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SPACE-LIKE EINSTEIN KÄHLER SUBMANIFOLDS IN AN INDEFINITE COMPLEX HYPERBOLIC SPACE

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ABSTRACT. The purpose of this paper is to study space-like Einstein Kähler submanifolds with restricted full immersions and parallel second fundamental forms in an indefinite complex hyperbolic space.

1. Introduction. The theory of semi-definite complex submanifolds of a semi-definite complex space form is one of the most interesting research subjects in differential geometry and it is studied by many geometers from the various points of view, see [**1**–**3, 10**–**12**] and [**14**], for instance.

As one of such studies, in their paper [**10**], Nakagawa and Takagi classified completely locally symmetric Kähler submanifolds of a complex projective space. In particular, it is seen that complex submanifolds whose second fundamental form are parallel of a complex projective space are all Einstein. Conversely, Einstein Kähler submanifolds of a complex space form do not satisfy necessarily the result that the second fundamental form is parallel, and it is seen in [**10**] that there exist many Einstein Kähler submanifolds of a complex projective space whose second fundamental form are not necessarily parallel. Furthermore, Romero [**13**] and Umehara [**15**] independently proved the indefinite version and they found that there exists a full holomorphic isomorphic immersion of an indefinite complex space form $M_s^n(c)$ into an indefinite complex space form $M_{s+t}^{n+p}(c')$.

On the other hand, Einstein Kähler submanifolds of a complex projective space whose second fundamental form are parallel were investigated by Nakagawa [**9**]. He proved the following

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Theorem A. *Let M be an $n(\geqq 2)$-dimensional Einstein Kähler submanifold immersed in a complex projective space $CP^{n+p}(c)$ of constant holomorphic sectional curvature c. If the immersion is full and if the second fundamental form of M is parallel, then the following hold*:

(1) *If $p < \frac{n}{2}$, then $p = 1$ and M is locally a complex quadric Q^n.*

(2) *If $p \geqq \frac{1}{2}n(n+1)$, then $p = \frac{1}{2}n(n+1)$ and M is locally $CP^n(\frac{c}{2})$.*

The purpose of this paper is to investigate the space-like version of Theorem A, namely to prove the following

Theorem. *Let M be an $n(\geqq 2)$-dimensional space-like Einstein Kähler submanifold of an indefinite complex space form $M_p^{n+p}(c)$ of constant holomorphic sectional curvature $c < 0$. If the immersion is full and if the second fundamental form of M is parallel, then the following hold*:

(1) *If $p < \frac{n}{2}$, then $p = 1$ and M is locally a complex quadric Q^n.*

(2) *If $p \geqq \frac{1}{2}n(n+1)$, then $p = \frac{1}{2}n(n+1)$ and M is locally $CH^n(\frac{c}{2})$.*

2. Indefinite Kähler manifolds. We begin by recalling basic formulas on indefinite Kähler manifolds. Let M be a complex $n(\geqq 2)$-dimensional connected semi-definite Kähler manifold equipped with the semi-definite Kähler metric tensor g and almost complex structure J. For the semi-definite Kähler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2s$, $0 \leqq s \leqq n$. In the case where s is contained in the range $0 < s < n$, the structure $\{g, J\}$ is said to be *indefinite Kähler structure* and M is called an *indefinite Kähler manifold*. In particular, in the case where $s = 0$ or n, it is said to be *Kähler structure*.

Let M' be a complex $(n+p)$-dimensional connected indefinite Kähler manifold of index $2p$, $n \geqq 2$, $p > 0$. Then we can choose a local field $\{E_A\} = \{E_1, \ldots, E_n, E_{n+1}, \ldots, E_{n+p}\}$ of unitary frames on a neighborhood of M'. This is a complex frame field on the neighborhood of M' which is orthonormal with respect to the indefinite Kähler metric g', that is, $g'(E_A, E_B) = \varepsilon_A \delta_{AB,}$ where

$$\varepsilon_A = 1 \text{ or } -1, \text{ according to whether } 1 \leqq A \leqq n \text{ or } n+1 \leqq A \leqq n+p.$$

Its dual field $\omega_0' = \{\omega_A\} = \{\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p}\}$ with respect to the unitary frame $\{E_A\}$ consists of complex-valued 1-forms of type (1,0) on M' such that $\omega_A(E_B) = \varepsilon_A \delta_{AB,}$ and $\omega_1, \dots, \omega_{n+p}, \bar{\omega}_1, \dots, \bar{\omega}_{n+p}$ are linearly independent, where $\bar{\omega}_A$ denotes the complex conjugate of $\omega_{A.}$ It is called the *canonical form* with respect to the unitary frame $\{E_A\}$. The indefinite Kähler metric g' of M' can be expressed as $g' = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_{A,}$ where the Latin capital indices A and B run over the range $1, \dots, n+p$. Associated with the frame field $\{E_A\}$, there exist complex-valued forms $\omega' = \{\omega_{AB}\}$ and $\Omega' = \{\Omega_{AB}\}$ the *connection form* and the *curvature form* on M', respectively. They satisfy the following structure equations of M'.

