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ON THE PRESERVATION OF DIRECTION CONVEXITY UNDER DIFFERENTIATION AND INTEGRATION

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ABSTRACT. For functions which are convex in one direction, we investigate to what extent this property is preserved under differentiation and integration.

1. Introduction. A domain $M \subset \mathbf{C}$ is said to be convex in the direction $e^{i\varphi}$ if for every $a \in \mathbf{C}$ the set

$$M \cap \left\{a + te^{i\varphi} : t \in \mathbf{R}\right\}$$

is either connected or empty. We denote by $C(\varphi)$ the family of univalent analytic functions f in the unit disk $\mathbf{D}$ with the property that $f(0) = 0$ and $f(\mathbf{D})$ is convex in the direction $e^{i\varphi}$. One of the interesting features about functions that are convex in one direction is that it is not in general so that $f \in C(\varphi)$ implies that $f(rz) \in C(\varphi)$, for $r < 1$. It was conjectured by Goodman and Saff [**2**], and later proved by Ruscheweyh and Salinas [**9**], that for $0 < r \leq \sqrt{2} - 1$ we have that $f \in C(\varphi)$ implies $f(rz) \in C(\varphi)$, but for $\sqrt{2} - 1 < r < 1$ this is not necessarily the case. In solving the Goodman-Saff conjecture, Ruscheweyh and Salinas introduced the class DCP, direction convexity preserving functions [**9**].

Definition 1.1. A function g, analytic in $\mathbf{D}$, is said to be direction convexity preserving, DCP, if for every $\varphi \in \mathbf{R}$, and every $f \in C(\varphi)$ we have $g * f \in C(\varphi)$. ($*$ denotes the Hadamard product.)

The problem of finding the largest r for which $f \in C(\varphi)$ implies $f(rz) \in C(\varphi)$ can then be formulated as finding the largest r for which the geometrical series $1/(1 - rz)$ is in DCP. Since we have $z/(1 - z)^2 * f(z) = zf'(z)$ and $\log(1 - z) * f(z) = \int_0^z (f(\zeta)/\zeta)d\zeta$, it is clear that if we can find the DCP-radius of the Koebe function

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and the logarithm, we will get results about direction convexity for the derivative and the integral of a given function in $C(\varphi)$. To check whether a function belongs to DCP we shall use a criterion developed in [**9**], but to state this criterion we need a definition.

Definition 1.2. Let u be a real, continuous, 2π-periodic function. The function u is said to be periodically monotone if there exist numbers $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that u increases on (θ_1, θ_2) and decreases on $(\theta_2, \theta_1 + 2\pi)$.

The following result was established in [**9**].

Theorem 1.1. (Ruscheweyh and Salinas). *Let g be nonconstant and analytic in $\mathbf{D}$, continuous on $\overline{\mathbf{D}}$ with $u(\theta) = Re\, g(e^{i\theta})$ three times continuously differentiable. Then $g \in DCP$ if and only if u is periodically monotone and satisfies*

$$u''(\theta)^2 - u'(\theta)u'''(\theta) \geq 0, \quad \theta \in \mathbf{R}. \tag{1.1}$$

2. The main results. In [**9**] it was also shown that $u_r(\theta) = \operatorname{Re} 1/(1 - re^{i\theta})$ satisfies the condition (1.1) for $0 < r \leq \sqrt{2} - 1$, and for no larger value of r, which settled the Goodman-Saff conjecture. We shall prove the following result.

Theorem 2.1. *Let $f \in C(\varphi)$. Then $zf'(rz) \in C(\varphi)$ for $0 < r \leq 2 - \sqrt{3} = 0.2679\ldots$, and the number $2 - \sqrt{3}$ is best possible.*

