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MONADS AND BUNDLES ON RATIONAL SURFACES

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ABSTRACT. A monad construction is presented for holomorphic bundles on an arbitrary blowup of $\mathbf{P}_2$ which have semi-stable direct image on $\mathbf{P}_2$. Three illustrative applications to different moduli problems are given.

1. Introduction. A *monad* M on a complex manifold X is a complex $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ of holomorphic vector bundles with $a(x)$ injective and $b(x)$ surjective at each $x \in X$; the *cohomology* of M is the vector bundle $E(M) = \operatorname{Ker} b / \operatorname{Im} a$. The utility of monads lies in the fact that, under certain auspicious conditions, a vector bundle (or family of such) can be described as the cohomology of a monad (or family of such) of a particularly simple kind.

Horrocks [**13**] was the first to introduce monads and used them to show that every holomorphic vector bundle on $\mathbf{P}_n$ can be described by monads with A, B, C all projectively trivial, i.e., trivial twisted by a line bundle. Barth [**3**] used this to classify stable bundles on $\mathbf{P}_2$ up to linear algebraic data, and this work was extensively generalized and developed in the book [**15**]. The monad description of bundles on $\mathbf{P}_3$ was used by Atiyah et al. [**2**] in their celebrated description of instantons (self-dual solutions of the Yang-Mills equations) on S^4, using the Ward correspondence [**17**] between holomorphic bundles on $\mathbf{P}_3$ and the instantons on S^4. The close relationship between complex analytic geometry and gauge theory has provided a rich source for applications of monads, particularly in the context of computing moduli spaces.

Methods similar to those used for the ADHM construction were used in [**7**] to describe the instantons on $\mathbf{CP}_2$; in this case, instantons correspond to certain holomorphic vector bundles on the flag manifold

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$\mathbf{F}_{1,2}(\mathbf{C}^3)$, and those bundles can again be described in terms of monads of a particularly simple nature (although in this instance, more complicated than for $\mathbf{P}_3$). These methods were subsequently modified in [**8**] to study moduli spaces of stable bundles on Hirzebruch surfaces, and it is reasonably self-evident from these two papers how the methods may be modified to deal with general flag manifolds, or more generally manifolds which are fibered over flag manifolds by flag manifolds.

Apart from classifying all instantons on S^4 up to linear algebraic data, the highly explicit nature of the monad descriptions of instantons on S^4 and $\mathbf{CP}_2$ has had several interesting applications. In [**4**] it was used to prove the $SU(2)$ version of the Atiyah-Jones conjecture. In [**10**], a *local* monad description was obtained to describe the behavior of a degenerating sequence of instantons on an arbitrary 4-manifold. In [**5**] the monad description was used to give a novel proof of Bott periodicity for the unitary groups, generalized to other classical groups in [**6**].

The monad descriptions are most effective when the bundles A, B, C are as simple as possible, projectively trivial or direct sums of such. This tends to be the case for manifolds "close to" projective spaces, flag manifolds and such like. In algebraic geometry, birationally equivalent manifolds are often regarded as "close," so there is some reason to expect that monad descriptions of vector bundles on manifolds which are birational to projective spaces should exist. In particular, it is reasonable to seek simple monad descriptions of bundles on rational surfaces. After blowing up sufficiently many times, an arbitrary rational surface is biholomorphic to a blowup of $\mathbf{P}_2$, so the classification problem for bundles on such a surface becomes that of classifying bundles on blowups of $\mathbf{P}_2$ subject to triviality constraints on certain components of the exceptional divisor.

In this paper monad descriptions for a large class of vector bundles on an arbitrary blowup of $\mathbf{P}_2$ will be presented, together with three different applications of these descriptions. The construction of the monads is loosely based on Atiyah's presentation [**1**] for bundles on $\mathbf{P}_3$ corresponding to instantons.

The bundles in this paper for which monad descriptions are obtained are those which have semi-stable direct image on $\mathbf{P}_2$. In Section 1 below, several such descriptions are presented, each having the desired property that the three bundles appearing in a monad are reasonably simple. However, without some extra assumptions, the monads do not enjoy all of the properties of similar such monads on $\mathbf{P}_n$ which make the latter so useful for explicit calculations. The principal source of the complications is the possible existence of multiple blowups, so although the general theory will be presented without any simplifying assumptions, the three examples illustrating the applications will all assume the absence of multiple blowups.

In Section 2 of this paper the monad construction is used to find an explicit description of moduli spaces $\mathcal{M}_n$ of stable holomorphic 2-bundles with $c_1 = 0$ and $c_2 = 2$ on the blowup of $\mathbf{P}_2$ at n distinct points. The result is that each point in $\mathbf{P}_2$ gives rise to a particular plane in $\mathcal{M}_0$ and $\mathcal{M}_n$ is essentially the blowup of $\mathcal{M}_0$ along the n planes corresponding to the points of $\mathbf{P}_2$ which have been blown up.

Holomorphic bundles on $\mathbf{P}_2$ which are trivial on the line at infinity are certainly semi-stable, and a theorem of Donaldson [**12**], which uses the ADHM construction, gives a correspondence between such bundles trivialized on this line and instantons on S^4 with a unitary trivialization at the point at infinity (*based* instantons). The analogous correspondence for instantons on $\mathbf{CP}_2$ and bundles on the blowup of $\mathbf{P}_2$ at one point was proved by King [**14**], and this correspondence was further generalized in [**9**] where it was shown that based instantons on a connected sum of n copies of $\mathbf{CP}_2$ correspond to holomorphic vector bundles on the blowup of $\mathbf{P}_2$ at n points trivialized on the line at infinity.

In [**5**] Bryan and Sanders used King's results to compute the topology of the moduli spaces of rank-stable SU instantons on $\mathbf{CP}_2$, this being the direct limit of the $SU(r)$ moduli spaces under the inclusions $SU(r) \hookrightarrow SU(r+1)$. They found that for $c_2 = k$, the moduli space has the homotopy type of $BU(k) \times BU(k)$, and they used this for their extraordinary proof of Bott periodicity. They conjectured that in the case of the n-fold connected sum, the spaces should have the homotopy type of the n-fold product $BU(k) \times \cdots \times BU(k)$, but using the monad description of Section 1 it will be shown here in Section 3 that the conjecture is not true.

In Section 4 the monad construction is restricted to the case of the blowup of $\mathbf{P}_2$ at a single point and to bundles which are trivial on a *neighborhood* of the line at infinity. This facilitates a characterization of isomorphism classes of bundles in a neighborhood of an exceptional line, which in turn can be used to provide a cut-and-paste description of vector bundles on the blowup of an arbitrary complex surface, as in [**11**].

In all three examples, many of the linear-algebraic details have been suppressed in the interests of brevity, but it is hoped that sufficient detail has been left for the interested reader to be able to reconstruct these calculations. Each section of the paper concludes with one or two open questions which such a reader may be tempted to tackle; the questions vary in difficulty but all appear resolvable with a sufficiently diligent analysis.

1. Construction of monads. Let $\pi : \widetilde{\mathbf{P}}_2 \to \mathbf{P}_2$ be a blowup of $\mathbf{P}_2$ consisting of n blowups and let L_∞ be a rational curve with self-intersection $+1$ in $\widetilde{\mathbf{P}}_2$ not meeting the exceptional divisor. For each blowup, there is a corresponding embedded rational curve E_i with negative self-intersection which is the proper transform in $\widetilde{\mathbf{P}}_2$ of the line introduced by the i-th blowup, and the classes $\{[L_\infty], [E_1], \ldots, [E_n]\}$ form a basis for $H^2(\widetilde{\mathbf{P}}_2, \mathbf{Z})$.

The intersection form of $\widetilde{\mathbf{P}}_2$ restricted to $[L_\infty]^\perp$ is negative definite and is isomorphic over $\mathbf{Z}$ to the standard negative definite form $(-1) \oplus \cdots \oplus (-1)$. Let $q_{ij} := [E_i] \cdot [E_j]$ so the matrix (q_{ij}) is unimodular symmetric negative definite; the inverse matrix is denoted (q^{ij}).

Let $\{\mathbf{h}, \mathbf{e}^1, \ldots, \mathbf{e}^n\}$ be the dual basis so the Poincaré dual of $[E_i]$ is $q_{ij}\mathbf{e}^j$, employing here and subsequently the summation convention on repeated upper and lower indices. The smooth complex line bundle on $\widetilde{\mathbf{P}}_2$ with first Chern class $\mathbf{e}^i$ restricts trivially to E_j for $j \neq i$ and trivially to L_∞ but restricts to the Hopf line bundle on E_i. Since $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}) = 0 = H^2(\widetilde{\mathbf{P}}_2, \mathcal{O})$, the holomorphic line bundles on $\widetilde{\mathbf{P}}_2$ are classified by $H^2(\widetilde{\mathbf{P}}_2, \mathbf{Z})$ so this line bundle has a unique holomorphic structure; henceforth, it will be denoted by $\mathcal{O}(\mathbf{e}^i)$ while $\mathcal{O}(\mathbf{h})$ will denote the pull-back of the Hopf line bundle on $\mathbf{P}_2$ to $\widetilde{\mathbf{P}}_2$. Using (2.1) of [**11**] it follows that $H^0(\widetilde{\mathbf{P}}_2, \mathcal{O}(\mathbf{e}^i)) = 0$ and $H^0(\widetilde{\mathbf{P}}_2, \mathcal{O}(-\mathbf{e}^i)) = \mathbf{C}$.

Let E be a holomorphic vector bundle on $\widetilde{\mathbf{P}}_2$ of rank r, first Chern class $a\mathbf{h} + a_i\mathbf{e}^i$ and second Chern class $k\mathbf{h}^2$. For $p, q_1, \ldots, q_n \in \mathbf{Z}$, set $\mathbf{q} := (q_1, \ldots, q_n)$, and let $E(p, \mathbf{q})$ be the tensor product of E with

$\mathcal{O}(p\mathbf{h}+q_i\mathbf{e}^i)=\mathcal{O}(p\mathbf{h})\otimes\mathcal{O}(q_1\mathbf{e}^1)\otimes\cdots\otimes\mathcal{O}(q_n\mathbf{e}^n)$. By the adjunction formula, the first Chern class of $\widetilde{\mathbf{P}}_2$ is $c_1(\widetilde{\mathbf{P}}_2)=3\mathbf{h}+n_i\mathbf{e}^i$ where $n_i=2+q_{ii}$. A straightforward calculation gives the Hirzebruch-Riemann-Roch formula for $E(p,\mathbf{q})$, namely

$$\begin{aligned}\chi(E(p,\mathbf{q}))=&-\Big[k-\frac{1}{2}a(a+3)-\frac{1}{2}q^{ij}a_i(a_j+n_j)\Big]\\ &+\frac{r}{2}\Big[(p+1)(p+2)+q^{ij}q_i(q_j+n_j)\Big]\\ &+[ap+q^{ij}a_iq_j].\end{aligned}\tag{1.1}$$

When there are no multiple blowups, this formula simplifies considerably, as in this case $q^{ij}=-\delta^{ij}$ and $n_i=1$; moreover, E_i is defined by a section of $\mathcal{O}(0,-\mathbf{e}^i)$.