$$d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \tag{2.2}$$

$$\Omega_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \tag{2.3}$$

where $R'_{\bar{A}BC\bar{D}}$ denotes the components of the Riemannian curvature tensor R' of M'. The equations (2.1) and (2.2) means that the skew-symmetry of $\Omega_{AB,}$ which is equivalent to the symmetric condition

$$R'_{\bar{A}BC\bar{D}} = \bar{R}'_{\bar{B}AD\bar{C}.} \tag{2.4}$$

By the exterior derivative of (2.1) and (2.3), the first Bianchi formula

$$\sum_B \varepsilon_B \Omega_{AB} \wedge \omega_B = 0$$

is given, which implies the further symmetric relations

$$R'_{\bar{A}BC\bar{D}} = R'_{\bar{A}CB\bar{D}} = R'_{\bar{D}CB\bar{A}} = R'_{\bar{D}BC\bar{A}.} \tag{2.5}$$

Now, relative to the frame field chosen above, the Ricci tensor S' of M' can be expressed as follows:

$$S' = \sum_{A,B} \varepsilon_A \varepsilon_B (S'_{A\bar{B}} \omega_A \otimes \bar{\omega}_B + S'_{\bar{A}B} \bar{\omega}_A \otimes \omega_B), \tag{2.6}$$

where $S'_{A\bar{B}} = \sum_C \varepsilon_C R'_{\bar{C}CA\bar{B}} = S'_{\bar{B}A} = \bar{S'}_{\bar{A}B}$. The scalar curvature r' of M is also given by

$$r' = 2\sum_A \varepsilon_A S'_{A\bar{A}}. \tag{2.7}$$

The indefinite Kähler manifold M' is said to be *Einstein* if the Ricci tensor S' is given by

$$S'_{A\bar{B}} = \frac{r'}{2(n+p)}\varepsilon_A \delta_{AB}. \tag{2.8}$$

Next, the components $R'_{\bar{A}BC\bar{D}E}$ and $R'_{\bar{A}BC\bar{D}\bar{E}}$ relative to the frame field $\{E_A\}$ of the covariant derivative of the Riemannian curvature tensor R' are obtained by

(2.9)

$$\begin{aligned}\sum_E \varepsilon_E (R'_{\bar{A}BC\bar{D}E}\omega_E + R'_{\bar{A}BC\bar{D}\bar{E}}\bar{\omega}_E) &= dR'_{\bar{A}BC\bar{D}} \\ &- \sum_E \varepsilon_E (R'_{\bar{E}BC\bar{D}}\bar{\omega}_{EA} + R'_{\bar{A}EC\bar{D}}\omega_{EB} + R'_{\bar{A}BE\bar{D}}\omega_{EC} + R'_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}).\end{aligned}$$

The second Bianchi formula is given by

$$R'_{\bar{A}BC\bar{D}E} = R'_{\bar{A}BE\bar{D}C}. \tag{2.10}$$

Let M be an n-dimensional semi-definite Kähler manifold of index $2s$, $0 \leqq s \leqq n$, with almost complex structure J. A plane section P of the tangent space T_xM of M at any point x is said to be *nondegenerate*, provided that the restriction of $g_x|_{T_xM}$ to P is nondegenerate. It is easily seen that P is nondegenerate if and only if it has a basis $\{X, Y\}$ such that $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$. The plane P is said to be *holomorphic* if it has a basis $\{X, JX\}$ for the plane P. It is also trivial that the plane P is nondegenerate if and only if it contains a vector X with $g(X,X) \neq 0$. For the non-degenerate plane P spanned by X and Y, the sectional curvature $K(X,Y)$ of P is usually defined by

$$K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

It is well known that the sectional curvature of the non-degenerate plane P is independent of the choice of the basis $\{X, Y\}$ for the plane. So, it is denoted by $K(X,Y) = K(P)$. Moreover, the sectional curvature $H(P)$ of the non-degenerate holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P) = K(P) = K(X, JX) = H(X)$ for any nonzero vector X in P.

The indefinite Kähler manifold M is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for any nondegenerate holomorphic plane P and any point on M. Then M is called an *indefinite complex space form*, which is denoted by $M_s^n(c)$, provided that it is of constant holomorphic sectional curvature c, of complex dimension n and of index $2s$, $0 < s < n$. It is seen in Barros and Romero [**4**] that the standard models of indefinite complex space forms are the following three kinds: the indefinite complex projective space $CP_s^n(c)$, the indefinite complex Euclidean space C_s^n or the indefinite complex hyperbolic space $CH_s^n(c)$, according to whether $c > 0$, $c = 0$ or $c < 0$. For any integer s, $0 < s < n$, it is also seen by [**4**] that they are complete simply connected indefinite complex space forms of dimension n and of index $2s$.