Proof. The result follows immediately from Lemma 3.1. □

There is one interesting observation to be made in connection with this result. If we denote by C and S^* the classes of convex and starlike univalent functions, respectively, it is well known that $f \in C$ if and only if $zf'(rz) \in S^*$ (Alexander's theorem [**1**, Theorem 2.12]). The radius of convexity in the whole class of normalized univalent functions is known to be $2 - \sqrt{3}$, and since this is sharp for the Koebe function the same number is also the radius of convexity in S^*. This was proved by Nevanlinna in 1919 [**6**]. These two results together give that if $f \in C$ then $zf'(rz) \in C$ for $0 < r \leq 2 - \sqrt{3}$, and the number $2 - \sqrt{3}$ is best possible. Hence, we have the same number occurring as both radius of convexity and radius of direction convexity for the derivative of a function that is convex, respectively convex in one direction. However, if we turn to the integral the situation becomes different. It is well known that if $f \in C$ then $\int_0^z (f(\zeta)/\zeta)d\zeta$ is also in C, but the same does not hold for $C(\varphi)$. In [**5**], see also [**8**], it is established that the function $\log(1 - rz)$ belongs to DCP for $0 < r \leq (4\sqrt{2} - \sqrt{5})/(3\sqrt{3})$ and for no larger value of r. This gives the following result which we include here for completeness.

Theorem 2.2 (Eva-Maria Nash). *Let $f \in C(\varphi)$. Then*

$$\int_0^{rz} (f(\zeta)/\zeta)\, d\zeta \in C(\varphi) \ \text{for}\ 0 < r \leq (4\sqrt{2} - \sqrt{5})/(3\sqrt{3}) = 0.6583\ldots,$$

and the number $(4\sqrt{2} - \sqrt{5})/(3\sqrt{3})$ is best possible.

A well-known class of univalent functions is the class of close-to-convex functions which we shall denote by K. Geometrically close-to-convexity of a function f means that the domain $f(|z| \leq r < 1)$ has the property that there are no parts of its boundary curve where the tangent turns backward through an angle greater than or equal to π. An equivalent characterization, due to Lewandowski [**3, 4**], is that the domain is linearly accessible, meaning that every point on the boundary can be reached from the outside by nonintersecting straight lines. It is clear that domains that are convex in one direction also are linearly accessible. Therefore the following result fits into the context of this paper.

Theorem 2.3. *Let $f \in K$. Then $zf'(z) \in C$ for $0 < r \leq 5 - \sqrt{24} = 0.1010\ldots$, and the number $5 - \sqrt{24}$ is best possible.*

The proof of Theorem 2.3 is in Section 4.

Corollary 2.4. *Let $f \in C(\varphi)$. Then $zf'(z) \in C$ for $0 < r \leq 5 - \sqrt{24}$, and the number $5 - \sqrt{24}$ is best possible.*

Proof. That the radius in this case is at least $5-\sqrt{24}$ follows from Theorem 2.3 and the aforementioned relation between K and $C(\varphi)$, and the sharpness follows from Lemma 3.1, part (2). $\square$

3. The Koebe function. For the rotated Koebe function

$$k_{\alpha,r}(z)=e^{i\alpha}\frac{z}{(1-rz)^2}$$

we get

$$u_\alpha(\varphi)=\operatorname{Re}k_{\alpha,r}(e^{i\varphi})=\frac{\cos\alpha(-2r+(1+r^2)\cos\varphi)-\sin\alpha(1-r^2)\sin\varphi}{(1+r^2-2r\cos\varphi)^2}.$$

In order to find the DCP-radius of $k_{\alpha,r}$ we should check that Theorem 1.1 can be applied to the function $u_\alpha(\varphi)$, compute $u_\alpha''(\varphi)^2-u_\alpha'(\varphi)u_\alpha'''(\varphi)$ and find the largest $r=r(\alpha)$ for which this expression is positive. This we are not able to do in general due to the fact that the computations become too involved, so we shall only discuss two cases, $\alpha=0$ and $\alpha=\pi/2$. Even these two special cases will involve heavy computation, and we will to a large extent omit the details.

Lemma 3.1. *Let $r(\alpha)$ denote the largest $r<1$ for which the function $k_{\alpha,r}$ belongs to DCP. Then the following holds.*

(1) $r(0)=2-\sqrt{3}$.

(2) $r(\pi/2)=5-\sqrt{24}$.