Assume now that π_*E is a normalized semi-stable torsion-free sheaf on $\mathbf{P}_2$, where *normalized* means $|a|<r$. Then $H^0(\widetilde{\mathbf{P}}_2,E(p,\mathbf{q}))=0=H^0(\widetilde{\mathbf{P}}_2,E^*(p,\mathbf{q}))$ for any $\mathbf{q}$ if $p<0$ so, by Serre duality, $H^2(\widetilde{\mathbf{P}}_2,E(p,\mathbf{q}))$ and $H^2(\widetilde{\mathbf{P}}_2,E^*(p,\mathbf{q}))$ also vanish for any $\mathbf{q}$ if $p=-1,-2$. The dimensions of the groups $H^1(\widetilde{\mathbf{P}}_2,E(p,\mathbf{q}))$ are therefore determined by the Riemann-Roch formula for these values of p.

Let $B_i:=H^1(E_i,E(-\mathbf{e}^i))$. Extensions Q_1 of the form

$$0\longrightarrow E\longrightarrow Q_1\longrightarrow\mathcal{B}_1\longrightarrow 0\quad\text{for }\mathcal{B}_1:=\textstyle\bigoplus_{i=1}^n B_i(1,\mathbf{e}^i)\tag{1.2}$$

are classified by $H^1(\widetilde{\mathbf{P}}_2,Hom(\mathcal{B}_1,E))=\bigoplus_i\operatorname{Hom}(B_i,H^1(\widetilde{\mathbf{P}}_2,E(-1,-\mathbf{e}^i)))$. The cokernel of the restriction map

$$H^1(\widetilde{\mathbf{P}}_2,E(-1,-\mathbf{e}^i))\to H^1(E_i,E(-\mathbf{e}^i))=B_i$$

is contained in $H^2(\widetilde{\mathbf{P}}_2,E(-1,-\mathbf{e}^i-q_{ij}\mathbf{e}^j))$, a group which vanishes by Serre duality and the semi-stability assumption. Therefore there is an extension class in $H^1(\widetilde{\mathbf{P}}_2,Hom(\mathcal{B}_1,E))$ mapping to the identity in $\operatorname{Hom}(B_i,B_i)$ under the composition of projection and restriction $H^1(\widetilde{\mathbf{P}}_2,Hom(\mathcal{B}_1,E))\to H^1(E_i,Hom(B_i(\mathbf{e}^i),E))=\operatorname{End}B_i$ for all $i=1,\dots,n$.

Let $A_i:=H^0(E_i,E(-\mathbf{e}^i))$. Extensions X_1 of the form

$$0\longrightarrow\mathcal{A}_1\longrightarrow X_1\longrightarrow E\longrightarrow 0\quad\text{for }\mathcal{A}_1:=\textstyle\bigoplus_{i=1}^n A_i(-1,-\mathbf{e}^i)\tag{1.3}$$

are classified by $H^1(\widetilde{\mathbf{P}}_2, Hom(E, \mathcal{A}_1))$, and as above the vanishing of $H^2(\widetilde{\mathbf{P}}_2, E^*(-1, -\mathbf{e}^i - q_{ij}\mathbf{e}^j))$ implies the existence of such an extension mapping to the identity in $\operatorname{End} A_i$ for each i under the projection and restriction $H^1(\widetilde{\mathbf{P}}_2, Hom(E, \mathcal{A}_1)) \to H^1(E_i, Hom(E, A_i(-\mathbf{e}^i))) = \operatorname{End} A_i$. Since $H^2(\widetilde{\mathbf{P}}_2, Hom(\mathcal{B}_1, \mathcal{A}_1)) = 0$, the map $H^1(\widetilde{\mathbf{P}}_2, Hom(Q_1, \mathcal{A}_1)) \to H^1(\widetilde{\mathbf{P}}_2, Hom(E, \mathcal{A}_1))$ is surjective so for each extension X_1 there is a compatible extension W_1 of Q_1 by $\mathcal{A}_1$. Compatibility of the extensions implies that there is a commutative diagram of the form

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \to & E & \to & Q_1 & \to & \mathcal{B}_1 & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & X_1 & \to & W_1 & \to & \mathcal{B}_1 & \to & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \mathcal{A}_1 & = & \mathcal{A}_1 & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{1.4}$$

i.e., the display for a monad $M : 0 \to \mathcal{A}_1 \to W_1 \to \mathcal{B}_1 \to 0$ on $\widetilde{\mathbf{P}}_2$ with cohomology E.

Restricting $M \otimes \mathcal{O}(-\mathbf{e}^i)$ to E_i and taking cohomology, it follows immediately from the construction of Q_1 and X_1 that $H^0(E_i, W_1(-\mathbf{e}^i)) = 0 = H^1(E_i, W_1(-\mathbf{e}^i))$ so $W_1|_{E_i}$ is trivial for all i; this implies that W_1 is the pull-back of a bundle from $\mathbf{P}_2$, namely $\pi_* W_1$ [**11**, Lemma 2.2]. The Chern classes of W_1 can be calculated in terms of the Chern classes of E and the dimension of A_i using the Riemann-Roch formula.

From the display (1.4), the semi-stability of $\pi_* E$ and the fact that $H^0(\widetilde{\mathbf{P}}_2, \mathcal{O}(\mathbf{e}^i)) = 0$ for all i it follows that $H^0(\mathbf{P}_2, \pi_* W_1(-1)) = 0 = H^0(\mathbf{P}_2, \pi_* W_1^*(-1))$, conditions which imply that $\pi_* W_1$ is the cohomology of a similar monad on $\mathbf{P}_2$: with $L_0 := H^1(\widetilde{\mathbf{P}}_2, W_1(-1, \mathbf{0}))$ and $K_0 := H^1(\widetilde{\mathbf{P}}_2, W_1^*(-1, \mathbf{0}))^*$, the same construction as above yields a monad $0 \to K_0(-1) \to W \to L_0(1) \to 0$ on $\mathbf{P}_2$ with cohomology $\pi_* W_1$ and with W trivial. (Triviality of W follows from the fact that $H^*(\mathbf{P}_2, W(-1)) = 0 = H^*(\mathbf{P}_2, W(-2))$ so that W is trivial on L_∞ and a trivialization there extends to a trivialization over $\mathbf{P}_2$.) Pulling this monad back to $\widetilde{\mathbf{P}}_2$, the obstruction to lifting the homomorphism $\mathcal{A}_1 \to W_1$ to a map $\mathcal{A}_1 \to \ker(W \to L_0(1, \mathbf{0}))$ lies in $H^1(\widetilde{\mathbf{P}}_2, Hom(\mathcal{A}_1, K_0(-1,\mathbf{0})))$. This cohomology class defines an extension $0 \to K_0(-1,\mathbf{0}) \to \mathcal{A} \to \mathcal{A}_1 \to 0$ and, by construction, there is an injection $\mathcal{A} \to \ker(W \to L_0(1,\mathbf{0}))$. Dually there is an extension $0 \to \mathcal{B}_1 \to \mathcal{B} \to L_0(1,\mathbf{0}) \to 0$ and an epimorphism $W \to \mathcal{B}$ such that composition $\mathcal{A} \to W \to \mathcal{B}$ is 0 and such that the cohomology is precisely E. To summarize:

Proposition 1.5. *Let E be a holomorphic bundle on $\widetilde{\mathbf{P}}_2$ such that $\pi_* E$ is normalized and semi-stable. Then E is the cohomology of a monad M on $\widetilde{\mathbf{P}}_2$ of the form*

$$M : 0 \longrightarrow \mathcal{A} \longrightarrow W \longrightarrow \mathcal{B} \longrightarrow 0 \tag{1.6}$$

where W is trivial and $\mathcal{A}, \mathcal{B}$ are respectively extensions of the form

$$\begin{gathered} 0 \longrightarrow K_0(-1,\mathbf{0}) \longrightarrow \mathcal{A} \longrightarrow \textstyle\bigoplus_{i=1}^n A_i(-1,-\mathbf{e}^i) \longrightarrow 0, \\ 0 \longrightarrow \textstyle\bigoplus_{i=1}^n B_i(1,\mathbf{e}^i) \longrightarrow \mathcal{B} \longrightarrow L_0(1,\mathbf{0}) \longrightarrow 0 \end{gathered} \tag{1.7}$$

for

$$\begin{aligned} A_i &= H^0(E_i, E(-1)), \qquad B_i = H^1(E_i, E(-1)), \\ K_0 &= H^1(\widetilde{\mathbf{P}}_2, E^*(-1,\mathbf{0}))^* \oplus \bigoplus_{i=1}^n A_i \otimes H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(0,\mathbf{e}^i))^* \oplus \bigoplus_{i=1}^n B_i \\ &\qquad \otimes H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(-2,-e^i))^*, \\ L_0 &= H^1(\widetilde{\mathbf{P}}_2, E(-1,\mathbf{0})) \oplus \bigoplus_{i=1}^n A_i \otimes H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(-2,-\mathbf{e}^i)) \oplus \bigoplus_{i=1}^n B_i \\ &\qquad \otimes H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(0,\mathbf{e}^i)). \end{aligned}$$

As can be seen from the discussion, there is in some general some choice involved in the construction of the monad M. Moreover, if E' is another normalized bundle on $\widetilde{\mathbf{P}}_2$, with semi-stable direct image on $\mathbf{P}_2$ and M' is a monad with cohomology E' constructed as above, a homomorphism $E \to E'$ cannot necessarily be lifted to a monad homomorphism $M \to M'$. However, if $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(-1,-\mathbf{e}^i))$ vanishes for all i, which occurs if and only if $\widetilde{\mathbf{P}}_2$ is constructed by never blowing up at the intersection of a pair of exceptional lines, an analysis as in the proof of Lemma II-4.1.3 of [**15**] shows that every homomorphism $E \to E'$ does lift to a monad homomorphism $M \to M'$. This lift need not be unique as the kernel of the epimorphism $\mathrm{Hom}(M, M') \to \mathrm{Hom}(E, E')$ is readily identified as $H^1(\widetilde{\mathbf{P}}_2, Hom(\mathcal{B}, \mathcal{A}'))$, a group which need not vanish.