The components $R'_{\bar{A}BC\bar{D}}$ of the Riemannian curvature tensor R' of the n-dimensional indefinite complex space form $M' = M_s^n(c)$ are given by

$$R'_{\bar{A}BC\bar{D}} = \frac{c}{2}\varepsilon_B\varepsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{2.11}$$

3. Space-like complex submanifolds. This section is concerned with space-like complex submanifolds of an indefinite Kähler manifold. First of all, the basic formulas for the theory of space-like complex submanifolds are prepared. Let (M', g') be an $(n+p)$-dimensional connected indefinite Kähler manifold of index $2p(> 0)$, and let M be an $n(\geqq 2)$-dimensional connected space-like complex submanifold of M'. Then M becomes the Kähler manifold endowed with the induced metric tensor g. We can choose a local field $\{E_A\} = \{E_i, E_x\} = \{E_1, \dots, E_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $E_1, \dots, E_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated.

$$A, B, \ldots = 1, \ldots, n, n+1, \ldots, n+p,$$
$$i, j, \ldots = 1, \ldots, n,$$
$$x, y, \ldots = n+1, \ \ldots \ , \ n+p.$$

With respect to the frame field $\{E_A\}$, let $\{\omega_A\} = \{\omega_i, \ \omega_x\}$ be its dual frame field. Then the indefinite Kähler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$, where $\varepsilon_i = 1$ and $\varepsilon_x = -1$. The canonical form $\{\omega_A\}$ and the connection form $\{\omega_{AB}\}$ with respect to the unitary frame field $\{E_A\}$ of the ambient space M' satisfy the structure equations

$$d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{AB} = 0, \tag{3.1}$$

$$d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \tag{3.2}$$

$$\Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \tag{3.3}$$

where $\{\Omega'_{AB}\}$, respectively $R'_{\bar{A}BC\bar{D}}$, denotes the curvature form, respectively the components of the indefinite Riemannian curvature tensor R', of M'.

Now restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.4}$$

and the induced Kähler metric tensor g of M is given by $g = 2\sum_i \varepsilon_i \omega_i \otimes \bar{\omega}_i$. Then $\{E_i\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_i\}$ is a canonical form with respect to $\{E_i\}$, which consists of complex valued 1-forms of type (1.0) on M. Moreover, $\omega_1, \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n$ are linearly independent. It follows from (3.4) and Cartan's lemma that the exterior derivatives of (3.4) give rise to

$$\omega_{xi} = \sum_j \varepsilon_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{3.5}$$

The quadratic form $\alpha = \sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', it follows that the structure equations for M are similarly given by

$$d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{3.6}$$

$$d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{3.7}$$

$$\Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l. \tag{3.8}$$

Moreover, the following relationships are obtained.

$$d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \tag{3.9}$$

$$\Omega_{xy} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{3.10}$$

where $\{\Omega_{xy}\}$ is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.5) and (3.8) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h^x_{jk} \bar{h}^x_{il}, \tag{3.11}$$

and by means of (3.5) and (3.10), we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j \varepsilon_j h^x_{kj} \bar{h}^y_{jl}. \tag{3.12}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_k \varepsilon_k R'_{\bar{k}ki\bar{j}} - h_{i\bar{j}}{}^2, \tag{3.13}$$

$$r = 2\Big(\sum_{j,k} \varepsilon_j \varepsilon_k R'_{\bar{j}jk\bar{k}} - h_2 \Big), \tag{3.14}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{k,x} \varepsilon_k \varepsilon_x h^x_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_j \varepsilon_j h_{j\bar{j}}{}^2$.

Next, the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of h^x_{ij} are given by

$$\tag{3.15} \sum_k \varepsilon_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) \\ = dh^x_{ij} - \sum_k \varepsilon_k (h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y \varepsilon_y h^y_{ij}\omega_{xy}.$$

Then, substituting dh^x_{ij} in this definition into the exterior derivative of (3.5) and using (3.1), (3.4), (3.5), (3.6), (3.7) and (3.15), we have

$$\tag{3.16} h^x_{ijk} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}$$

from the coefficients of $\omega_j \wedge \omega_k$ and $\omega_j \wedge \bar{\omega}_k$.

Similarly, the components h^x_{ijkl} and $h^x_{ijk\bar{l}}$, respectively $h^x_{ij\bar{k}l}$ and $h^x_{ij\bar{k}\bar{l}}$, of the covariant derivative of h^x_{ijk}, respectively $h^x_{ij\bar{k}}$, can be defined by

$$\tag{3.17} \sum_l \varepsilon_l (h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) \\ = dh^x_{ijk} - \sum_l \varepsilon_l (h^x_{ljk}\omega_{li} + h^x_{ilk}\omega_{lj} + h^x_{ijl}\omega_{lk}) + \sum_y \varepsilon_y h^y_{ijk}\omega_{xy},$$

and

$$\tag{3.18} \sum_l \varepsilon_l (h^x_{ij\bar{k}l}\omega_l + h^x_{ij\bar{k}\bar{l}}\bar{\omega}_l) \\ = dh^x_{ij\bar{k}} - \sum_l \varepsilon_l (h^x_{lj\bar{k}}\omega_{li} + h^x_{il\bar{k}}\omega_{lj} + h^x_{ij\bar{l}}\bar{\omega}_{lk}) + \sum_y \varepsilon_y h^y_{ij\bar{k}}\omega_{xy}.$$

Differentiating (3.15) exteriorly and using the properties $d^2 = 0$, (3.6), (3.7), (3.9), (3.13), (3.15) and (3.16), we have the following Ricci formula for the second fundamental form on M.

$$\tag{3.19} h^x_{ijkl} = h^x_{ijlk}, \qquad h^x_{ij\bar{k}\bar{l}} = h^x_{ij\bar{l}\bar{k}}$$

from the coefficients of $\omega_k \wedge \omega_l$ and $\bar{\omega}_k \wedge \bar{\omega}_l$, respectively, and

$$\tag{3.20} h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} = \sum_r \varepsilon_r (R_{\bar{l}ki\bar{r}}h^x_{rj} + R_{\bar{l}kj\bar{r}}h^x_{ri}) - \sum_y \varepsilon_y R_{\bar{l}ky\bar{x}}h^y_{ij}$$

from the coefficients of $\omega_k \wedge \bar{\omega}_l$.

