ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

BIFURCATIONS OF BOUNDED SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

YU SHU-XIANG

ABSTRACT. In this paper, using the notion of an isolated invariant set and an isolating block, an existence criterion of bifurcation points of nonstationary bounded solutions for planar systems depending on a parameter is given.

1. Introduction. Consider the one-parameter family of differential systems in R^n

$$\frac{dx}{dt} = F(x, \lambda). \tag{1.1}$$

Let $F : R^n \times R \to R^n$ be continuous and assume that, for each $\lambda \in R$, the solution of an initial value problem is unique.

Each zero of F is called a stationary solution of (1.1). Clearly, if (x_0, λ_0) satisfies $F(x_0, \lambda_0) = 0$, then x_0 is a critical point of the $\lambda = \lambda_0$ system (1.1). In this paper we shall investigate bifurcation points of nonstationary bounded solutions of (1.1), where a bounded solution means that it is bounded both in the forward and backward time directions.

Definition 1.1 [**3**]. A point $(x_0, \lambda_0) \in R^n \times R$ is said to be a bifurcation point of nonstationary bounded solutions of the system (1.1) if for any open neighborhood U of (x_0, λ_0) there is a nonstationary solution of (1.1) included in U.

It follows directly from Definition 1.1 that if (x_0, λ_0) is a bifurcation point, then x_0 has to be a critical point of the $\lambda = \lambda_0$ system (1.1).

AMS *Mathematics Subject Classification.* 34C.
Key words and phrases. Bifurcations, bounded solutions, connecting orbits, isolating blocks.
Research supported by the National Science Foundation of China.
Received by the editors on May 2, 2001, and in revised form on February 18, 2002.

Therefore, without loss of generality we assume that $(0,0) \in R^n \times R$ is a stationary solution of (1.1), i.e., $F(0,0) = 0$, and $x_0 = 0$ is a critical point of the $\lambda = 0$ system (1.1). A problem is to find the conditions which guarantee that the origin $(0,0) \in R^n \times R$ is a bifurcation point of nonstationary bounded solutions of (1.1). Recently, Izydorek and Rybicki in [**3**] gave some sufficient conditions which guarantee that a bifurcation point of stationary solutions of (1.1) is also a bifurcation point of nonstationary bounded solutions. In the present paper, using the concept of an isolated invariant set and a generalization of a result about the existence of connecting orbits in [**4**], we shall give an existence criterion of bifurcation points of nonstationary bounded solutions for planar systems depending on a parameter.

2. The existence of connecting orbits. Consider the differential system defined in the plane

$$\begin{aligned} \frac{dx}{dt} &= X(x,y), \\ \frac{dy}{dt} &= Y(x,y). \end{aligned} \tag{2.1}$$

Let $V = (X, Y)$ be a C^1-vector field and $f(p,t)$ be a flow generated by V. Let B be the closure of a bounded, connected open set in R^2 with the boundary ∂B consisting of n mutually disjoint components $L_1, \dots, L_n$, each of which is a simple closed curve. Denote by L_1 the external boundary. We define three subsets b^+, b^-, τ as follows:

$$\begin{aligned} b^+ &= \{p \in \partial B \mid \exists\, \varepsilon > 0 \text{ with } f(p, (-\varepsilon, 0)) \cap B = \varnothing\}, \\ b^- &= \{p \in \partial B \mid \exists\, \varepsilon > 0 \text{ with } f(p, (0, \varepsilon)) \cap B = \varnothing\}, \\ \tau &= \{p \in \partial B \mid V \text{ is tangent to } \partial B \text{ at } p\}. \end{aligned}$$

Definition 2.1 [**2**]. If $b^+ \cap b^- = \tau$ and $b^+ \cup b^- = \partial B$, then B is called an isolating block for the flow defined by (2.1).

Definition 2.2. If a simple closed curve C is the union of alternating nonclosed whole trajectories and critical points, and it is contained in the ω(or α)-limit set of some trajectory, then we say that C is a singular closed trajectory.

In this paper we shall prove the following theorem which is a generalization of Theorem 1 in [**4**].

Theorem 2.1. *If the system* (2.1) *admits an isolating block B such that the following two conditions are satisfied in B*:

(i) *there are precisely n critical points $O_1, \dots, O_n$, $n \geq 2$, and at least one of them, say O_1, is a repeller*;

(ii) *there are no closed trajectories and singular closed trajectories*;

then there must be a trajectory in B running from O_1 to another critical point.