The sequences (1.7) do not split in general: for any i such that $B_i \neq 0$ and $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(0, \mathbf{e}^i)) \neq 0$, the corresponding term in (1.7) cannot be split from $\mathcal{B}$. The groups $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(\mathbf{0}, \mathbf{e}^i))$ all vanish if and only if there are no multiple blowups, in which case the sequences (1.7) do have unique splittings. Also in this case the group $H^1(\widetilde{\mathbf{P}}_2, Hom(\mathcal{B}, \mathcal{A}'))$ is easily computed to be $\oplus_{i=1}^n \mathrm{Hom}(B_i, A_i')$.

The simplifications resulting from the assumption that there are no multiple blowups can be summarized as follows:

Corollary 1.8. *Under the hypotheses of Proposition* 1.5*, assume* $\widetilde{\mathbf{P}}_2$ *is the blowup of* $\mathbf{P}_2$ *at* n *distinct points. Then* E *is the cohomology of a monad on* $\widetilde{\mathbf{P}}_2$ *of the form*

(1.9)
$$M: \qquad 0 \longrightarrow \begin{matrix} K_0(-1, \mathbf{0}) \\ \oplus \\ \bigoplus_{i=1}^n A_i(-1, -\mathbf{e}^i) \end{matrix} \longrightarrow W \longrightarrow \begin{matrix} L_0(1, \mathbf{0}) \\ \oplus \\ \bigoplus_{i=1}^n B_i(1, \mathbf{e}^i) \end{matrix} \longrightarrow 0\,,$$

(*with* $A_i = H^0(E_i, E(-1))$, $B_i = H^1(E_i, E(-1))$, $K_0 = H^1(\widetilde{\mathbf{P}}_2, E^*(-1,0))^*$ *and* $L_0 = H^1(\widetilde{\mathbf{P}}_2, E(-1,0))$ *as before*)*. If* E' *is another normalized bundle on* $\widetilde{\mathbf{P}}_2$ *with semi-stable direct image on* $\mathbf{P}_2$ *and* M' *is a monad of the form* (1.9) *with* $E(M') \simeq E'$*, there is an exact sequence*

$$0 \to \bigoplus_{i=1}^n \mathrm{Hom}(B_i, A_i') \to \Gamma(\widetilde{\mathbf{P}}_2, Hom(M, M')) \to \Gamma(\widetilde{\mathbf{P}}_2, Hom(E, E')) \to 0.$$

Proposition 1.5 provides a useful monad description of a single bundle E, but for purposes of describing families of bundles of the same topological type, it has the drawback that the dimensions of the vector spaces A_i and B_i can jump: only $\dim A_i - \dim B_i = a_i$ is constant. This difficulty can be overcome by twisting the bundles under consideration by appropriate powers of the line bundles $\mathcal{O}(\mathbf{e}^i)$: By Proposition 2.8

of [**11**], a bound on the *charge* $C(E) = [c_2 - \frac{r-1}{2r}c_1^2](E)$ gives bounds on the splitting type of E over any curve E_i, so it is always possible to find a line bundle L which is trivial off the exceptional divisor such that $H^1(E_i, E \otimes L) = 0$ for any r-bundle E of specified charge such that $\pi_* E$ is semi-stable. When $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(-1, \mathbf{e}^i))$ vanishes for every i, the hypotheses of Lemma II-4.1.3 of [**15**] are fulfilled so this approach has the added advantage that isomorphism classes of such bundles are in one-to-one correspondence with isomorphism classes of the corresponding monads.

Unfortunately, except in cases of small charge, for explicit calculations this approach can become quite unwieldy as the dimensions of the other vector spaces in the monads grow rapidly. There is, however, a less redundant description, at least when there are no multiple blowups. In this case, $H^1(\widetilde{\mathbf{P}}_2, \mathcal{O}(-2, -2\mathbf{e}^i)) \simeq \mathbf{C}$, so there is an extension $0 \to \mathcal{O}(-1, -\mathbf{e}^i) \to W_0 \to \mathcal{O}(1, \mathbf{e}^i) \to 0$ on $\widetilde{\mathbf{P}}_2$ corresponding to $1 \in \mathbf{C}$, and it is easy to verify that W_0 is trivial. By adding this short exact sequence to the complex (1.9), a new monad is obtained with the same cohomology E but with A_i, B_i and W increased in dimension by 1, 1 and 2 respectively; more generally, any number of such short exact sequences can be added to (1.9) without changing the cohomology of the monad.

On taking cohomology, the sequence $0 \to E(-1, \mathbf{0}) \to E(-1, -\mathbf{e}^i) \to E|_{E_i}(-1) \to 0$ gives the exact sequence $0 \to A_i \to H^1(\widetilde{\mathbf{P}}_2, E(-1, \mathbf{0})) \to H^1(\widetilde{\mathbf{P}}_2, E(-1, -\mathbf{e}^i)) \to B_i \to 0$, with $h^1(\widetilde{\mathbf{P}}_2, E(-1, \mathbf{0}))$ and $h^1(\widetilde{\mathbf{P}}_2, E(-1, -\mathbf{e}^i))$ determined by the Riemann-Roch formula in terms of the Chern classes of E. If $V_i := \ker H^1(\widetilde{\mathbf{P}}_2, E(-1, -\mathbf{e}^i)) \to B_i$, the exact sequence $0 \to V_i(-1, -\mathbf{e}^i) \to V_i \otimes W_0 \to V_i(1, \mathbf{e}^i) \to 0$ can be added to (1.9) to yield another monad with cohomology E. Using (noncanonical) isomorphisms $V_i \oplus A_i \simeq H^1(\widetilde{\mathbf{P}}_2, E(-1, \mathbf{0}))$ and $B_i \oplus V_i \simeq H^1(\widetilde{\mathbf{P}}_2, E(-1, -\mathbf{e}^i))$, on setting $\mathbf{e}^0 := \mathbf{0}$, the following description is thus obtained:

Proposition 1.10. *If $\widetilde{\mathbf{P}}_2$ is the blowup of $\mathbf{P}_2$ at n distinct points, a holomorphic bundle E on $\widetilde{\mathbf{P}}_2$ with $\pi_* E$ normalized and semi-stable is the cohomology of a monad of the form*

$$(1.11)\quad M : 0 \longrightarrow \bigoplus_{i=0}^{n} K_i(-1, -\mathbf{e}^i) \longrightarrow W \longrightarrow \bigoplus_{i=0}^{n} L_i(1, \mathbf{e}^i) \longrightarrow 0$$

where W is trivial, $K_0 = H^1(\widetilde{\mathbf{P}}_2, E^*(-1,\mathbf{0}))^*$, $K_i = H^1(\widetilde{\mathbf{P}}_2, E(-1,\mathbf{0}))$ for $i = 1,\dots,n$ and $L_i = H^1(\widetilde{\mathbf{P}}_2, E(-1,-\mathbf{e}^i))$ for $i = 0,1,\dots,n$. If E' is the cohomology of a monad M' of the same form, every homomorphism $E \to E'$ is induced by a homomorphism of monads $M \to M'$, and the set of such monad homomorphisms inducing the zero bundle homomorphism is isomorphic to $\oplus_{i=1}^n \mathrm{Hom}(L_i, K_i')$.

The dimensions of the vector spaces appearing in Proposition 1.10 are easily computed using the Riemann-Roch theorem: If E has rank r, first and second Chern classes $a\mathbf{h} + a_i\mathbf{e}^i$ and $k\mathbf{h}^2$, respectively, K_0 has dimension $k - (1/2)a(a-1) + (1/2)\sum_{i=1}^n a_i(a_i - 1)$, L_0 has dimension $l_0 := k - (1/2)a(a+1) + (1/2)\sum_{i=1}^n a_i(a_i+1)$, and for $i > 0$, K_i has dimension l_0 and L_i has dimension $l_0 - a_i$.

When $n = 1$, this description was given by King in [**14**] in the case of bundles which are trivial on the line at infinity; it can also be deduced without difficulty from the results of [**8**].

Some effort was spent attempting to find a construction of monads whereby bundles on the blowup $\widetilde{X}$ of a rational surface X at a single point could be described in terms of monads on X, but no significant progress was made. Such a description would lend itself to inductive proofs and hopefully to be able to better analyze bundles in the case of multiple blowups. It is also rather difficult to work with the line bundles $\mathcal{O}(\mathbf{e}^i)$ in the presence of multiple blowups, and it would be useful to have monad descriptions which make direct use of the line bundles with first Chern classes of square -1 introduced with each blowup.

2. Stable 2-bundles with $c_1 = 0$, $c_2 = 2$. The first application of the monad descriptions in the previous section is that of the construction of moduli spaces for stable 2-bundles with $c_1 = 0$ and $c_2 = 2$. As mentioned in the introduction, for simplicity it will be assumed throughout that there are no multiple blowups, so $\widetilde{\mathbf{P}}_2$ is the blowup of $\mathbf{P}_2$ at a finite set of points $\{p_1,\dots,p_n\}$. This case is already sufficiently complicated to serve the purposes of illustration.

Stability on $\widetilde{\mathbf{P}}_2$ is with respect to a metric of the form $\omega_{\epsilon\alpha} = \pi^*\omega + \epsilon\alpha^i\rho_i$ where ω denotes $1/2\pi$ times the Fubini-study metric on $\mathbf{P}_2$, ρ_i is a closed $(1,1)$-form restricting to $1/2\pi$ times the Fubini-study metric on $E_i = \pi^{-1}(p_i)$, α^i is a positive constant and $\epsilon > 0$ is sufficiently small that $\omega_{\epsilon\alpha}$ is positive on $\widetilde{\mathbf{P}}_2$. According to Proposition 3.5 of [**11**], for each fixed $\alpha = (\alpha^1, \ldots, \alpha^n)$, the moduli space of $\omega_{\epsilon\alpha}$-stable bundles of fixed topological type is independent of ϵ for ϵ sufficiently small where "sufficiently small" can be specified precisely in terms of α and the Chern classes of E. In particular, if $\|\alpha\|^2 := \sum_i (\alpha^i)^2 < 1/(k+1)$, a 2-bundle E with $c_1(E) = 0$ and $c_2(E) = k > 0$ which is ω_α-stable is $\omega_{\epsilon\alpha}$-stable for all ϵ such that $0 < \epsilon\|\alpha\| < 1/\sqrt{1+k}$, and moreover $\pi_* E$ is semi-stable.

Assume henceforth that α as above is fixed with $\|\alpha\|^2 < 1/3$. If E is a stable 2-bundle on $\widetilde{\mathbf{P}}_2$ with $c_1(E) = 0$ and $c_2(E) = 2$, the stability condition implies $H^0(\widetilde{\mathbf{P}}_2, E) = 0 = H^0(\widetilde{\mathbf{P}}_2, E(0, \mathbf{e}^i - \mathbf{e}^j))$ for any i, j such that $\alpha^i \geq \alpha^j$. Conversely, it is straightforward to check that these conditions imply that E is ω_α-stable.