In particular, let the ambient space M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c')$ of constant holomorphic sectional curvature c' and of index $2p(>0)$. Then we get

$$R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\varepsilon_j\varepsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \varepsilon_x h^x_{jk}\bar{h}^x_{il}, \tag{3.21}$$

$$S_{i\bar{j}} = \frac{c'}{2}(n+1)\varepsilon_i\delta_{ij} - h_{i\bar{j}}{}^2, \tag{3.22}$$

$$r = c'n(n+1) - 2h_2, \tag{3.23}$$

$$h^x_{ij\bar{k}} = 0 \tag{3.24}$$

and

$$\begin{aligned} h^x_{ijk\bar{l}} = {} & \frac{c'}{2}(\varepsilon_k h^x_{ij}\delta_{kl} + \varepsilon_i h^x_{jk}\delta_{il} + \varepsilon_j h^x_{ki}\delta_{jl}) \\ & - \sum_{r,y}\varepsilon_r\varepsilon_y(h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}. \end{aligned} \tag{3.25}$$

4. Examples of space-like Einstein Kähler submanifolds. We give in this section some examples of space-like Einstein Kähler submanifolds of an indefinite complex hyperbolic space $CH_p^{n+p}(c)$, $c < 0$, whose second fundamental forms are parallel or not parallel.

Example 4.1. For an indefinite complex hyperbolic space $CH_1^{n+1}(c)$, if $\{z_1, \dots, z_{n+2}\}$ is the usual homogeneous coordinate system of $CH_1^{n+1}(c)$, then the equation $z_{n+2} = 0$ defines a totally geodesic space-like complex hypersurface identifiable with $CH^n(c)$. So, it is Einstein and it is trivial that its second fundamental form is parallel.

A semi-definite complex hyperbolic space $CH_s^n(-c)$, $c > 0$, is obtained from a semi-definite complex projective space $CP_{n-s}^n(c)$ by reversing the sign of its semi-definite Kähler metric. By taking into account the fact, the previous discussion shows that $CH_s^n(-c)$ is totally geodesic complex hypersurface of both $CH_s^{n+1}(-c)$ and $CH_{s+1}^{n+1}(-c)$.

Example 4.2. For the homogeneous coordinate system $\{z_1, \dots, z_{n+2}\}$ of CP_s^{n+1} (c), an indefinite complex quadric Q_s^n $0 < s < n$, is an indefinite complex hypersurface of $CP_s^{n+1}(c)$ defined by the equation

$$-\sum_{j=1}^{s} {z_j}^2 + \sum_{k=s+1}^{n+2} {z_k}^2 = 0.$$

Then Q_s^n is a complete complex hypersurface of index $2s$, and moreover, in the similar way to Kobayashi and Nomizu [**7**, Chapter 11, Example 10.6], it is Einstein and then the Ricci tensor S satisfies

$$S = \frac{c}{2} ng, \qquad {h_{i\bar{j}}}^2 = -\frac{c}{2}\varepsilon_i \delta_{ij}.$$

Note that Q^n can be also considered as a complete space-like Einstein complex hypersurface of $CH_1^{n+1}(c)$, $c < 0$. In this case, in particular, the second fundamental form of Q^n is parallel because it is of codimension one.

Remark 4.1. In his paper [**14**], Smyth showed that a complete Einstein complex hypersurface M of a complex space form $M^{n+1}(c)$ is totally geodesic or $c > 0$ and M is the complex quadric Q^n.

Remark 4.2. An indefinite Einstein complex hypersurface of an indefinite complex space form is investigated in detail by Montiel and Romero [**8**].

The following example was also given by them.

Example 4.3. Let us consider an indefinite complex hypersurface $M = M_n^{2n}$ of $CP_{n+1}^{2n+1}(c)$ defined by the equation

$$\sum_{j=1}^{n+1} z_j z_{n+1+j} = 0$$

in the usual homogeneous coordinate system of $CP_{n+1}^{2n+1}(c)$. It is a complete complex hypersurface of index $2n$, which is denoted by Q_n^*.

It is easily seen that the Ricci tensor S satisfies

$$S = c(n+1)g, \qquad {h_{i\bar{j}}}^2 = -\frac{c}{2}\varepsilon_i \delta_{ij}$$

and hence it is Einstein, so the second fundamental form is parallel.

A same discussion as that in Example 4.3 shows that it is also an indefinite complete Einstein complex hypersurface of $CH_n^{2n+1}(-c)$, whose second fundamental form is parallel.