Proof of Theorem 2.1. The proof is completely analogous to the proof of Theorem 1 in [**4**]. Consider any trajectory $\gamma(p)$ originating from O_1. As $t \to +\infty$, $\gamma(p)$ may tend to a critical point, or approach to infinity, or tend to a set containing ordinary points. In the third case $\gamma(p)$ cannot be a separatrix, see [**4**]. Let G be the region of negative attraction of the critical point O_1, i.e., $G = \{p \in R^2 \mid \lim_{t\to-\infty} f(p,t) = O_1\}$. We distinguish two cases:

(I) Each of the trajectories originating from O_1 is not a separatrix.

(II) There is at least one trajectory originating from O_1 being a separatrix.

The first case can be excluded in a similar way as shown in the proof of Theorem 1 in [**4**].

Consider now the second case. Since O_1 is a repeller, we know that there is in a small neighborhood of O_1 a simple closed curve transversal to the flow. Let ρ be such a curve. Suppose that there are no trajectories joining O_1 and O_k, $k = 2, \dots, n$. Then, using the same argument used in [**4**], it can be proved that every trajectory $\gamma(p)$, where $p \in \rho$, must meet ∂B for increasing time and ρ is mapped topologically onto ∂B by trajectories. Thus B is filled by these trajectories originating from O_1 together with O_1. But this contradicts the fact that the set $\{O_2, \dots, O_n\} \subset B$. Hence Theorem 2.1 is proved.

3. The existence of bifurcation points. Consider now the differential system depending on a parameter

$$\begin{aligned}\frac{dx}{dt} &= X(x,y,\lambda),\\ \frac{dy}{dt} &= Y(x,y,\lambda).\end{aligned}\tag{3.1}$$

Suppose the system (3.1) is defined in the region $\Omega = R^2 \times R$ and that $X, Y \in C^1$ in x, y and λ in Ω.

Definition 3.1. A point $(x_0, y_0, \lambda_0) \in \Omega$ is called a bifurcation point of stationary solutions of (3.1) if (x_0, y_0) is an isolated critical point of (3.1) for $\lambda = \lambda_0$ and there are at least two branches of stationary solutions of (3.1) emanating from it in the halfspace $\lambda > \lambda_0$ or in the halfspace $\lambda < \lambda_0$.

Remark 3.1. By Definition 3.1 the point $(x, y, \lambda) = (0, 0, 0)$ is a bifurcation point of stationary solutions of (3.1) means that the origin $(x, y) = (0, 0) = O$ is an isolated critical point of (3.1) for $\lambda = 0$ and there is a bounded connected region G in the xy-plane containing the origin in its interior and a value $\lambda_1 > 0$ such that for $0 < \lambda \leq \lambda_1$, G contains precisely n critical points $P_i(\lambda)$, $i = 1, \dots, n$, $n \geq 2$, and these critical points move continuously with λ finally coalescing at $\lambda = 0$, i.e., $P_1(0) = \cdots = P_n(0) = (0, 0) = O$.

There is an example, e.g., [**3**, p. 268], which shows that a bifurcation point of stationary solutions is not necessarily a bifurcation point of nonstationary bounded solutions. In the present paper we shall give the conditions such that a bifurcation point of stationary solutions of (3.1) is also a bifurcation point of nonstationary bounded solutions, see Theorem 3.1 below.

We shall prove the following theorem.

Theorem 3.1. *Suppose that the system* (3.1) *satisfies the following two conditions*:

(i) *the point* $(x, y, \lambda) = (0, 0, 0)$ *is a bifurcation point of stationary solutions*;

(ii) *at least one of the points* $P_i(\lambda)$, $\lambda = 1, \dots, n$, *is a repeller*;

then the point $(x, y, \lambda) = (0, 0, 0)$ *is a bifurcation point of nonstationary bounded solutions of* (3.1).