Since $\pi_* E$ is semi-stable, the monad descriptions provided by Corollary 1.8 and Proposition 1.10 are applicable. Although the monad (1.11) has a pleasing symmetry not enjoyed by (1.6), for detailed calculations it turns out to be easier to work with the description furnished by Proposition 1.5 using the method described in the remark following the statement of that proposition.

Since $H^0(\widetilde{\mathbf{P}}_2, E) = 0 = H^2(\widetilde{\mathbf{P}}_2, E)$, the Riemann-Roch formula shows that $H^1(\widetilde{\mathbf{P}}_2, E)$ also vanishes. This implies that $H^1(E_i, E) = 0$ for each i, so E can split on E_i only as $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. The same applies to any rational curve of degree ≤ 2 in $\widetilde{\mathbf{P}}_2$ for similar reasons and, indeed, by the theorem of Grauert and Mülich [**15**, p. 206], the restriction of E to the generic linearly embedded line is trivial. Setting $\mathbf{1} := (1, \ldots, 1)$ and applying Proposition 1.5 to $E(0, \mathbf{1})$, the vector spaces B_i are all 0, the spaces A_i are all 2-dimensional as is K_0 and L_0 has dimension $2n+2$. Since E has rank 2, W has dimension $4n+6$ and, with K denoting a fixed 2-dimensional complex vector space, Proposition 1.5 implies that $E(0, \mathbf{1})$ is described as the cohomology of a monad on $\widetilde{\mathbf{P}}_2$ of the form

$$M : 0 \longrightarrow \bigoplus_{i=0}^n K(-1, -\mathbf{e}^i) \xrightarrow{A} W \xrightarrow{B} L(1, \mathbf{0}) \longrightarrow 0. \tag{2.1}$$

where K, W and L are respectively vector spaces of dimension 2, $4n+6$ and $2n+2$ and where $\mathbf{e}^0 := \mathbf{0}$.

The moduli space of $\omega_{\epsilon\alpha}$-stable bundles is thus identified with the space of maps (A, B) satisfying the monad condition $BA = 0$, the nonsingularity criteria (i.e., $A(x)$ is injective and $B(x)$ is onto at each $x \in \widetilde{\mathbf{P}}_2$) and the conditions implying $E(M)$ is stable, all modulo the group of monad automorphisms which in this case is $\mathcal{G} = \operatorname{Aut}(\oplus_{i=0}^n K(-1,-\mathbf{e}^i)) \times GL(W) \times GL(L)$ which acts freely modulo the scalar multiples of the identity via $(g_{\mathcal{A}}, g_W, g_L) \cdot (A, B) = (g_W A g_{\mathcal{A}}^{-1}, g_L B g_W^{-1})$. Since $H^0(\widetilde{\mathbf{P}}_2, \mathcal{O}(\mathbf{e}^i - \mathbf{e}^j)) = 0$ for $i \neq j$, the automorphisms $g_{\mathcal{A}}$ have the form

$$g_{\mathcal{A}} = \begin{bmatrix} \lambda_0 g_{00} & 0 & 0 & \cdots & 0 \\ \lambda_1 g_{10} & g_{11} & 0 & \cdots & 0 \\ \lambda_2 g_{20} & 0 & g_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n g_{n0} & 0 & 0 & \cdots & g_{nn} \end{bmatrix} \tag{2.2}$$

for some $g_{ii} \in GL(K)$ and $g_{i0} \in \operatorname{End}(K)$, where $\lambda_i \in \Gamma(\widetilde{\mathbf{P}}_2, \mathcal{O}(-\mathbf{e}^i))$ is a fixed section defining E_i.

Fix homogeneous coordinates $z^a = (z^0, z^1, z^2)$ in $\mathbf{P}_2$ such that $L_\infty = \{z^2 = 0\}$, and let p_i have inhomogeneous coordinates $p_i^a = (p_i^A, 1)$, $A = 0, 1$. Thus $z^A - p_i^A z^2 = \lambda_i w_i^A$ for some uniquely determined sections w_i^A in $H^0(\widetilde{P}_2, \mathcal{O}(\mathbf{h} + \mathbf{e}^i))$, these restricting to homogeneous coordinates on E_i. In the sequel it is useful to write $w_{i0} := -w_i^1$ and $w_{i1} := w_i^0$, so $w_i^A w_{iA} \equiv 0$.

More generally, this convention will be adopted for any pair of objects such as matrices or vectors; for example, if m_A ($A = 0, 1$) are endomorphisms of some vector space V, $m_A m^A = m_0 m_1 - m_1 m_0$. In the same vein it is sometimes convenient to denote by V_A the vector space $V \oplus V$ so that, for example, $(m_0, m_1) : V \to V \oplus V$ can be written as $m_A : V \to V_A$. This notation, which is essentially Penrose's "abstract index notation," greatly facilitates and makes transparent many of the linear-algebraic calculations required to refine the monad descriptions. The convention that summation over repeated upper and lower indices is assumed remains in force so, for example, the map $V \oplus V \to V$ given by $(v_0, v_1) \mapsto m^0 v_0 + m^1 v_1$ would be written $m^A : V_A \to V$. In this framework a vector $p \in \mathbf{C}^3$ is identified with its three components p^a, $a = 0, 1, 2$, and it is consistent to write, for example, $p = p^a \in \mathbf{C}^a$. Further details can be found in Chapter 2 of [**16**].

As noted earlier, if $D \subset \mathbf{P}_2$ is a line not meeting any of the points p_i, the restriction of E to (the proper transform of) D satisfies

$H^1(D,E)=0$; in particular, $H^1(L_\infty,E)=0$. If the map B of (2.1) has the form $B(z)=B_a z^a$ for some $B_a\in \mathrm{Hom}(W,L)$, the condition that $H^1(L_\infty,E)=0$ is then equivalent to the condition that $B_A: W\to L_A$ be surjective.

For simplicity assume from now on that E is trivial on L_∞; since E is trivial on the generic line in $\mathbf{P}_2$ not meeting any point p_i, this is not a great restriction and can easily be removed later. Using the monad condition together with this assumption, it is then straightforward to deduce that bases can be found for L and W so that $L=\oplus_{i=0}^n K$, $W=\begin{matrix}\oplus_{i=0}^n K_A\\ \oplus\\ K\end{matrix}$ and in terms of these decompositions the maps A and B have the forms

$$\text{(2.3a)}\qquad A(z)=\begin{bmatrix} z_A+a_{00A}z^2 & 0 & 0 & \cdots & 0\\ a_{10A}z^2 & w_{1A} & 0 & \cdots & 0\\ a_{20A}z^2 & 0 & w_{2A} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n0A}z^2 & 0 & 0 & \cdots & w_{nA}\\ bz^2 & 0 & 0 & \cdots & 0\end{bmatrix}$$

(2.3b)

$$B(z)=\begin{bmatrix} z^A+a_{00}^A z^2 & 0 & 0 & \cdots & 0 & d_0z^2\\ a_{10}^A z^2 & z^A-p_1^A z^2 & 0 & \cdots & 0 & d_1 z^2\\ a_{20}^A z^2 & 0 & z^A-p_2^A z^2 & \cdots & 0 & d_2 z^2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n0}^A z^2 & 0 & 0 & \cdots & z^A-p_n^A z^2 & d_n z^2\end{bmatrix}.$$

Using the fact that $(z^A-z^2p_i^A)w_{iA}=\lambda_i w_i^A w_{iA}=0$, the condition $BA\equiv 0$ is rapidly found to be equivalent to

$$\text{(2.4)}\qquad a_{00}^A a_{00A}+d_0 b=0,\quad a_{i0}^A(a_{00A}+p_{iA}\mathbf{1}_K)+d_i b=0\quad \text{for } i>0.$$

The elements of $\mathcal{G}$ which preserve this form themselves have the form

(2.5)

$$\left(g_{\mathcal{A}},\begin{bmatrix} g_L\delta_B^A & 0\\ 0 & g_K\end{bmatrix},g_L\right),\quad \text{for } g_L=\begin{bmatrix} g_{00} & 0 & 0 & \cdots & 0\\ g_{10} & g_{11} & 0 & \cdots & 0\\ g_{20} & 0 & g_{22} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ g_{n0} & 0 & 0 & \cdots & g_{nn}\end{bmatrix}$$

with $g_{\mathcal{A}}$ as in (2.2), with the action given explicitly by

$$\begin{aligned} a_{00A} &\longmapsto g_{00}a_{00A}g_{00}^{-1},\ b \longmapsto g_K b g_{00}^{-1},\ d_0 \longmapsto g_{00}d_0 g_K^{-1}, \\ a_{i0A} &\longmapsto g_{ii}(a_{i0A} + g_{ii}^{-1}g_{i0}a_{00A})g_{00}^{-1}, \\ d_i &\longmapsto g_{ii}(d_i + g_{ii}^{-1}g_{i0}d_0)g_K^{-1} \quad (i>0). \end{aligned} \tag{2.6}$$

By direct calculation from the display for the monad (2.1), the condition that $H^0(\widetilde{\mathbf{P}}_2, E)$ should vanish is equivalent to the condition that d_0 in (2.3b) should be an isomorphism. Hence, by choosing g_{i0} in (2.6) appropriately, the map B can be placed in the form (2.3b) with $d_i = 0$ for $i > 0$ and with $d_0 = \mathbf{1}_K$; once this has been achieved, the automorphisms of (2.5) must satisfy $g_{00} = g_K$ and $g_{i0} = 0$ for $i > 0$ and b in (2.3a) is given by $b = -a_{00}^{A}a_{00A}$.

Determining the conditions under which $H^0(\widetilde{\mathbf{P}}_2, E(0, \mathbf{e}^i - \mathbf{e}^j)) \neq 0$ is considerably more difficult to read directly from the display for M. An element of this group is a section in $H^0(\widetilde{\mathbf{P}}_2, E(0, -\mathbf{e}^j))$ which vanishes on E_i and, by chasing the display for the monad, $H^0(\widetilde{\mathbf{P}}_2, E(0, -\mathbf{e}^j))$ is identified with the kernel of $a_{00}^{A} + p_j^{A}\mathbf{1}_K : K \to K^{A}$. Further diagram chasing then reveals that the section of $E(0, -\mathbf{e}^j)$ corresponding to an element k_0 in this kernel vanishes on E_i if and only if $a_{i0}^{A}k_0 = 0$.