Example 4.4. For the homogeneous coordinate systems $\{z_1, \cdots, z_s, z_{s+1} \cdots, z_{n+1}\}$ of $CP_s^n(c)$ and $\{w_1, \cdots, w_t, w_{t+1} \cdots, w_{m+1}\}$ of $CP_t^m(c)$, a mapping f of $CP_s^n(c) \times CP_t^m(c)$ into $CP_{R(n,m,s,t)}^{N(n,m)}(c)$ with

$$N(n,m) = n + m + nm, \quad R(n,m,s,t) = s(m-t) + t(n-s) + s + t$$

is defined by

$$f(z,w) = (z_a w_{u,} z_r w_{x,} z_b w_{y,} z_s w_v),$$

where

$$a, b, \ldots = 1, \ldots, s; \quad r, s, \ldots = s+1, \ldots, n+1,$$
$$x, y, \ldots = 1, \ldots, t; \quad u, v, \ldots = t+1, \ldots, m+1.$$

Then f is a well-defined holomorphic mapping and it is seen that f is also an isomorphic imbedding, which is called an *indefinite Segre imbedding*. In the case of $n = m$, it is Einstein and the Ricci tensor S satisfies

$$S = \frac{c}{2}(n+1)g.$$

The second fundamental form is not necessarily parallel. In particular, if $s = t = 0$, then f is a classical Segre imbedding, see Nakagawa and Takagi [**10**]. This example is due to Ikawa, Nakagawa and Romero [**6**]. If $n \neq m$, then it is not Einstein, but its Ricci tensor is parallel. So, the second fundamental form is not parallel.

As the simple case in the definite product ones, $CP^1(c) \times CP^1(c)$ is the complex quadric Q^2 in $CP^3(c)$. In the indefinite case, however, we can consider two product manifolds $CP_1^1(c) \times CP_1^1(c)$ and $CP_1^1(c) \times CP^1(c)$ which are mutually different complex quadric in $CP_2^3(c)$. In fact, it is seen in Montail and Romero [**8**] that they are denoted by Q_2^2 and Q_1^*, respectively.

By using the fact that an indefinite complex hyperbolic space $CH_s^n(-c)$ is obtained from $CP_{n-s}^n(c)$ by changing the Kähler metric to its negative. Another indefinite Segre imbedding

$$f : CH_s^n(-c) \times CH_t^m(-c) \longrightarrow CH_{S(n,m,s,t)}^{N(n,m)}(-c)$$

is given, where

$$S(n,m,s,t) = (n-s)(m-t) + st + s + t.$$

In the case where $n = m$, it is Einstein and the Ricci tensor S satisfies $S = -\frac{c}{2}(n+1)g$. In particular, for $s = t = 0$, we have a holomorphic isometric imbedding f of $CH^n(-c) \times CH^m(-c)$ into $CH^{N(n,m)}_{nm}(-c)$.

It is easily seen that the Ricci tensor on the complex submanifold with parallel second fundamental form is also parallel. We give here some examples complex submanifolds of a complex projective space whose second fundamental forms are parallel. These submanifolds are completely classified by Nakagawa and Takagi [**10**] and their geometric properties are also completely determined.

Example 4.5. Let M be an n-dimensional compact irreducible Hermitian symmetric space with Kähler metric under the canonical imbedding into a complex projective space $CP^{n+p}(c)$. Then the degree of the imbedding coincides with the rank of M as a symmetric space. This shows that the following six kinds of compact irreducible Hermitian symmetric spaces:

$$\begin{gathered} CP^n (= SU(n+1)/S(U(n) \times U(1)), \\ Q^n (= SO(n+2)/SO(n) \times SO(2)), \quad n \geqq 3, \\ SU(s+2)/S(U(s) \times U(2)), \quad s \geqq 3, \\ SO(10)/U(5), \\ E_6/Spin(10) \times T, \\ E_7/E_6 \times T \end{gathered}$$

admit Kähler imbeddings with parallel second fundamental form into $CP^{n+p}(c)$, where $U(n)$, $SU(n)$ and $SO(n)$ denote the unitary group, the special unitary group and the special orthogonal group, respectively, and E_6, $\mathrm{Spin}\,(10)$ and T denote the exceptional group, the spin group and the torus group, respectively. The above six spaces are Einstein, their dimensions are $n, n, 2s, 10, 16$ and 27, respectively, and their scalar curvatures are given by $cn(n+1), cn^2, 2cs(s+2), 80c, 192c$ and $486c$, respectively. If the imbedding is full, then the codimension p is $0, 1, \frac{1}{2}(s^2 - s), 5, 10$ and 28, respectively.

Example 4.6. We give another example of complex submanifold of an N-dimensional complex projective space $CP^N(c)$ of constant holomorphic sectional curvature c. Define a mapping f of $CP^{n_1}(c_1) \times \cdots \times CP^{n_r}(c_r)$ into $CP^N(c)$ by

$$\begin{aligned}(z_0^1, \dots, z_{n_1}^1, \dots, z_0^r, \dots, z_{n_r}^r)&\\ \longrightarrow (z_0^1 \cdots z_0^r, \dots, z_{i_1}^1 \cdots z_{i_r}^r, \dots, z_{n_1}^r \cdots z_{n_r}^r),&\\ i_\alpha = 0,\ 1,\ \dots, n_\alpha, \quad \alpha = 1, \dots, r,&\end{aligned}$$

where $N = (n_1+1) \ \cdots \ (n_r+1) - 1$ and $(z_0^\alpha, \dots, z_{n_\alpha}^\alpha)$ are complex homogeneous coordinates of $CP^{n_\alpha}(c_\alpha)$. Then it is easy to see that f induces a Kähler imbedding of a Kähler manifold $CP^{n_1}(c_1) \times \cdots \times CP^{n_r}(c_r)$ into $CP^N(c)$ if and only if $c_1 = \cdots = c_r = c$.

