *P is a regular point of D if and only if $\mathrm{rk}(Q_P) = 4$;*

(iii) *P is a singular point of D if and only if $\mathrm{rk}(Q_P) = 3$.*

Proof. To $\mathcal{N}$ we can associate naturally a quadric bundle (see [**5**], Lemma 6.1) and D is its discriminat curve (see [**5**], 6.2). Then we can apply Proposition 1.2 of [**5**]. $\square$

Each morphism $\pi: C \to E$ of degree 2 onto an elliptic curve induces

$$\pi^*: W^1_2(E) \cong E \hookrightarrow W^1_4(C)$$

where, for each smooth curve Γ, the symbol $W^r_d(\Gamma)$ denotes the subvariety of $\mathrm{Pic}^d(\Gamma)$ parametrizing the complete linear series on Γ of degree d and dimension at least r.

It follows that $W^1_4(C)$ must contain the elliptic curve $\pi^* E$. Since the g^1_4's on $\widetilde{C}$ are cut out by the rulings of the quadrics of rank at most 4 through $\widetilde{C}$, there exists a two-to-one morphism ε from the variety $W^1_4(C)$ onto D. Such a morphism ε is ramified exactly at the points of $W^1_4(C)$ corresponding to the quadrics of rank 3 in $\mathcal{N}$. Moreover the images of the points of ramification of ε are the singularities of D. It follows from the Hurwitz formula, that D contains a line L (namely $\varepsilon(\pi^* E)$) and then it is the union of L and a quartic F.

Proposition 1.2. *Let C be a bielliptic curve of genus $g = 5$. Then there is a bijective correspondence between lines $L \subseteq D$ and bielliptic structure on C.*

Proof. We have shown above that each bielliptic structure on C induces a line $L \subseteq D$. For the proof see exercises F-11 and F-12 of Chapter VI of [**2**]. $\square$

Now assume that C carries n bielliptic structures. Then the discriminant curve D splits as the union of n lines, say $L_0, \ldots, L_{n-1}$, and a plane curve F of degree $5-n$. In particular we then have $n = 1, 2, 3, 5$. Since the case $n = 1$ has been described in [**6**] from now on we will always assume that $n = 2, 3, 5$.

Let $\widehat{D} \xrightarrow{d} D$ be the normalization of D. Notice that $\widehat{D}$ is the disjoint union of the normalization $\widehat{F}$ of F and of the lines $L_0, \dots, L_{n-1}$.

We can define a map $s\colon \widehat{D} \to \mathbf{P}^4_{\mathbf{C}}$ associating to $P \in \widehat{D}$ the vertex $s(P)$ of the corresponding quadric $Q_P \in \mathcal{N}$. We have

$$s^*\mathcal{O}_{\mathbf{P}^4_{\mathbf{C}}}(1) \cong \omega_{\widehat{D}} \otimes \theta$$

where θ is an invertible sheaf on $\widehat{D}$ such that $\theta^{\otimes 2} \cong \omega_{\widehat{D}}$ and $h^0\big(\widehat{D}, \theta\big) = 0$, i.e. a non-effective theta characteristic on $\widehat{D}$ (see [**5**], 6.12 and Lemma 6.12). Since each theta characteristic on $\mathbf{P}^1_{\mathbf{C}}$ is $\mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(-1)$ then the datum of θ is equivalent to the datum of a non-effective theta characteristic η on $\widehat{F}$. In this way we can associate to C a unique triple $(F, \eta, \cup_{i=0}^{n-1} L_i)$.

Let C' be another bielliptic curve of genus 5 and $(F', \eta', \cup_{i=0}^{n'-1} L'_i)$ its associated triple. The class $[C]$ determines the canonical model of C up to projective isomorphisms: it follows that if $C' \cong C$, then there exists $\varphi \in \mathrm{PGL}_3$ such that $\varphi(F) = F'$, $\varphi(\cup_{i=0}^{n-1} L_i) = \cup_{i=0}^{n'-1} L'_i$ and $\widehat{\varphi}^*(\eta') = \eta$ ($\widehat{\varphi}$ is the extension of φ to the normalizations).