To determine the nonsingularity conditions for M, note first that A and B are automatically injective and surjective respectively at each point of L_∞. Moreover, by inspection $B(z) : W \to L \otimes \mathcal{O}(1,0)_z$ is onto at any point $z \notin \{p_1, \ldots, p_n\}$ and has nontrivial cokernel at $z = (p_i^{A}, 1)$ if and only if $a_{i0}^{A} : K_A \to K$ is not onto. The nonsingularity criteria for the map A are equally straightforward to determine and the result is that $A(x)$ is not injective if $\pi(x) = z \neq p_i$ for all i and $z^{A}\mathbf{1}_K + z^2 a_{00}^{A} : K \to K^{A}$ has a nontrivial kernel or, if $z = p_i$ and $(p_i^{A}\mathbf{1}K_K + a_{00}^{A})k_0 = 0 = w_{iA}a_{i0}^{A}k_0$ for some nonzero $k_0 \in K$ and some nonzero $w_{iA} \in \mathbf{C}_A$.

Writing $a_i^{A} := a_{i0}^{A}$ and $g_i := g_{ii}$ for $i \geq 0$, the results so far are summarized as follows.

Proposition 2.7. *For $\|\alpha\|^2 < 1/3$, the moduli space $\mathcal{M}_n = \mathcal{M}(\widetilde{\mathbf{P}}_2, r = 2, c_1 = 0, c_2 = 2)$ of ω_α-stable 2-bundles on $\widetilde{\mathbf{P}}_2$ with $c_1 = 0$ and $c_2 = 2$ with trivial restriction to L_∞ is identified with the set of*

$(n+1)$-tuples $(a_0^A, \dots, a_n^A) \in \oplus_{i=0}^n \mathbf{C}^A \otimes \operatorname{End}(K)$ *satisfying the monad condition*

$$a_i^A(a_{0_A} + p_{iA}\mathbf{1}_K) = 0, \quad i = 1, \dots, n, \tag{2.8}$$

the stability condition

$$[a_0^A + p_j^A \mathbf{1}_K \quad a_i^A] : K \to K^A \oplus K^A \text{ is injective for } 0 < i, j \le n \text{ such that } \alpha^i \ge \alpha^j \text{ and } i \ne j \tag{2.9}$$

and the nonsingularity conditions

$$a_i^A : K_A \to K \quad \text{is onto for all } i > 0; \tag{2.10a}$$

$$z^2 a_0^A + z^A \mathbf{1}_K : K \to K^A \quad \text{is injective for } z^A \ne p_i^A z^2 \text{ and all } i > 0; \tag{2.10b}$$

$$[a_0^A + p_i^A \mathbf{1}_K \quad w_A a_i^A] : K \to K^A \oplus K \text{ is injective for all } i > 0 \text{ and } w_A \in \mathbf{C}_A \setminus \{0\} \tag{2.10c}$$

modulo the action of $\times_{i=0}^n GL(K)$ *given by*

$$a_i^A \longmapsto g_i a_i^A g_0^{-1} \quad \text{for } g_i \in GL(K), \quad i = 0, \dots, n. \tag{2.11}$$

When $n = 0$ it is well known (see, e.g., [**15**, pp. 349–354]) that the moduli space of stable 2-bundles on $\mathbf{P}_2$ with $c_1 = 0$ and $c_2 = 2$ is canonically identified with the projectivized space of symmetric 3×3 matrices $\mathbf{P}(\mathbf{C}^{(ab)})$ minus the hypersurface consisting of the singular such matrices; (the parentheses around the indices here indicate symmetrization). The bundle associated with a symmetric matrix A^{ab} is nontrivial on the line $q_a z^a = 0$ if and only if $A^{ab} q_a q_b = 0$ so, in particular, the subset consisting of those which are trivial on L_∞ can be identified with pairs $(A^{AB}, A^A) \in \mathbf{C}^{(AB)} \oplus \mathbf{C}^A$, where $A^A = A^{A2}$ and $A^{22} = 1$. The description provided by Proposition 2.7 is related to this via $a_0^A \longmapsto (\det(a_0)^{AB}, (1/2)\operatorname{tr} a_0^A)$, where the determinant of a *pair* of $k \times k$ matrices a^0, a^1 is by definition the (coefficients of the) degree k polynomial in z_0, z_1 given by $\det(a^A z_A)$. The nonsingularity condition (2.10b) is equivalent to the condition that a_0^0 and a_0^1 should have no common eigenvector, and this in turn is equivalent to the condition

$(\det(a_0)^{AB} - (1/4)\mathrm{tr}\, a_0^A \mathrm{tr}\, a_0^B)(\det(a_0)_{AB} - (1/4)\mathrm{tr}\, a_{0_A} \mathrm{tr}\, a_{0_B}) \neq 0$, this expression being twice the determinant of the corresponding symmetric 3×3 matrix.

With $a_0^A, \dots, a_n^A$ as in Proposition 2.7, a direct calculation reveals that equation (2.8) implies $\det(a_0)^{AB} + \mathrm{tr}\, a_0^{(A} p_i^{B)}$ $(= \det(a_0 - p_i\mathbf{1})^{AB})$ is a scalar multiple of $\det(a_i)^{AB}$ when the latter is nonzero. A lengthy analysis along the lines of that presented in Section 4 of [**8**] and which will be omitted here shows that the map

$$(a_0^A, \dots, a_n^A) \longmapsto (\det(a_0)^{AB}, \mathrm{tr}\, a_0^A, [\det(a_1)^{AB}], \dots, [\det(a_n)^{AB}])$$

realizes the moduli space of stable 2-bundles on $\widetilde{\mathbf{P}}_2$ with $c_1 = 0$, $c_2 = 2$ which are trivial on L_∞ as a blowup of the corresponding moduli space for $\mathbf{P}_2$.

It is a simple matter now to remove the assumption of triviality on L_∞. The final result is that the moduli space of all stable 2-bundles on $\widetilde{\mathbf{P}}_2$ with $c_1 = 0$ and $c_2 = 0$ has the following complete description:

Proposition 2.12. *Let $\overline{\mathcal{M}}_0 := \mathbf{P}(\mathbf{C}^{(ab)})$, $S_0 := \{[v^{ab}] \in \overline{\mathcal{M}}_0 \mid \det(v^{ab}) = 0\}$ and, for $v^a \in \mathbf{C}^a \setminus \{0\}$, let $P(v^a)$ be the 2-plane in $\overline{\mathcal{M}}_0$ given by $P(v^a) := \{[v^{(a}w^{b)}] \mid w^b \in \mathbf{C}^b \setminus \{0\}\} \subset S_0$. If $\alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^n > 0$, for all $\epsilon > 0$ sufficiently small the moduli space of $\omega_{\epsilon\alpha}$-stable 2-bundles with $c_1 = 0$ and $c_2 = 2$ on $\widetilde{\mathbf{P}}_2$ is isomorphic to $\overline{\mathcal{M}}_n \setminus (S_n \cup T_n)$ where $\overline{\mathcal{M}}_n \stackrel{\pi_n}{\to} \overline{\mathcal{M}}_{n-1}$ is the blowup of $\overline{\mathcal{M}}_{n-1}$ along the proper transform $\widetilde{P}(p_n^a)$ of $P(p_n^a)$ in $\overline{\mathcal{M}}_{n-1}$, S_n is the proper transform of S_{n-1} and T_n is the proper transform of $\cup_{i:\alpha^i=\alpha^n} \widetilde{P}(p_i^a) \cap \widetilde{P}(p_n^a)$.*

(The points of T_n correspond to the extensions $0 \to \mathcal{O}(\mathbf{e}^i - \mathbf{e}^n) \to E \to \mathcal{O}(\mathbf{e}^n - \mathbf{e}^i) \to 0$.)

The explicit construction of the moduli space of 2-bundles E on $\widetilde{\mathbf{P}}_2$ with $\pi_* E$ semi-stable, $c_1(E) = -\mathbf{h}$ and $c_2(E) = 2\mathbf{h}^2$ can be approached along similar lines to those described above; the problem is in some respects simpler since any such bundle is automatically $\omega_{\epsilon\alpha}$-stable for all ϵ sufficiently small, regardless of α.

It would be very interesting to have an explicit description akin to that above in the case of stable 2-bundles with $c_2 = 2\mathbf{h}^2$ and $c_1 = -\mathbf{h}$

or 0 when there are multiple blowups. These would be expected to be limiting cases of the descriptions in the absence of multiple blowups.

3. Bundles trivialized on L_∞. According to the results of [**9**], for a certain class of metrics on $n\mathbf{CP}_2$ (the n-fold connected sum of $\mathbf{CP}_2$) with itself, the moduli spaces of self-dual solutions of the Yang-Mills equations (instantons) on a $U(r)$ bundle over this space with a fixed unitary trivialization at some point x_0 are in one-to-one correspondence with holomorphic r-bundles E on $\widetilde{\mathbf{P}}_2$ with a fixed holomorphic trivialization on L_∞. For $n = 0$, this was proved by Donaldson [**12**] using the ADHM description of instantons on S^4 [**2**] and, for $n = 1$ it was proved by King [**14**] using the analogous construction for instantons on $\mathbf{CP}_2$ [**7**]. The moduli spaces are all smooth spaces of real dimension $4rC(E)$ where $C(E) = c_2(E) - ((r-1)/2r)c_1(E)^2$.

In this section the monads of Proposition 1.10 will be used to obtain a reasonably simple description of holomorphic bundles trivialized on L_∞, and using this description it will be possible to show that the conjecture of Bryan and Sanders mentioned in the introduction cannot be correct in its current form.

Let $\widetilde{\mathbf{P}}_2$ be the blowup of $\mathbf{P}_2$ at n distinct points, none of these points lying on L_∞. Let E be a holomorphic r-bundle on $\widetilde{\mathbf{P}}_2$ with $c_1(E) = a_i\mathbf{e}^i$ which is trivial on L_∞ and which has a given trivialization on that line. By Proposition 1.10, there are vector spaces K_i, W, L_i such that E is described as the cohomology of a monad of the form

$$(3.1) \quad M : 0 \longrightarrow \bigoplus_{i=0}^n K_i(-1, -\mathbf{e}^i) \xrightarrow{A} W \xrightarrow{B} \bigoplus_{i=0}^n L_i(1, \mathbf{e}^i) \longrightarrow 0$$

where K_0 has dimension $k + (1/2)\sum_{i=1}^n a_i(a_i - 1)$, L_0 has dimension $l_0 = k + (1/2)\sum_{i=1}^n a_i(a_i + 1)$, and for $i > 0$, K_i and L_i have dimensions l_0 and $l_0 - a_i$, respectively.