In particular, the Kähler manifold $CP^{n_1}(c) \times \cdots \times CP^{n_r}(c)$ is Einstein if and only if $n_1 = \cdots = n_r = n$. The scalar curvature r of $CP^{n_1}(c) \times CP^{n_2}(c)$ is given by

$$r = c\{n_1(n_1+1) + n_2(n_2+1)\}.$$

And moreover, we see $h_2 = cn_1n_2$ and $N = (n_1+1)(n_2+1) - 1$.

In their paper [10], Nakagawa and Takagi proved the following classification theorem.

Theorem 4.1. *Let M be an n-dimensional complete complex submanifold imbedded into an N-dimensional complex projective space $CP^N(c)$ with parallel second fundamental form. If M is irreducible, then M is congruent to one of six kinds of complex submanifolds imbedded into $CP^N(c)$ with parallel second fundamental form given in the above Example* 4.5. *If M is reducible, then M is congruent to $(CP^{n_1} \times CP^{n_2}, f)$ given in Example* 4.6 *for some n_1 and n_2 with $n = n_1 + n_2$. The corresponding local version is true.*

Example 4.7. Calabi [**5**] classified completely a Kähler imbedding of simply connected complex space forms into complete simply connected space forms. He gave a full Kähler imbedding of $CP^n(c)$ into $CP^{N(p)}(pc)$ by

$$(z_0, \dots, z_n) \ \to \ \left({z_0}^p, \dots, \sqrt{\frac{p!}{p_0! \cdots p_n!}}\ {z_0}^{p_0} \cdots {z_n}^{p_n}, \dots, {z_n}^p\right),$$

where $N(p) = {}_{n+p}C_p - 1$, ${}_nC_m$ denotes the number of possible combinations of n objects taken m at a time, $(z_0, \dots, z_n)$ are homogeneous coordinates of $CP^n(c)$ and $p_0, \dots, p_n$ range over all nonnegative integers with $p_0 + \dots + p_n = p$, which is called a *p-canonical imbedding.*

On the other hand, Romero [**13**] and Umehara [**15**] independently proved the indefinite version and they found that there exists a full holomorphic isometric immersion of an indefinite complex space form $M_s^n(c)$ into an indefinite complex space form $M_{s+t}^{n+p}(c')$.

Aiyama, Nakagawa and Suh [**3**] obtained the following local property of the above immersion.

$$\sum_x h^x_{i_1 \cdots i_k} \bar{h}^x_{j_1 \cdots j_l} = \begin{cases} 0 & \text{for all } k \neq l, \\ \dfrac{1}{2^{k-1}} \Pi_{r=1}^{k-1} (c' - cr) \varepsilon_{i_1} \cdots \varepsilon_{i_k} & \\ \quad \sum_\tau \delta_{\tau(i_1) j_1} \cdots \delta_{\tau(i_k) j_k} & \text{for } k = l, \end{cases}$$

where $\sum_\tau$ denotes the summation on all permutations τ with respect to the indices $i_1, \dots, i_k$. By this formula, it is easily seen that the 2-canonical imbedding of $CP^n(c)$ into $CP^{N(2)}(2c)$ has the parallel second fundamental form but the second fundamental form of $CP^n(c)$ is not parallel for the $p(\geqq 3)$-canonical imbedding.

5. Parallel second fundamental forms. Let M be an $n(\geqq 2)$-dimensional space-like complex Einstein submanifold of an indefinite complex space form $M' = M_p^{n+p}(c)$ of constant holomorphic sectional curvature c. Assume that the second fundamental form is parallel. We denote by A the $p \times p$-matrix defined by (A_y^x) and H the $p \times \frac{1}{2} n(n+1)$-matrix defined by $(h^x_{(jk)})_{j \leqq k}$, where $A_y^x = \sum_{i,j} h^x_{ij} \bar{h}^y_{ij}$. Under the above assumption, making use of (3.23), we can simplify the equation (3.25) as follows:

$$AH = \frac{1}{2n}(cn^2 - 2r)H \tag{5.1}$$

because $\varepsilon_i = 1$ and $\varepsilon_x = -1$. By the definition of the matrices A and H, we see that $A = HH^*$, where the symbol * denotes the complex conjugate and transpose operator, namely, $H^* = {}^t\bar{H}$. From (5.1), we have

$$A^2 = \frac{1}{2n}(cn^2 - 2r)A. \tag{5.2}$$

Since the matrix A is a positive semi-definite Hermitian one, it implies that $cn^2 - 2r \geqq 0$. This means that the matrix A has at most two different eigenvalues 0 and $\frac{1}{2n}(cn^2 - 2r)$.

We investigate here a property concerning the rank of matrices A and H. We denote by $\operatorname{rank} A$ the rank of the matrix A. At any point x in M we put $q(x) = \operatorname{rank} A(x)$. Then the following result is verified.