Proof. *The case* $\alpha=0$. We easily verify that for $r\le 2-\sqrt{3}$ the function u_0 is periodically monotone, so Theorem 1.1 can be applied. When we compute $u_0''(\varphi)^2-u_0'(\varphi)u_0'''(\varphi)$ we end up with an expression which is positive if and only if a polynomial of degree six in $x=\cos\varphi$ is positive. This polynomial is

$$p_1(r,x)=a_6(r)x^6+a_5(r)x^5+a_4(r)x^4+a_3(r)x^3+a_2(r)x^2+a_1(r)x+a_0(r),$$

where

$$\begin{aligned}
a_6(r) &= 128r^4(1+r^2)^2\\
a_5(r) &= 32r^3(1-11r^2-11r^4+r^6)\\
a_4(r) &= -16r^2(1+r^2+r^6+r^8)\\
a_3(r) &= -8r(1-13r^2+12r^4+12r^6-13r^8+r^{10})\\
a_2(r) &= 16r^2(1-14r^2+34r^4-14r^6+r^8)\\
a_1(r) &= 8r(1-13r^2+20r^4+20r^6-13r^8+r^{10})\\
a_0(r) &= 1-18r^2+175r^4-380r^6+175r^8-18r^{10}+r^{12}.
\end{aligned}$$

We shall find the largest r for which $p_1(r,x) \geq 0$, $x \in [-1,1]$. Our strategy is to compare this polynomial with other polynomials in order to successively obtain polynomials of lower degree which are easier to analyze. We see that

$$\begin{aligned}
p_1(r,-1) &= (1+r)^8(1-4r+r^2)^2\\
p_1(r,1) &= (1+r)^8(1+4r+r^2)^2;
\end{aligned}$$

hence, $p_1(2-\sqrt{3},-1)=0$, so the DCP-radius is at most $2-\sqrt{3}$. Our aim is to prove that $p_1(r,x) \geq 0$ for all $r \in [0, 2-\sqrt{3}]$ and all $x \in [-1,1]$. Define

$$q_1(r,x) = p_1(r,1)\frac{(x+1)^2}{4}.$$

Then $q_1(r,-1)=0$, $q_1(r,1)=p_1(r,1)$ and $q_1(r,x) \geq 0$, $x \in [-1,1]$. This means that

$$p_2(r,x) = \frac{p_1(r,x)-q_1(r,x)}{1-x}$$

will be a polynomial of degree five in x. If we can show that $p_2(r,x) \geq 0$ for all $r \in [0,2-\sqrt{3}]$ and all $x \in [-1,1]$, we are done. The coefficients of p_2 are listed below.

$$\begin{aligned}
b_5(r) &= -128r^4(1+r^2)^2\\
b_4(r) &= -32r^3(1+4r-11r^2+8r^3-11r^4+4r^5+r^6)\\
b_3(r) &= 16r^2(1-2r-7r^2+22r^3-16r^4+22r^5-7r^6-2r^7+r^8)\\
b_2(r) &= 8r(1+2r-17r^2-14r^3+56r^4-32r^5+56r^6-14r^7\\
&\qquad -17r^8+2r^9+r^{10})\\
b_1(r) &= (1+32r-18r^2-512r^3+511r^4+1504r^5-2780r^6+1504r^7\\
&\qquad +511r^8-512r^9-18r^{10}+32r^{11}+r^{12})/4\\
b_0(r) &= (3-54r^2-32r^3+637r^4+288r^5-1940r^6+288r^7\\
&\qquad +637r^8-32r^9-54r^{10}+3r^{12})/4.
\end{aligned}$$

Here we see that

$$\begin{aligned}
p_2(r,-1) &= (1+r)^8(1-4r+r^2)^2/2\\
p_2(r,1) &= (1-r)^2(1+4r+r^2)(1+14r-8r^2-238r^3+206r^4\\
&\qquad\qquad -238r^5-8r^6+14r^7+r^8).
\end{aligned}$$

A numerical calculation, using Mathematica, of the roots of $p_2(r,1)$ shows that the smallest positive root is $r = 0.277819\cdots > 2-\sqrt{3}$. Now we define

$$q_2(r,x) = p_2(r,1)\frac{(x+1)^3}{8}.$$

Then $q_2(r,-1)=0$, $q_2(r,1)=p_2(r,1)$ and $q_2(r,x)\geq 0$, $x\in[-1,1]$ and $r\leq 2-\sqrt{3}$. This means that

$$p_3(r,x) = \frac{p_2(r,x)-q_2(r,x)}{1-x}$$

will be a polynomial of degree four in x, and again it will be enough to show that this is positive. The coefficients of p_3 are listed below.