Using the condition that E is trivialized on L_∞ and employing the same notation as in the previous section, a basis can be chosen for W

so that $W = \begin{matrix} \bigoplus_{i=0}^{n} L_{iA} \\ \oplus \\ \mathbf{C}^r \end{matrix}$ and B has the form

$$B = \begin{bmatrix} z^A + b_{00}^A z^2 & b_{01}^A z^2 & b_{02}^A z^2 & \cdots & b_{0n}^A z^2 & dz^2 \\ 0 & w^{1A} & 0 & \cdots & 0 & 0 \\ 0 & 0 & w^{2A} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & w^{nA} & 0 \end{bmatrix}$$

for some $b_{0i}^A \in \mathrm{Hom}(L_i, L_0)$ and $d \in \mathrm{Hom}(\mathbf{C}^r, L_0)$. The map A of (3.1) must then have the form

$$A = \begin{bmatrix} a_{00} z_A + a_{00A} z^2 & a_{01} w_{1A} & a_{02} w_{2A} & \cdots & a_{0n} w_{nA} \\ a_{10} \lambda_1 w_{1A} & a_{11} w_{1A} & 0 & \cdots & 0 \\ a_{20} \lambda_2 w_{2A} & 0 & a_{22} w_{2A} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} \lambda_n w_{nA} & 0 & \cdots & \cdots & a_{nn} w_{nA} \\ c_0^B z_B + c z^2 & c_1^B w_{1B} & c_2^B w_{2B} & \cdots & c_n^B w_{nB} \end{bmatrix}$$

for some $a_{ij} \in \mathrm{Hom}(K_i, L_j)$, $a_{0iA} \in \mathrm{Hom}(K_0, L_i)$, $c_i^B \in \mathrm{Hom}(K_i, \mathbf{C}^r)$, $c \in \mathrm{Hom}(K_0, \mathbf{C}^r)$. The monad condition $BA = 0$ is then

$$-a_{00}^A + b_{00}^A a_{00} + \sum_{i=1}^{n} b_{0i}^A a_{i0} + d c_0^A = 0 \tag{3.2a}$$

$$(p_i^A + b_{00}^A) a_{0i} + b_{0i}^A a_{ii} + d c_i^A = 0 \quad \text{for } i > 0 \tag{3.2b}$$

$$b_{00}^A a_{00A} - \sum_{i=1}^{n} b_{0i}^A a_{i0} p_{iA} + dc = 0. \tag{3.2c}$$

The condition that E be trivial on L_∞ is equivalent to the condition that the matrix

$$a = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ a_{20} & 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & 0 & 0 & \cdots & a_{nn} \end{bmatrix} : \begin{matrix} K_0 \\ \oplus \\ \vdots \\ \oplus \\ K_n \end{matrix} \to \begin{matrix} L_0 \\ \oplus \\ \vdots \\ \oplus \\ L_n \end{matrix}$$

should be nonsingular and, given this condition, replacing A and B by $g_W A$ and Bg_W^{-1} respectively for $g_W = \left[\begin{smallmatrix} \delta_B^A & 0 \\ -c^A a^{-1} & \delta_B^A \end{smallmatrix}\right]$ where $c^A := [c_0^A \; c_1^A \; \cdots \; c_n^A]$ replaces c_i^A by 0 in (3.2) for $i \geq 0$.

Let a_{0*} and a_{*0} denote the first row and column of a, respectively, and let $a^A := [a_{00}^A \; 0 \; \cdots \; 0]$, $b^A := [b_{00}^A \; b_{01}^A \; \cdots \; b_{0n}^A]$ and $p^A := \operatorname{diag}[0, p_1^A, \ldots, p_n^A]$. Equations (3.2a) and (3.2b) can be written in matrix form as $b^A a + a_{0*} p^A = a^A$ so $b^A = (a^A - a_{0*} p^A) a^{-1}$ is determined by a and a_{00}^A. Equation (3.2c) can be written as $b_{00}^A a_{00A} - b^A p_A a_{*0} + dc = 0$, or

$$b_{00}^A a_{00A} + a_{0*} p^A a^{-1} p_A a_{*0} - a^A a^{-1} p_A a_{*0} + dc = 0. \tag{3.3}$$

By calculating the commutator of p^A and a and substituting for b_{00}^A, it is easily found that (3.3) can be written in the relatively simple form

$$(q^A a^{-1} q_A)_{00} + dc = 0 \tag{3.4}$$

where

$$q^A = \begin{bmatrix} -a_{00}^A & p_1^A a_{01} & p_2^A a_{02} & \cdots & p_n^A a_{0n} \\ p_1^A a_{10} & p_1^A a_{11} & 0 & \cdots & 0 \\ p_2^A a_{20} & 0 & p_2^A a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n^A a_{n0} & 0 & 0 & \cdots & p_n^A a_{nn} \end{bmatrix}$$

and the subscripts 00 indicate the 00 entry.

The data for the monad M is determined by (a, q^A, c, d) satisfying (3.4) and the requirements that M be nonsingular. If M' is another monad of the same form as (3.1) determined by (a', q'^A, c', d') and E' is its cohomology, there is an isomorphism $E \to E'$ preserving trivializations on L_∞ if and only if there are matrices

$$g = \begin{bmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0n} \\ 0 & g_{11} & 0 & \cdots & 0 \\ 0 & 0 & g_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{nn} \end{bmatrix} \in \operatorname{Aut} \begin{pmatrix} L_0 \\ \oplus \\ \vdots \\ \oplus \\ L_n \end{pmatrix},$$

$$h = \begin{bmatrix} h_{00} & 0 & 0 & \cdots & 0 \\ h_{10} & h_{11} & 0 & \cdots & 0 \\ h_{20} & 0 & h_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n0} & 0 & 0 & \cdots & h_{nn} \end{bmatrix} \in \operatorname{Aut} \begin{pmatrix} K_0 \\ \oplus \\ \vdots \\ \oplus \\ K_n \end{pmatrix}$$

such that

$$(a', q'^A, c', d') = (g, h) \cdot (a, q^A, c, d) := (gah, gq^A h, ch_{00}, g_{00} d).$$

Explicitly, this action on (a, a_{00}^A) is given by

$$\begin{aligned}
& a_{00} \longmapsto g_{00} a_{00} h_{00} + \sum_{i=1}^{n} (g_{0i} a_{i0} h_{00} + g_{00} a_{0i} h_{i0} + g_{0i} a_{ii} h_{i0}) \\
& a_{ii} \longmapsto g_{ii} a_{ii} h_{ii}, \quad a_{0i} \longmapsto g_{00} a_{0i} h_{ii} + g_{0i} a_{ii} h_{ii}, \\
& a_{i0} \longmapsto g_{ii} a_{i0} h_{00} + g_{ii} a_{ii} h_{i0}, \quad i > 0 \\
& a_{00}^A \longmapsto g_{00} a_{00}^A h_{00} - \sum_{i=1}^{n} (g_{0i} a_{i0} h_{00} + g_{00} a_{0i} h_{i0} + g_{0i} a_{ii} h_{i0}) p_i^A.
\end{aligned} \tag{3.5}$$

The set of pairs (g, h) fixing (a, q^A, c, d) are those of the form

$$g = 1 + am, \quad h = (1 + ma)^{-1} \text{ for} \tag{3.6}$$

$$m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & m_{11} & 0 & \cdots & 0 \\ 0 & 0 & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{nn} \end{bmatrix} \in \mathrm{Hom} \begin{pmatrix} L_0 & & K_0 \\ \oplus & & \oplus \\ \vdots & , & \vdots \\ \oplus & & \oplus \\ L_n & & K_n \end{pmatrix}.$$

Since a is nonsingular, after fixing isomorphisms $K_i \simeq L_0$, it is possible to choose g, h as above so that $a_{0i} = 1$ for $i > 0$. To preserve this form, h must then be of the above form with $h_{ii} = (g_{00} + g_{0i} a_{ii})^{-1}$ for each such i. With a of this form, every such matrix g, with $g_{00} + g_{0i} a_{ii}$ nonsingular, can be factored as $g = D(1 + am)$ where D is "diagonal" and m is as in (3.6) so the moduli space can now be identified with the set of nonsingular matrices of the form of a with $a_{0i} = 1$ for $i > 0$ together with a_{00}^A, c, d all satisfying the monad condition (3.4), modulo the free action of the group of nonsingular matrices $g = \mathrm{diag}\,[g_{00}, \ldots, g_{nn}]$ and h with $h_{ii} = g_{00}^{-1}$ for $i > 0$. This action is now explicitly described by

$$\begin{aligned}
& a_{00} \longmapsto g_{00} \left(a_{00} + \sum_{i=1}^{n} h_{i0} h_{00}^{-1} \right) h_{00} \\
& a_{ii} \longmapsto g_{ii} a_{ii} g_{00}^{-1}, \quad a_{i0} \longmapsto g_{ii} (a_{i0} + a_{ii} h_{i0} h_{00}^{-1}) h_{00}, \quad i > 0 \\
& a_{00}^A \longmapsto g_{00} \left(a_{00}^A - \sum_{i=1}^{n} h_{i0} h_{00}^{-1} p_i^A \right) h_{00}, \quad c \longmapsto c h_{00}, \quad d \longmapsto g_{00} d.
\end{aligned} \tag{3.7}$$

For $n = 0$, $a = a_{00}$ is nonsingular, so after choosing h_{00} appropriately, it can be assumed that $a_{00} = 1$. Then the moduli space is identified with the triples (a_{00}^A, c, d) satisfying the nonsingularity conditions together with the identity $a_{00}^A a_{00A} + dc = 0$ modulo the action of $GL(K_0)$ given by $(a_{00}^A, c, d) \mapsto (g a_{00}^A g^{-1}, c g^{-1}, g d)$ for $g \in GL(K_0)$.

For $n = 1$, by choosing h_{10} appropriately, it can be assumed that $a_{00} = 0$ and then the nonsingularity of a is equivalent to the condition that $a_{10} : K_0 \to L_1$ be an isomorphism. After having fixed once and for all an isomorphism between K_0 and L_1, by then choosing g_{11} appropriately it can be assumed that $a_{10} = 1$. The space of monad data is thus identified with the quadruples (a_{00}^A, a_{11}, c, d) satisfying the nonsingularity conditions and $a_{00}^A a_{11} a_{00A} + dc = 0$ modulo the action of $GL(K_0) \times GL(K_1)$ given by $(a_{00}^A, a_{11}, c, d) \mapsto (g a_{00}^A h, h^{-1} a_{11} g^{-1}, ch, gd)$.