Lemma 5.1. *Let M be an $n(\geqq 2)$-dimensional space-like complex Einstein submanifold of $M_p^{n+p}(c)$. If the second fundamental form on M is parallel and if M is not totally geodesic, then for any point x in M, we have*

$$q(x) = \operatorname{rank} H(x) = \frac{n}{2r - cn^2}\{cn(n+1) - r\}. \tag{5.3}$$

Proof. Suppose that there exists a geodesic point x in M, namely, there is a point x at which all eigenvalues are zero. The assumption that the second fundamental form is parallel implies that the scalar curvature r is constant, and hence we have $r = cn(n+1)$ on M, from which together with (3.23) again it follows that M is totally geodesic. Accordingly, by the assumption of this lemma, there do not exist geodesic points. In other words, the matrix A has at least one positive eigenvalue $\lambda = \frac{1}{2n}(cn^2 - 2r)$. Since any point x in M is not a geodesic one, we see that $\operatorname{rank}(AH) = \operatorname{rank} H$ by (5.1), which yields that $\operatorname{rank} H \leqq \operatorname{rank} A$. On the other hand, because of $A = HH^*$, we have $\operatorname{rank} A \leqq \operatorname{rank} H$. Thus we obtain $\operatorname{rank} H = \operatorname{rank} A$. The first equality of the formula in Lemma 5.1 follows from this property. In fact, since a positive eigenvalue $\lambda(x)$ of the matrix A at point x is given by

$$\lambda(x) = \frac{1}{2n}(cn^2 - 2r)$$

and the scalar curvature r is constant on M, the eigenvalue λ is constant on M, so that the multiplicity q of λ is constant on M. Thus we have the rank q of A is also constant on M and it is easily seen that the rank of the matrix A is equal to that of the matrix H at any point in M and it satisfies

$$q\lambda = \mathrm{Tr}\, A = -h_2 = -\frac{1}{2}\{cn(n+1) - r\}$$

by (3.23).

This completes the proof. □

Now we give an information for the range of the scalar curvature r on M. Since the second fundamental form of M is parallel, putting $k = l$ in (3.25) and then summing up with respect to k, we have

$$c(n+2)h^x_{ij} - 2\Big\{\sum_r (h_{i\bar{r}}{}^2 h^x_{rj} + h_{j\bar{r}}{}^2 h^x_{ri}) + \sum_y \varepsilon_y A^x_y h^y_{ij}\Big\} = 0. \tag{5.4}$$

Transvecting $\varepsilon_x \bar{h}^x_{ij}$ to this equation and then summing up with respect to i, j and x, we get

$$c(n+2)h_2 - 4h_4 - 2\mathrm{Tr}\, A^2 = 0, \tag{5.5}$$

where $h_4 = \sum_{i,j} h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$. The matrix $(h_{j\bar{k}}{}^2)$ is a negative semi-definite Hermitian one, whose eigenvalues λ_js are nonpositive real valued functions on M. This yields that

$$h_4 = \sum_j {\lambda_j}^2 \geqq \frac{1}{n}\big(\sum_j \lambda_j\big)^2 = \frac{1}{n}\big(\mathrm{Tr}\, H\big)^2 = \frac{1}{n}{h_2}^2, \tag{5.6}$$

where the equality holds if and only if $\lambda = \lambda_j$ for any index j, namely, we have

$$h_{j\bar{k}}{}^2 = \lambda\delta_{jk}. \tag{5.7}$$

This means that the equality (5.6) holds on M if and only if M is Einstein. On the other hand, the matrix A is a positive semi-definite Hermitian one of order p. Thus its eigenvalues μ_xs are all nonnegative real-valued functions on M and hence we have

$$\mathrm{Tr}\, A^2 = \sum_x {\mu_x}^2 \geqq \frac{1}{p}\Big(\sum_x \mu_x\Big)^2 = \frac{1}{p}\big(\mathrm{Tr}\, A\big)^2 = \frac{1}{p}{h_2}^2, \tag{5.8}$$

where the equality holds if and only if $\mu = \mu_x$ for any index x, that is, the matrix A satisfies $A = \mu I_p$, where I_p denotes the identity matrix of order p.

Making use of these properties, we can prove

Lemma 5.2. *Let M be a space-like complex submanifold of $M_p^{n+p}(c)$. If the second fundamental form on M is parallel and if M is not totally geodesic, then the scalar curvature r satisfies*

$$r \leqq \frac{c}{n+2p} n^2(n+p+1), \tag{5.9}$$

where the equality holds if and only if M is Einstein.

Proof. By (5.5), (5.6) and (5.8), we have

$$c(n+2)h_2 - \frac{4}{n}{h_2}^2 - \frac{2}{p}{h_2}^2 \geqq 0,$$

where the equality holds if and only if M is Einstein because if the equality holds at a point, then the fact that the squared norm h_2 is constant implies that it holds on M and the matrix A satisfies $A = \mu I_p$. Hence we obtain

$$\{cnp(n+2) - 2(n+2p)h_2\}h_2 \geqq 0.$$

The squared norm h_2 of the second fundamental form α on M is negative constant because the second fundamental form α on M is parallel and M is not totally geodesic. So, we have

$$h_2 \geqq \frac{c}{2(n+2p)} np(n+2), \tag{5.10}$$

where the equality holds if and only if M is Einstein. Hence the assertion (5.9) is derived by (3.23) and (5.10). $\square$

Next, we give another information about the restriction of the scalar curvature on M.