$$\begin{aligned}
c_4(r) &= 128r^4(1+r^2)^2\\
c_3(r) &= 32r^3(1+8r-11r^2+16r^3-11r^4+8r^5+r^6)\\
c_2(r) &= (1+16r-114r^2+176r^3+2751r^4-4032r^5+3940r^6-4032r^7\\
&\qquad +2751r^8+176r^9-114r^{10}+16r^{11}+r^{12})/8\\
c_1(r) &= (1-50r^2+64r^3+767r^4-704r^5-156r^6-704r^7+767r^8\\
&\qquad +64r^9-50r^{10}+r^{12})/2\\
c_0(r) &= (5-16r-122r^2+272r^3+1467r^4-1024r^5-1676r^6-1024r^7\\
&\qquad +1467r^8+272r^9-122r^{10}-16r^{11}+5r^{12})/8.
\end{aligned}$$

Numerical calculations show that the smallest positive root of $c_0(r)$ is $r = 0.73053\cdots > 2-\sqrt{3}$, and all the other polynomials $c_i(r)$ are strictly positive for r between 0 and 1. Therefore we can conclude that $p_3(r,x) > 0$ for $x \in [0,1]$, and the r-values in question. It remains to investigate $x \in [-1,0)$.

Define

$$q_3(r,x) = p_3(r,1)\frac{(x+1)^4}{16}.$$

Then $q_3(r,-1) = 0$, $q_3(r,1) = p_3(r,1)$ and $q_3(r,x) \geq 0$, $x \in [-1,1]$ and $r \leq 2-\sqrt{3}$. This means that

$$p_4(r,x) = \frac{p_3(r,x) - q_3(r,x)}{1-x}$$

will be a polynomial of degree three in x, and we shall proceed to prove that this is greater or equal to zero for all $r \in [0, 2-\sqrt{3}]$ and all $x \in [-1,0]$. The coefficients of p_4 are as follows.

$$\begin{aligned}
d_3(r) = (&5 - 218r^2 + 480r^3 - 3013r^4 - 5344r^5 - 12492r^6 - 5344r^7 \\
&- 3013r^8 + 480r^9 - 218r^{10} + 5r^{12})/64 \\
d_2(r) = (&25 - 1090r^2 + 352r^3 + 1319r^4 - 4192r^5 - 29692r^6 - 4192r^7 \\
&+ 1319r^8 + 352r^9 - 1090r^{10} + 25r^{12})/64 \\
d_1(r) = (&47 - 128r - 1486r^2 + 1824r^3 + 10385r^4 - 4000r^5 - 37860r^6 \\
&- 4000r^7 + 10385r^8 + 1824r^9 - 1486r^{10} - 128r^{11} + 47r^{12})/64 \\
d_0(r) = (&35 - 128r - 758r^2 + 1696r^3 + 6557r^4 - 2848r^5 - 17300r^6 \\
&- 2848r^7 + 6557r^8 + 1696r^9 - 758r^{10} - 128r^{11} + 35r^{12})/64
\end{aligned}$$

The smallest positive root of $d_0(r)$ is seen to be $0.55922\cdots > 2-\sqrt{3}$. Therefore $p_4(r,0) > 0$ for the r-values in question. Hence, if we define the second degree polynomial $p_5(r,x)$ by

$$p_4(r,x) - p_4(r,0)(x+1)^2 = -64xp_5(r,x)$$

our result will be established if we can prove that $p_5(r,x) \geq 0$ for

$x \in [-1,0]$ and $r \le 2-\sqrt{3}$. The coefficients of $p_5(r,x)$ are

$$\begin{aligned} e_2(r) &= -5+218r^2-480r^3+3013r^4+5344r^4+12492r^6+5344r^7\\ &\quad+3013r^8-480r^9+218r^{10}-5r^{12}\\ e_1(r) &= 10-128r+332r^2+1344r^3+5238r^4+1344r^5+12392r^6\\ &\quad+1344r^7+5238r^8+1344r^9+332r^{10}-128r^{11}+10r^{12}\\ e_0(r) &= 23-128r-30r^2+1568r^3+2729r^4-1696r^5+3260r^6\\ &\quad-1696r^7+2729r^8+1568r^9-30r^{10}-128r^{11}+23r^{12}. \end{aligned}$$

We see that $p_5(r,-1) = 8(1+r)^8(1-4r+r^2)^2$ which is greater than or equal to zero for $r \le 2-\sqrt{3}$. The derivative with respect to x of $p_5(r,x)$ is

$$p_6(r,x) = e_1(r) + 2e_2(r)x.$$

We see that

$$\begin{aligned} p_6(r,-1) &= 4(1+r)^2(1-4r+r^2)U(r)\\ p_6(r,0) &= e_1(r), \end{aligned}$$

where

$$U(r) = 5-22r-40r^2+374r^3+262r^4+374r^5-40r^6-22r^7+5r^8.$$

Both $U(r)$ and $e_1(r)$ can be seen to have no real roots. Therefore we must have $p_5(r,x) \ge p_5(r,-1) \ge 0$ for $-1 \le x \le 0$, $0 < r \le 2-\sqrt{3}$.