For $n > 1$, the simplest possible case is that for which $c_1(E) = 0$ and $c_2(E) = 1$. Then the monad condition is simply $dc = 0$ and the nonsingularity conditions are easily found to be $c \neq 0 \neq d$. The moduli space has the homotopy type of a fibration over the flag manifold $\mathbf{F}_{1,r-1}$ of 1-planes in $r-1$ planes in $\mathbf{C}^r$, where the fiber is an open subset of the n-fold cartesian product $\mathbf{P}_1 \times \cdots \times \mathbf{P}_1$; for $n = 2$ this subset is the complement of the diagonal and therefore has the homotopy type of $\mathbf{P}_1$. Under the direct limit induced by the inclusions $\mathbf{C}^r \hookrightarrow \mathbf{C}^{r+1}$, the rank stable moduli space for $n = 2$ has the homotopy type of a $\mathbf{P}_1$ bundle over $\mathbf{P}_\infty \times \mathbf{P}_\infty$, so the conjecture of Bryan and Sanders cannot hold.

The obvious open problem at this point is to determine precisely the homotopy type of the rank stable moduli spaces, for which the monad descriptions provide a direct method of attack.

Note added in proof: Two recent preprints of J.P. Santos are also directly related to this question, and also demonstrate the invalidity of the conjecture of Bryan and Sanders: see `math.AG/0212176` and `math.AG/0301158`.

4. Bundles trivial on a neighborhood of L_∞. The third and final application of the monad descriptions is to vector bundles on the blowup of $\mathbf{P}_2$ which are trivial on a neighborhood of L_∞. As discussed at the end of Section 2 of [**11**], vector bundles on the blowup of a complex surface at a point can be described in terms of bundles on the surface glued to bundles on a neighborhood of the exceptional line, for which the gluing operation requires a choice of trivialization for the latter bundle away from the exceptional line. By gluing such objects to the trivial bundle on $\mathbf{P}_2$, an effective description of them is obtained by classifying the holomorphic bundles on the blowup of $\mathbf{P}_2$ which are trivialized on a neighborhood of L_∞; this is the objective of this section.

Let x_0 be a point in $\mathbf{P}_2$ not on L_∞, let $\widetilde{\mathbf{P}}_2$ be the blowup of $\mathbf{P}_2$ at x_0, and let $L_0 \subset \widetilde{\mathbf{P}}_2$ be the exceptional line. In the notation of previous sections, $n = 1$, $x_0 = p^1$ and $L_0 = E_1$. Let $\mathbf{e} := \mathbf{e}^1$, so L_0 is defined by a section of $\mathcal{O}(-\mathbf{e}) = \mathcal{O}(0,-1)$. For ease of calculation, x_0 will be taken to have homogeneous coordinates $(0,0,1)$.

Let E be a holomorphic r-bundle on $\widetilde{\mathbf{P}}_2$ with $c_1(E) = a\mathbf{e}$ and $c_2(E) = k\mathbf{h}^2$. If E is trivial on L_∞ it has semi-stable direct image on $\mathbf{P}_2$ so Propositions 1.5 and 1.10 are applicable. The latter proposition shows that E is the cohomology of a monad of the form

(4.1)
$$M: \quad 0 \longrightarrow \begin{matrix} K_0(-1,\ 0) \\ \oplus \\ K_1(-1,-1) \end{matrix} \quad \begin{matrix} {\scriptstyle A_a z^a} \\ \longrightarrow \\ {\scriptstyle B_A w^A} \end{matrix} \quad W \quad \begin{matrix} {\scriptstyle C_A w^A} \\ \longrightarrow \\ {\scriptstyle D_a z^a} \end{matrix} \quad \begin{matrix} K_0(1,1) \\ \oplus \\ K_1(1,0) \end{matrix} \longrightarrow 0$$

where $w^A \in \Gamma(\widetilde{\mathbf{P}}_2, \mathcal{O}(1,1))$ restrict to homogeneous coordinates on L_0, so $z^A w_A = 0$ for $(w_0, w_1) = (-w^1, w^0)$ as before. Here K_0, K_1 and W are vector spaces of dimensions $k+(1/2)a(a-1)$, $k+(1/2)a(a+1)$ and $r+4k+2a^2$, respectively.

Using the fact that E is trivial on L_∞, it is quickly found in the usual way that M is isomorphic to a monad of the form

$$(4.2) \qquad 0 \longrightarrow \begin{matrix} K_0(-1,0) \\ \bigoplus \\ K_1(-1,-1) \end{matrix} \longrightarrow \begin{matrix} K_0^A \\ \bigoplus \\ K_1^A \\ \bigoplus \\ R \end{matrix} \longrightarrow \begin{matrix} K_0(1,1) \\ \bigoplus \\ K_1(1,0) \end{matrix} \longrightarrow 0$$

where R is an r-dimensional vector space and the two maps of this monad have the form

$$(4.3) \qquad \begin{bmatrix} z^A - ca^A z^2 & 0 \\ a^A z^2 & w^A \\ a_2 z^2 & 0 \end{bmatrix}, \qquad \begin{bmatrix} w_A & cw_A & 0 \\ a_A z^2 & z_A & d_2 z^2 \end{bmatrix}$$

for some $a^A \in \mathrm{Hom}(K_0, K_1)$, $c \in \mathrm{Hom}(K_1, K_0)$, $a_2 \in \mathrm{Hom}(K_0, R)$, $d_2 \in \mathrm{Hom}(R, K_1)$. The monad condition on these maps is that they must satisfy

$$a_A c a^A = d_2 a_2, \tag{4.4}$$

together with nondegeneracy conditions to ensure that the first map of (4.3) is injective and the second is surjective at each point of $\widetilde{\mathbf{P}}_2$. The only freedom in this description is $(a^A, c, a_2, d_2) \mapsto (g_1 a^A g_0^{-1}, g_0 c g_1^{-1}, g_R a_2 g_0^{-1}, g_1 d_2 g_R^{-1})$ for some $g_0 \in GL(K_0)$, $g_1 \in GL(K_1)$, $g_R \in GL(R)$. A trivialization for E on L_∞ is determined by a choice of basis for R, so if this is fixed the automorphism g_R must be the identity.

A line L in $\mathbf{P}_2$ not passing through x_0 is given by an equation of the form $z^2 = \alpha_A z^A$ for some $\alpha_A \in \mathbf{C}^2$, and the bundle E is trivial on this line if and only if $\begin{bmatrix} 1 & c \\ \alpha_A a^A & 1 \end{bmatrix} \in \mathrm{End}\begin{pmatrix} K_0 \\ \oplus \\ K_1 \end{pmatrix}$ is an isomorphism; equivalently, if and only if $\chi_\alpha := 1 - \alpha_A a^A c \in \mathrm{End}\,(K_1)$ (or $1 - \alpha_A c a^A \in \mathrm{End}\,(K_0)$) is an isomorphism. Thus, E is trivial on *all* lines not passing through x_0 if and only if $\alpha_A c a^A$ is nilpotent for any $\alpha_A \in \mathbf{C}^2$. Note that this condition implies that the monad is nonsingular at each point in $\widetilde{\mathbf{P}}_2 \setminus L_0$.

If E is trivial on a line $L_\alpha \neq L_\infty$ in $\mathbf{P}_2$ not meeting x_0, a trivialization on L_∞ determines a trivialization on L_α by propagating from the point $L_\infty \cap L_\alpha$; for any isomorphism $\tau : \mathbf{C}^r \to R$, the corresponding r-dimensional subspace of W is the image of $\begin{bmatrix} -\alpha^A c \chi_\alpha^{-1} d_2 \tau \\ \alpha^A \chi_\alpha^{-1} d_2 \tau \\ \tau \end{bmatrix}$. For $\alpha, \beta \in \mathbf{C}^2$, $L_\alpha \cap L_\beta$ does not lie on L_∞ if and only if $\beta \cdot \alpha := \beta_B \alpha^B = -\alpha \cdot \beta$ is nonzero; i.e., if and only if α, β are linearly independent. If E is trivialized along both L_α and L_β by propagating from $L_\infty \cap L_\alpha$ and $L_\infty \cap L_\beta$ respectively, the two trivializations agree at $L_\alpha \cap L_\beta$ if and only if there exists $\kappa_0 \in \mathrm{Hom}(\mathbf{C}^r, K_0)$, $\kappa_1 \in \mathrm{Hom}(\mathbf{C}^r, K_1)$ such that

$$\begin{aligned} &\begin{bmatrix} -\alpha^A c \chi_\alpha^{-1}\, d_2 \tau \\ \alpha^A \chi_\alpha^{-1}\, d_2 \tau \\ \tau \end{bmatrix} - \begin{bmatrix} -\beta^A c \chi_\beta^{-1}\, d_2 \tau \\ \beta^A \chi_\beta^{-1}\, d_2 \tau \\ \tau \end{bmatrix} \\ &\qquad\qquad = \begin{bmatrix} \alpha^A - \beta^A - \beta \cdot \alpha c a^A & 0 \\ \beta \cdot \alpha a^A & \alpha^A - \beta^A \\ \beta \cdot \alpha a_2 & 0 \end{bmatrix} \begin{bmatrix} \kappa_0 \\ \kappa_1 \end{bmatrix}. \end{aligned}$$

Nonsingularity of the monad implies that κ_0 and κ_1 are uniquely determined, and the system is equivalent to the pair of equations

$$\begin{gathered} -\beta \cdot \alpha a^A c\kappa_1 + (\alpha^A - \beta^A)\kappa_1 = \alpha^A \chi_\alpha^{-1}\, d_2\tau - \beta^A \chi_\beta^{-1}\, d_2\tau \\ a_2 c\kappa_1 = 0. \end{gathered} \tag{4.5}$$

Since $\alpha^A\chi_\alpha^{-1} - \beta^A\chi_\beta^{-1} = \chi_\alpha^{-1}(\alpha^A - \beta^A - \beta\cdot\alpha a^A c)\chi_\beta^{-1}$, the first of these equations is equivalent to

$$(\alpha^A - \beta^A)(\kappa_1 - \chi_\alpha^{-1}\chi_\beta^{-1}\, d_2\tau) - \beta\cdot\alpha(a^A c\kappa_1 - \chi_\alpha^{-1} a^A c \chi_\beta^{-1}\, d_2\tau) = 0. \tag{4.6}$$

Since α, β are linearly independent, (4.6) is equivalent to the pair of equations obtained by contracting with α and β in turn, namely,

$$\beta\cdot\alpha\chi_\alpha[\kappa_1 - \chi_\alpha^{-1}\chi_\beta^{-1}\, d_2\tau] = 0 = \beta\cdot\alpha\chi_\beta[\kappa_1 - \chi_\beta^{-1}\chi_\alpha^{-1}\, d_2\tau],$$

and these in turn are equivalent to the condition that $\chi_\alpha^{-1}, \chi_\beta^{-1}$ should commute on the image of d_2. Thus the trivializations agree if and only if $\chi_\alpha^{-1}\chi_\beta^{-1}\, d_2 = \chi_\beta^{-1}\chi_\alpha^{-1}\, d_2$ and $a_2 c\chi_\alpha^{-1}\chi_\beta^{-1}\, d_2 = 0$.