Let $S^n|\mathcal{O}_N(1)|$ be the n^{th}-symmetric product of $|\mathcal{O}_N(1)|$ (i.e. $S^n|\mathcal{O}_N(1)| := |\mathcal{O}_N(1)|^{\times n}/\mathfrak{S}_n$). Following [**7**] we denote by $\overline{|\mathcal{O}_N(5-n)|}$ the variety of pairs (F, η) where F and η are as above and set $\overline{X}_n := \overline{|\mathcal{O}_N(5-n)|} \times S^n|\mathcal{O}_N(1)|$. The above construction shows, when $n = 2, 3, 5$, that $C \to (F, \eta, \cup_{i=0}^{n-1} L_i)$ induces a rational map

$$\mathfrak{m}\colon \mathfrak{M}_5^{be,n} \dashrightarrow \overline{X}_n / \mathrm{PGL}_3 \,.$$

Proposition 1.3. *For each $n = 2, 3, 5$ the map $\mathfrak{m}$ is birational. Moreover $\mathfrak{M}_5^{be,n}$ is irreducible.*

Proof. The open set

$$\mathcal{U} := \{\ (F, \eta, \cup_{i=0}^{n-1} L_i) \in \overline{X}_n \mid F \cup L_0 \cup \dots \cup L_{n-1} \text{ is stable } \}$$

is non-empty. Then it turns out that the map $\mathfrak{m}$ is onto $\mathcal{U}$ by Proposition 6.23 of [**5**]. It is also injective on $\mathcal{U}$ by proposition 6.19 of [**5**].

Finally its inverse induces a surjection $\mathcal{U} \to \mathfrak{M}_5^{be,n}$, whence the irreducibility of $\mathfrak{M}_5^{be,n}$ follows. $\square$

2. The proof of the main theorem.

2.1. The rationality of $\mathfrak{M}_5^{be,2}$. If $n = 2$, then F is a cubic. In particular the general F carries exactly three non-effective theta-characteristics.

Lemma 2.1.1. *There exists a* PGL_3*-equivariant birational map*

$$h\colon |\mathcal{O}_N(3)| \dashrightarrow \overline{|\mathcal{O}_N(3)|}.$$

Proof. The map h is defined in 5.7 of [**7**]. It assigns to a plane cubic the Hessian invariant of the net of polar cubics. For the proof of the lemma see Theorem 5.7.1 and Remark 5.7.3 of [**7**]. $\square$

The above lemma yields the existence of a PGL_3-equivariant birational map

$$H\colon \overline{X}_2 \dashrightarrow X_2 := |\mathcal{O}_N(3)| \times S^2|\mathcal{O}_N(1)|$$

whence $\overline{X}_2/\mathrm{PGL}_3 \approx X_2/\mathrm{PGL}_3$. We conclude that we have to prove

Lemma 2.1.2. *The quotient* X_2/PGL_3 *is rational of dimension* 5.

Proof. Let $V := H^0\big(N, \mathcal{O}_N(3)\big)$, $U := \{\, q \in H^0\big(N, \mathcal{O}_N(2)\big) \mid \mathrm{rk}(q) = 2\,\}$, $G := \mathrm{GL}_3 \times \mathbf{C}^*$, $H := \{\, (\omega I, \omega) \mid \omega^3 = 1\,\} \subseteq G$. $\overline{G} := G/H$ is a group acting on $\mathbf{E} := V \times U$ as follows: GL_3 acts in the natural way both on V and U, $\mathbf{C}^*$ acts on U via homotheties. It is clear that $X_2/\mathrm{PGL}_3 \approx \mathbf{E}/G \cong \mathbf{E}/\overline{G}$. Consider the $\overline{G}$-equivariant morphism of vector bundles $\mu\colon H^0\big(N, \mathcal{O}_N(1)\big) \times U \to \mathbf{E}$ sending $(\ell, q) \mapsto (\ell q, q)$. In this way we obtain a new $\overline{G}$-invariant vector bundle over U, namely

$\mathbf{E}' := \mathbf{E}/\operatorname{im}(\mu)$. The fibre of $\mathbf{E}'$ over $q \in U$ is $V/qH^0\big(N, \mathcal{O}_N(1)\big) \cong \mathbf{C}^7$, hence $\dim(\mathbf{E}') = 12$.

The natural quotient projection $\pi\colon \mathbf{E} \to \mathbf{E}'$ is $\overline{G}$-equivariant too and it induces on $\mathbf{E}$ a structure of vector bundle on $\mathbf{E}'$ with fibre $\mathbf{C}^3$.

Moreover notice that $\dim(\overline{G}) = \dim(G) = 10$.