Proof of Theorem 3.1. The following proof proceeds by reduction to absurdity. Suppose that the point $(x, y, \lambda) = (0, 0, 0)$ is not a bifurcation point of nonstationary bounded solutions of (3.1). Then by Definition 1.1 we know that there is an open neighborhood U of the point $(0, 0, 0)$ such that there are no nonstationary bounded solutions of (3.1) included in U. This implies that there is an open neighborhood $G_1(\subset G)$ of the origin $O = (0, 0)$ and a value $\lambda_2(\leq \lambda_1)$ such that $G_1 \times [0, \lambda_2] \subset U$, hence for $0 \leq \lambda \leq \lambda_2$ there are no nonstationary bounded trajectories of (3.1) in G_1. In particular, for $0 \leq \lambda \leq \lambda_2$ there are no closed trajectories and singular closed trajectories of (3.1) in G_1. Since the origin $O = (0, 0)$ is an isolated critical point of the $\lambda = 0$ system (3.1), there is an open neighborhood $G_2(\subset G_1)$ of the origin O such that O is its unique critical point in $\overline{G_2}$, where $\overline{G_2}$ denotes the closure of G_2. By $\overline{G_2} \subset G_1$, it follows that the orbit $\gamma(p)$ of the $\lambda = 0$ system (3.1) originating from each point $p \in \partial G_2$, where ∂G_2 denotes the boundary of G_2, cannot be contained completely in $\overline{G_2}$. Therefore, $\overline{G_2}$ is an isolating neighborhood of the $\lambda = 0$ system (3.1) [**1**, pp. 3–4]. Since $\overline{G_2}$ contains no nonstationary bounded orbits of the $\lambda = 0$ system (3.1), the critical point O is the maximal invariant set contained in $\overline{G_2}$. Moreover, there is a value $\lambda_3(\leq \lambda_2)$ such that $\overline{G_2}$ is also an isolating neighborhood of (3.1) for $0 \leq \lambda \leq \lambda_3$ [**1**, p. 4]. It is easy to see that $\overline{G_2} \times [0, \lambda_3] \subset G_1 \times [0, \lambda_2] \subset U$. By the condition (i) of Theorem 3.1 and Remark 3.1, we know that there is a value $\lambda_4(\leq \lambda_3)$ such that for $0 < \lambda \leq \lambda_4$, n critical points $P_i(\lambda)$, $i = 1, \dots, n$, lie in the interior of $\overline{G_2}$. Let S_λ denote the maximal invariant set of (3.1) for $0 < \lambda \leq \lambda_4$ in $\overline{G_2}$. Clearly, we have $P_i(\lambda) \in S_\lambda$, $i = 1, \dots, n$. We now can prove that, for any $\lambda \in (0, \lambda_4]$ the system (3.1) has a nonstationary bounded orbit in $\overline{G_2}$. In fact, by [**2**, p. 53], we know that for the isolated invariant set S_λ and the isolating neighborhood $\overline{G_2}$, one can construct an isolating block B_λ for S_λ which lies in $\overline{G_2}$. Obviously, $B_\lambda \subset \overline{G_2} \subset G_1$. As stated above, for $0 \leq \lambda \leq \lambda_2$, there are no closed trajectories and singular closed trajectories of (3.1) in G_1. Thus, for $0 \leq \lambda \leq \lambda_4$, there are no closed trajectories and singular closed trajectories of (3.1) in B_λ. By $S_\lambda \subset B_\lambda$ together with the condition (ii) of Theorem 3.1, it follows that B_λ contains precisely n

critical points $P_i(\lambda)$, $i = 1, \dots, n$, $n \geq 2$, and there is at least one, say $P_1(\lambda)$, that is a repeller. Therefore, Theorem 2.1 implies that there must be a trajectory T_λ in B_λ running from $P_1(\lambda)$ to another critical point. T_λ is just a nonstationary bounded orbit in $\overline{G_2}$. Clearly $T_\lambda \subset U$. This is a contradiction. Hence, Theorem 3.1 is proved.

Remark 3.2. If we replace assuming that there is at least one repeller in Theorem 3.1 by assuming that there is at least one attractor, then the conclusion of Theorem 3.1 also holds.

In fact, if we make a change $t \to -t$ in the system (3.1), then we get a new system $(3.1)'$ which satisfies the conditions of Theorem 3.1. It follows that the point $(x, y, \lambda) = (0, 0, 0)$ is a bifurcation point of nonstationary bounded solutions of $(3.1)'$. But the new system $(3.1)'$ and the old system (3.1) have the same bifurcation points of nonstationary bounded solutions. Hence the conclusion of Remark 3.2 follows.

Remark 3.3. In the existence theorem of [**3**] it is required that the function $F(x, \lambda)$ is analytic and the critical points $P_i(\lambda)$, $i = 1, \dots, n$, are nondegenerate, i.e., where the linearized system has a nonzero Jacobian. However, Theorem 3.1 does not have this restriction in the present paper.

Acknowledgments. The author sincerely thanks the referee for very valuable suggestions.

REFERENCES

1. C. Conley, *Isolated invariant sets and the Morse index*, Conf. Board Math. Sci., vol. 38, Amer. Math. Soc., Providence, 1978.

2. C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc. **158** (1971), 35–61.

3. M. Izydorek and S. Rybicki, *Bifurcations of bounded solutions of* 1*-parameter ODE's*, J. Differential Equations **130** (1996), 267–276.

4. Yu Shu-Xiang, *The existence of trajectories joining critical points*, J. Differential Equations **66** (1987), 230–242.

Institute of Mathematics, Academia Sinica, 100080 Beijing, China
E-mail address: ysx@math03.math.ac.cn