Suppose now that E is trivial *on a neighborhood of* L_∞. Since $\chi_\alpha = 1 - \alpha\cdot ac$ with $\alpha\cdot ac$ nilpotent, $\chi_\alpha^{-1} = 1 + \alpha\cdot ac + (\alpha\cdot ac)^2 + \cdots + (\alpha\cdot ac)^{k_1}$ and the condition that $a_2 c\chi_\alpha^{-1}\chi_\beta^{-1}\, d_2 = 0$ for all $\alpha, \beta \in \mathbf{C}^2$ is equivalent to the vanishing of $a_2 c a^{(A_1} c \cdots a^{A_m)} c a^{(B_1} c \cdots a^{B_n)} c d_2$ for all $m, n = 0, 1, \ldots$, where the parentheses indicate symmetrization. Using (4.4) it follows easily that this is equivalent to the condition

$$a_2 c d_2 = 0 = a_2 c a^{A_1} c \cdots a^{A_m}\, c d_2, \quad m = 1, 2, \ldots. \tag{4.7}$$

Thus E is trivial on a neighborhood of L_∞ if and only if both (4.4) and (4.7) hold, subject to the constraints that the monad be nonsingular at each point of the exceptional line.

At this point the analysis is considerably simplified if it is assumed that $H^0(L_0, E(-1)) = 0$ from now on. By Lemma 2.2 (a) of [**11**], this implies that $\pi_* E$ is locally free and therefore a trivial bundle on $\mathbf{P}_2$ since it is trivial off x_0. Thus $H^0(\widetilde{\mathbf{P}}_2, E) \to H^0(L_\infty, E) = R$ is an isomorphism. The exact sequence $0 \to \mathcal{O}(-1,0) \to \mathcal{O}(-1,-1) \to \mathcal{O}_{L_0}(-1) \to 0$ induces the exact sequence $0 \to H^0(L_0, E(-1)) \to K_1 \to K_0 \to H^1(L_0, E(-1)) \to 0$, and the map $K_1 \to K_0$ is easily seen to be the map c. Moreover, $H^0(\widetilde{\mathbf{P}}_2, E) = \ker d_2 : R \to K_1$ so the assumption that $H^0(L_0, E(-1)) = 0$ implies that $c : K_1 \to K_0$ is injective and that $d_2 \equiv 0$. Hence the conditions on the monad are now simply that (a^A, a_2, c) should satisfy

$$a_A c a^A = 0, \quad c : K_1 \to K_0 \text{ is injective.} \tag{4.8}$$

Thus $ca^A \in \mathrm{End}\, K_0$ (and also $a^A c \in \mathrm{End}\, K_1$) for $= 0, 1$ are commuting nilpotent endomorphisms. The nonsingularity criterion is automatically satisfied at every point of $\widetilde{\mathbf{P}}_2 \setminus L_0$, and at points of L_0 it reduces to

$$\begin{gathered}(a^A, a_2) : K_0 \to K_1^A \oplus R \quad \text{is injective;} \\ w_A a^A : K_0 \to K_1 \quad \text{is surjective } \forall\, w_A \in \mathbf{C}^2 \setminus \{0\}.\end{gathered} \tag{4.9}$$

The former of these conditions is straightforward to verify, and the latter follows by using the fact that since E has r sections which are independent at each point off the exceptional line, $H^1(L, E(-1))$ must vanish for any line L which is a fiber of the canonical projection $p : \widetilde{\mathbf{P}}_2 \to L_0$.

A pair of monads M, M' of this form defined by (a^A, c, a_2), (a'^A, c', a'_2) have isomorphic cohomologies E, E' if and only if there are automorphisms $g_0 \in \mathrm{Aut}\,(K_0)$, $g_1 \in \mathrm{Aut}\,(K_1)$ and $g_R \in \mathrm{Aut}\,(R)$ satisfying

$$a'^A = g_1 a^A g_0^{-1}, \quad c' = g_0 c g_1^{-1}, \quad a'_2 = g_R a_2 g_0^{-1} \tag{4.10}$$

and if trivializations are fixed on L_∞, g_R must be the identity.

If (4.9) is satisfied, a^A defines the data for a bundle A on $\mathbf{P}_1$ with $\mathrm{rank}\,(A) = -a$, $c_1(A) = -k_0$ and $H^0(\mathbf{P}_1, A) = 0$, given by the exact sequence $0 \to A \to K_0(-1) \overset{a^A w_A}{\to} K_1 \to 0$, where w_A are now homogeneous coordinates on $\mathbf{P}_1$. An isomorphic bundle is obtained under the first transformation of (4.10).

A quick diagram chase reveals that the vanishing of $a_A c a^A$ implies that the map $c : H^1(\mathbf{P}_1, A) \to H^1(\mathbf{P}_1, A(-1))$ is induced by a map $A \to A(-1)$, also denoted by c for convenience. The homomorphism c is injective on H^1 and the second transformation of (4.10) is the appropriate rule determined by the first transformation. Another diagram chase shows that the kernel of $A \to A(-1)$ is $U(-1)$, where

$U \subset K_0$ is the joint kernel of $a^A : K_0 \to K_1^A$, so from the first statement of (4.9), it follows that there is an exact sequence

$$0 \longrightarrow A \xrightarrow[c]{a_2} \begin{matrix} R(-1) \\ \oplus \\ A(-1) \end{matrix} \longrightarrow Q \longrightarrow 0 \tag{4.11}$$

where $a_2 \in H^0(\mathbf{P}_1, Hom(A, R(-1))) = \mathrm{Hom}(K_0, R)$ is as above and Q is the r-bundle on $\mathbf{P}_1$ defined by the sequence.

It is straightforward to identify the bundles A and Q in terms of E: if $p : \widetilde{\mathbf{P}}_2 \to \mathbf{P}_1$ is the projection which identifies $\widetilde{\mathbf{P}}_2$ as the projectivization of $\mathcal{O} \oplus \mathcal{O}(-1)$ over $\mathbf{P}_1$, $p^*O(1) = \mathcal{O}(1,1)$ and the fibers of p are the proper transforms of the lines in $\mathbf{P}_2$ passing through x_0. For any such fiber $L = p^{-1}(y)$, $\mathcal{O}(1,0)|_L = \mathcal{O}_L(1)$ and $\mathcal{O}(0,1)|_L = \mathcal{O}_L(-1)$. The second statement of (4.9) is equivalent to $H^1(L, E(-1,0)) = 0$ for every fiber L of p, so $p_*E(-1,0)$ is a bundle of rank $-a$ on $\mathbf{P}_1$; this is A. The vanishing of $H^1(L, E(-1,0))$ implies that of $H^1(L, E(-1,-1))$ so $p_*E(-1,-1)$ is a bundle of rank $r-a$ on $\mathbf{P}_1$. Taking direct images of the sequence

$$0 \longrightarrow E(-2,-1) \xrightarrow{z^2} E(-1,-1) \longrightarrow E(-1,-1)|_{L_\infty} \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow A(-1) \longrightarrow p_*E(-1,-1) \longrightarrow R(-1) \longrightarrow 0,$$

here identifying $E|_{L_\infty}$ with the trivial r-bundle R. The obstruction to splitting this sequence is a class in $H^1(\mathbf{P}_1, Hom(R, A)) = \mathrm{Hom}(R, K_1)$, and it is easily checked that, up to a sign, this is just the homomorphism d_2 above; thus, the sequence splits uniquely, and direct images of the sequence

$$0 \longrightarrow E(-1,0) \xrightarrow{\lambda} E(-1,-1) \longrightarrow E(-1,-1)|_{L_0} \longrightarrow 0$$

(where $\lambda \in H^0(\widetilde{\mathbf{P}}_2, \mathcal{O}(0,-1))$ defines L_0) gives the sequence (4.11), identifying Q with $E|_{L_0}(-1)$.

Conversely, the bundle E on $\widetilde{\mathbf{P}}_2$ can be recovered from the sequence

$$0 \longrightarrow (p^*A)(0,1) \xrightarrow[\lambda 1 - z^2 c]{a_2 z^2} \begin{matrix} R \\ \oplus \\ p^*A \end{matrix} \longrightarrow E \longrightarrow 0 \tag{4.12}$$

where the map $R \to E$ is given by the extension of the trivialization on L_∞ to $\widetilde{\mathbf{P}}_2$ and the map $p^*A \to E$ is z^2 times the canonical map $p^*A = p^*p_*E(-1,0) \to E(-1,0)$. Thus r-bundles E on $\widetilde{\mathbf{P}}_2$ trivialized on a neighborhood of L_∞ and satisfying $H^0(L_0, E(-1)) = 0$ are classified by (isomorphism classes of) short exact sequences on $\mathbf{P}_1$ of the form (4.11), where $Q(1) = E|_{L_0}$. To summarize:

Proposition 4.13. *Let $\widetilde{\mathbf{P}}_2$ be the blowup of $\mathbf{P}_2$ at $x_0 \in \mathbf{P}_2 \setminus L_\infty$, and let L_0 be the exceptional line. There is a one-to-one correspondence between isomorphism classes of holomorphic r-bundles E on $\widetilde{\mathbf{P}}_2$ with $c_1(E) = a\mathbf{e}$, $c_2(E) = k\mathbf{h}^2$ which are trivialized in a neighborhood of L_∞ and satisfy $H^0(L_0, E) = 0$ and isomorphism classes of triples (A, c, a_2) where A is a holomorphic vector bundle on L_0 of rank $-a$ and first Chern class $-(k + (1/2)a(a-1))$, $c \in \Gamma(L_0, Hom(A, A(-1))$ and $a_2 \in \Gamma(L_0, Hom(A, \mathcal{O}^r(-1)))$ with these maps satisfying $c : H^1(L_0, A) \to H^1(L_0, A(-1))$ is injective and* $\operatorname{Ker} c(x) \cap \operatorname{Ker} a_2(x) = 0$ *at each $x \in L_0$.*

Again there are several directions worthy of further exploration in the context of this example. Although the characterization of bundles on a neighborhood of a blown-up point in terms of bundles on $\mathbf{P}_1$ equipped with certain homomorphisms provides an easy way to construct the former such bundles with prescribed numerical invariants, a clear picture of the spaces of these bundles nevertheless remains somewhat elusive. The same remains true for individual strata of the moduli space determined by the splitting types of the bundles on L_0. For bundles of rank 2, there are some simplifications which enable further analysis; in this case at least there is some evidence to believe that an effective and useful description will emerge.

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