Claim 2.1.2.1. *The action of $\overline{G}$ over $\mathbf{E}'$ is almost free.*

Assuming the claim we obtain that $\mathbf{E}/\overline{G}$ is a vector bundle over $\mathbf{E}'/\overline{G}$. Since the last quotient is unirational of dimension 2 it follows from a theorem of Castelnuovo that it is actually rational. It follows that $X_2/\operatorname{PGL}_3 \approx \mathbf{E}/\overline{G}$ is rational too.

Proof of Claim 2.1.2.1. Choose a general element $e := ([f], q) \in \mathbf{E}'$. With a proper choice of the homogeneous coordinates ν_0, ν_1, ν_2 in $N \cong \mathbf{P}^2_{\mathbf{C}}$ we can assume that $q = \nu_0\nu_1$. By the very definition of $\mathbf{E}'$ we can assume that

$$f(\nu_0, \nu_1, \nu_2) = f_0\nu_0^3 + f_1\nu_1^3 + f_2\nu_0^2\nu_2 + f_3\nu_1^2\nu_2 + f_4\nu_0\nu_2^2 + f_5\nu_1\nu_2^2 + f_6\nu_2^3.$$

Since each element of the stabilizer G_e of e inside G must fix $\nu_0\nu_1$ then $G_e \subseteq \langle\mu\rangle \cdot G_{e,0} \subseteq G$ where

$$G_{e,0} := \left\langle \left(\begin{pmatrix} \alpha_{0,0} & 0 & 0 \\ 0 & \alpha_{1,1} & 0 \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} \end{pmatrix}, \alpha_{0,0}^{-1}\alpha_{1,1}^{-1} \right) \right\rangle, \quad \mu := \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right).$$

Assume $\alpha \in G_{e,0}$. Then a direct substitution yields the system

$$\begin{cases} f_0\alpha_{0,0}^3 + f_2\alpha_{0,0}^2\alpha_{2,0} + f_4\alpha_{0,0}\alpha_{2,0}^2 + f_6\alpha_{2,0}^3 = f_0 \\ f_1\alpha_{1,1}^3 + f_3\alpha_{1,1}^2\alpha_{2,1} + f_5\alpha_{1,1}\alpha_{2,1}^2 + f_6\alpha_{2,1}^3 = f_1 \\ f_2\alpha_{0,0}^2\alpha_{2,2} + 2f_4\alpha_{0,0}\alpha_{2,0}\alpha_{2,2} + 3f_6\alpha_{2,0}^2\alpha_{2,2} = f_2 \\ f_3\alpha_{1,1}^2\alpha_{2,2} + 2f_5\alpha_{1,1}\alpha_{2,1}\alpha_{2,2} + 3f_6\alpha_{2,1}^2\alpha_{2,2} = f_3 \\ f_4\alpha_{0,0}\alpha_{2,2}^2 + 3f_6\alpha_{2,0}\alpha_{2,2}^2 = f_4 \\ f_5\alpha_{1,1}\alpha_{2,2}^2 + 3f_6\alpha_{2,1}\alpha_{2,2}^2 = f_5 \\ f_6\alpha_{2,2}^3 = f_6. \end{cases}$$

Let f be general. The last equation yields $\alpha_{2,2}^3 = 1$. Then from the fifth one $\alpha_{2,0} = f_4(\alpha_{2,2} - \alpha_{0,0})/3f_6$ hence, from the third one, $(3f_2f_6 - f_4^2)(\alpha_{0,0} - \alpha_{2,2})(\alpha_{0,0} + \alpha_{2,2}) = 0$, whence we get $\alpha_{0,0} = \pm\alpha_{2,2}$. If $\alpha_{0,0} = -\alpha_{2,2}$, substituting the expression for $\alpha_{2,0}$ in the first equation, we finally obtain $27f_0f_6^2 - 9f_2f_4f_6 + 2f_4^3 = 0$, i.e. f would not be general: thus $\alpha_{0,0} = \alpha_{2,2}$ hence $\alpha_{2,0} = 0$.