The case $\alpha = \pi/2$. It is straightforward to verify that $u_{\pi/2}$ is periodically monotone, and we omit the details. The expression $u''_{\pi/2}(\varphi)^2 - u'_{\pi/2}(\varphi)u'''_{\pi/2}(\varphi)$ gives rise to a polynomial of degree five,

$$p_1(r,x) = a_5(r)x^5 + a_4(r)x^4 + a_3(r)x^3 + a_2(r)x^2 + a_1(r)x + a_0(r)$$

with

$$\begin{aligned} a_5(r) &= 32r^3(1+r^2)\\ a_4(r) &= 16r^2(1-7r^2+r^4)\\ a_3(r) &= 2r(4-12r^2-r^4+4r^6)\\ a_2(r) &= -16r^2(5-6r^2+5r^4)\\ a_1(r) &= -4r(1-29r^2-29r^4+r^6)\\ a_0(r) &= 1+20r^2-154r^4+20r^6+r^8. \end{aligned}$$

Here we get

$$p_1(r,-1) = (1+r)^6(1-10r+r^2)$$
$$p_1(r,1) = (1-r)^6(1+10r+r^2)$$

which shows that the DCP-radius is at most $5-\sqrt{24}$. From Theorem 2.3 and the relation between K and $C(\varphi)$ we know that for $f \in C(\varphi)$ and $r \le 5-\sqrt{24}$ we have $zf'(rz) \in C$ and therefore in particular $izf'(rz) \in C(\varphi)$. Hence, $k_{\pi/2,r} \in \mathrm{DCP}$ for $0 < r \le 5-\sqrt{24}$. $\square$

4. Proof of Theorem 2.3. Let $f \in K$ and set $F(z) = zf'(z)$. We shall find the largest $r = |z|$ such that

$$\operatorname{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} > 0 \tag{4.1}$$

with

$$\mathcal{T} = \left\{\frac{1+xz}{(1-z)^3},\ |x|=1,\ z \in \mathbf{D}\right\}$$

and

$$\mathcal{V} = \{f' \mid f \in K\}.$$

it is well known [**7**, Theorem 1.9] that

$$\mathcal{T} \subset \mathcal{V} \subset du(\mathcal{T}),$$

where $du(\mathcal{T})$ denotes the dual hull of $\mathcal{T}$. (Here we assume the additional normalization $f'(0) = 1$.) Using the duality principle [**7**, Corollary 1.1] we can therefore conclude that if (4.1) holds for $F(z) = z(1+xz)/(1-z)^3$, it holds for the whole class K. With $z = \mathrm{r}e^{i\theta}$ and $x = e^{i\varphi}$ we find, after some computation that (4.1) holds for this particular F if and only if

$$\begin{aligned} 1 + 16r^2 + r^4 + 8r(1+r^2)\cos\theta + 18r^2\cos\varphi + 2r^2\cos 2\theta \\ +6r(1+r^2)\cos(\varphi+\theta) + 6r^2\cos(\varphi+2\theta) \ge 0, \end{aligned} \tag{4.2}$$

$\theta, \varphi \in [0, 2\pi)$. With $r = 5-\sqrt{24}$ we have $1+r^2 = 10r$ and $1+2r^2+r^4 = 100r^2$, in which case (4.2), after cancelling the factor $2r^2$, turns into

$$57 + 40\cos\theta + 9\cos\varphi + \cos 2\theta + 30\cos(\varphi+\theta) + 3\cos(\varphi+2\theta) \ge 0. \tag{4.3}$$

Using calculus one can verify that, except for $\varphi = 0$ and $\theta = \pi$ where we have equality, strict inequality holds in (4.3). Hence, (4.2) holds for $r \le 5 - \sqrt{24}$. $\square$

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