Working now with the even equations we also get $\alpha_{1,1} = \alpha_{2,2}$, $\alpha_{2,1} = 0$. Since $\alpha_{2,2}^3 = 1$, we conclude $\alpha \in H$, whence $G_e \subseteq \langle\mu\rangle \cdot H$ and an easy computation shows that $\mu \cdot H \cap G_e = \varnothing$, then $G_e = H$, i.e. the action of $\overline{G}$ is almost free. $\square$

The proof of the above claim concludes the proof of the lemma. $\square$

Remark 2.1.3. Let $\mathfrak{X}$ be the locus of points $[C] \in \mathfrak{M}_5$ for which D splits as the union of a cubic and a conic. Then it is clear that $\mathfrak{M}_5^{be,n} \subseteq \mathfrak{X}$ for each $n \geq 2$. Notice that $\mathfrak{X}$ contains all the points $[C]$ representing non-trigonal curves C carrying an involution $i \in \mathrm{Aut}(C)$ such that C/i is a smooth curve of genus 3 (see [**2**], Exercise F-23 of Chapter VI).

Then the map h defined in Lemma 2.1.1 induces a birational equivalence $\mathfrak{X} \approx |\mathcal{O}_N(3)| \times |\mathcal{O}_N(2)|/\mathrm{PGL}_3$. The quotient on the right is rational, since it is birationally equivalent to $\mathfrak{M}_4^{be}$ (see [**4**]).

2.2. The rationality of $\mathfrak{M}_5^{be,3}$.

If $n = 3$, then F is a, necessarily non-singular, conic. In particular $F \cong \mathbf{P}^1_{\mathbf{C}}$ hence $\omega_F \cong \eta^{\otimes 2}$ yields $\eta = \mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(-1)$, thus $\overline{|\mathcal{O}_N(2)|} = |\mathcal{O}_N(2)|$, hence $\overline{X}_3 \cong X_3 := |\mathcal{O}_N(2)| \times S^3|\mathcal{O}_N(1)|$.

We conclude that the proof of the main theorem in this case is then equivalent to prove

Lemma 2.2.1. *The quotient X_3/PGL_3 is rational of dimension* 3.

Proof. Let $Y := \{(F, \{\nu_0\nu_1\nu_2 = 0\})\} \subseteq X_3$. If $\alpha \in \mathrm{PGL}_3$ is such that $\alpha(Y) \subseteq Y$ then α must fix $\{\nu_0\nu_1\nu_2 = 0\}$, hence α is represented by a 3×3 matrix which is the product of a diagonal matrix and a permutation matrix. In particular

$$\alpha \in H := \mathfrak{S}_3 \cdot PD \cong \langle \sigma, \tau \rangle \cdot PD \subseteq \mathrm{PGL}_3$$

where PD is the image via $\mathrm{GL}_3 \to \mathrm{PGL}_3$ of the torus $D \subseteq \mathrm{GL}_3$ of diagonal matrices and σ, τ are the classes of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

respectively. Since Y is obviously PGL_3 dense inside X_3 it follows that Y is a (PGL_3, H)-section of X_3 (see Section 2.8 of [**9**] for the definition and properties of relative sections).

Since $PD \trianglelefteq H$ it suffices to show that

$$\mathbf{C}(Y)^H \cong (\mathbf{C}(Y)^{PD})^{H/PD} \cong (\mathbf{C}(H^0(N, \mathcal{O}_N(2)))^D)^{\mathfrak{S}_3}$$

is rational.

First we describe $\mathbf{C}(H^0(N, \mathcal{O}_N(2)))^D$. If $q(\nu_0, \nu_1, \nu_2) := \sum_{i,j} p_{i,j} \nu_0^{2-i-j} \nu_1^i \nu_2^j$ and $t := (t_0, t_1, t_2) \in D$, then

$$t(q)(\nu_0, \nu_1, \nu_2) = \sum_{i,j} (t_0^{2-i-j} t_1^i t_2^j p_{i,j}) \nu_0^{2-i-j} \nu_1^i \nu_2^j$$

Since D leaves the space generated by each monic monomial invariant, then the field above is generated by D-invariant fractional monomials. It is easy to see that $M := \prod_{i,j} p_{i,j}^{\alpha_{i,j}}$ is D-invariant if and only if $(\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2})$ is a solution of the system

$$\begin{cases} 2\alpha_{0,0} + \alpha_{0,1} + \alpha_{0,2} = 0 \\ \alpha_{0,1} + 2\alpha_{1,1} + \alpha_{1,2} = 0 \\ \alpha_{0,2} + \alpha_{1,2} + 2\alpha_{2,2} = 0. \end{cases}$$

Let A be the matrix of the above system and set

$$U := \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad V := \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since

$$UAV = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

it follows that $\mathbf{C}(H^0(N, \mathcal{O}_N(2)))^D = \mathbf{C}(X_1, X_2, X_3)$ where $X_1 := p_{0,0}p_{1,1}/p_{0,1}^2$, $X_2 := p_{0,0}p_{1,2}/p_{0,1}p_{0,2}$, $X_3 := p_{0,0}p_{2,2}/p_{0,2}^2$.

Let $Y_1 := X_1/X_2$, $Y_2 := X_3/X_1$, $Y_3 := X_2$ so that $\mathbf{C}(X_1, X_2, X_3) = \mathbf{C}(Y_1, Y_2, Y_3)$. We have $\mathbf{C}(Y)^H \cong \mathbf{C}(Y_1, Y_2, Y_3)^{\mathfrak{S}_3}$ and $\sigma(Y_1, Y_2, Y_3) = (Y_2, Y_1, Y_3)$, $\tau(Y_1, Y_2, Y_3) = (Y_2, Y_3, Y_1)$.

It follows that $\langle Y_1, Y_2, Y_3 \rangle \subseteq \mathbf{C}(Y_1, Y_2, Y_3)$ is the usual linear representation of $\mathfrak{S}_3$ via permutation: then the action on $\langle Y_1, Y_2, Y_3 \rangle$ is generated by pseudoreflections, whence $\mathbf{C}[Y_1, Y_2, Y_3]^{\mathfrak{S}_3} \cong \mathbf{C}[Z_1, Z_2, Z_3]$ for suitable $\mathfrak{S}_3$-invariant elements $Z_i \in \mathbf{C}[Y_1, Y_2, Y_3] \subseteq \mathbf{C}(Y_1, Y_2, Y_3)$ (see [**9**], Theorem 8.1).

On the other hand the group of characters of $\mathfrak{S}_3$ is finite, hence $\mathbf{C}(Y_1, Y_2, Y_3)^{\mathfrak{S}_3}$ is exactly $\mathbf{C}(Z_1, Z_2, Z_3)$, thus it is rational. □

2.3. The rationality of $\mathfrak{M}_5^{be,4} = \mathfrak{M}_5^{be,5}$. Again $\eta = \mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(-1)$, thus $\overline{|\mathcal{O}_N(1)|} = |\mathcal{O}_N(1)|$, hence $\overline{X}_4 = \overline{X}_5 \cong X_5 := S^5|\mathcal{O}_N(1)|$.

Again the proof of the main theorem for $n = 4, 5$ in this case is equivalent to

Lemma 2.3.1. The quotient X_5/PGL_3 is rational of dimension 2.

Proof. X_5/PGL_3 is a unirational of dimension 2, hence it is rational from a well known theorem of Castelnuovo. □

We are now ready to give the

Proof of the Main Theorem. The main theorem now follows from Proposition 1.3 and Lemmas 2.1.2, 2.2.1, 2.3.1. □

REFERENCES

1. R. Accola, *Topics in the theory of Riemann surfaces*, Lecture Notes in Math., vol. 1595, Springer, New York, 1977.

2. E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves*, vol. I, Springer, New York, 1985.

3. E. Ballico and A. Del Centina, *Ramification points of double coverings of curves and Weierstrass points*, Ann. Mat. Pura Appl. **177** (1999), 293–313.

4. F. Bardelli and A. Del Centina, *The moduli space of genus four double covers of elliptic curves is rational*, Pacific J. Math **144** (1990), 219–227.

5. A. Beauville, *Variété de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. **10** (1977), 309–391.

6. G. Casnati, A. Del Centina, *The rationality of the moduli space of bielliptic curves of genus five*, Bull. London Math. Soc. **28** (1996), 356–362.

7. I.V. Dolgachev and V. Kanev, *Polar covariants of plane cubics and quartics*, Adv. Math. **98** (1993), 216–301.

8. E. Izadi, *Chow rings of moduli space of curves of genus* 5, in *The moduli space of curves* (R. Dijkgraaf, C. Faber and G. van der Geer, eds.), Progr. Math., vol. 129, Birkhäuser, Boston, 1995.

9. V.L. Popov and E.B. Vinberg, *Invariant theory*, in *Algebraic geometry IV* (A.N. Parshin and I.R. Shafarevich, eds.), Encyclopaedia Math. Sci., vol. 55, Springer, Berlin, 1991.

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