Lemma 5.3. *Let M be a space-like complex submanifold of $M_p^{n+p}(c)$, $c<0$. If the second fundamental form on M is parallel, then the scalar curvature r satisfies*

$$r \leqq \frac{c}{2}n(n+1), \tag{5.11}$$

where the equality holds if and only if M is a complex space form $M^n(\frac{c}{2})$.

Proof. First, we introduce a tensor field F of type (0,4) with components $F_{\bar{i}jk\bar{l}}$ defined by

$$F_{\bar{i}jk\bar{l}} = \sum_x \varepsilon_x h^x_{jk}\bar{h}^x_{il} - \frac{c}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).$$

Then we get

$$\sum_{i,j,k,l} F_{\bar{i}jk\bar{l}}\bar{F}_{\bar{i}jk\bar{l}} = \operatorname{Tr} A^2 + c \operatorname{Tr} A + \frac{c^2}{8}n(n+1) \geqq 0, \tag{5.12}$$

where the equality holds on M if and only if we have

$$\sum_x \varepsilon_x h^x_{jk}\bar{h}^x_{il} = \frac{c}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})$$

on M. By (2.11) and (3.21), if the above equality holds on M, then it is seen that M is a complex space form $M^n(\frac{c}{2})$ of constant holomorphic sectional curvature $\frac{c}{2}$. From (5.6) and (5.12), we can eliminate the term $\operatorname{Tr} A^2$ in (5.5) and then we obtain the following inequality

$$\{4h_2 - cn(n+1)\}(4h_2 + cn) \leqq 0,$$

where the equality holds if and only if M is a complex space form $M^n(\frac{c}{2})$ because h_2 is constant. Since the holomorphic sectional curvature of the ambient space is assumed to be negative, we see $4h_2 + cn < 0$, so we have

$$h_2 \geqq \frac{c}{4}n(n+1),$$

where the equality holds if and only if M is a complex space form $M^n(\frac{c}{2})$. Hence the assertion (5.11) is proved by (3.23) and the above equation. □

6. Proof of Theorem. Let M be an $n(\geqq 2)$-dimensional space-like Einstein Kähler submanifold of an indefinite complex space form $M' = M_p^{n+p}(c)$, $c < 0$. Assume that the second fundamental form α on M is parallel. Under this assumption, we give an information for the range of the scalar curvature r on M.

Lemma 6.1. *Let M be an $n(\geqq 2)$-dimensional space-like Einstein Kähler submanifold of $M_p^{n+p}(c)$, $c < 0$. If M is not totally geodesic and if the second fundamental form on M is parallel, then we have*

$$r \leqq cn^2 \quad \text{or} \quad r \geqq \frac{c}{4}n(3n+2). \tag{6.1}$$

Proof. Since M is Einstein, the Ricci tensor S satisfies

$$S_{j\bar{k}} = \frac{r}{2n}\delta_{jk}, \tag{6.2}$$

where r is the scalar curvature, it follows from $\varepsilon_i = 1$, (3.22) and (6.2) that

$$h_{j\bar{k}}{}^2 = \sum_{l,x} \varepsilon_x h^x_{jl}\bar{h}^x_{lk} = \frac{1}{2n}\{cn(n+1) - r\}\delta_{jk}. \tag{6.3}$$

On the other hand, we have by (3.25)

$$c(\varepsilon_k h^x_{ij}\delta_{kl} + \varepsilon_i h^x_{jk}\delta_{il} + \varepsilon_j h^x_{ki}\delta_{jl}) \\ - 2\sum_{r,y} \varepsilon_y (h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl} = 0$$

because the second fundamental form of M is parallel. Transvecting $\varepsilon_x \bar{h}^x_{im}\bar{h}^z_{jk}$ to this equation and then summing up with respect to x, i, j and k, we obtain

$$c\Big(2\sum_j h_{j\bar{m}}{}^2\bar{h}^z_{jl} + \sum_x \varepsilon_x \bar{h}^x_{ml}A^x_z\Big) - 2\sum_{j,y} \varepsilon_y h_{j\bar{m}}{}^2 A^y_z \bar{h}^y_{jl} \\ - 4\sum_{i,j,k,r,x,y} \varepsilon_x\varepsilon_y \bar{h}^x_{mr}h^y_{ri}\bar{h}^z_{ij}h^x_{jk}\bar{h}^y_{kl} = 0.$$

We denote by H^x a symmetric matrix defined as (h^x_{jk}) of order n. Then we can reform the above equation as follows:

$$\sum_{x,y} \varepsilon_x \varepsilon_y \bar{H}^x H^y \bar{H}^z H^x \bar{H}^y = \frac{1}{8n^2}\{2r^2 - cn(3n+2)r + c^2 n^2(n^2+2n+2)\}\bar{H}^z, \tag{6.4}$$

where we have used (5.1) and (6.3).

Now we define a tensor field G with components G^{xyz} by

$$\begin{aligned} G^{xyz} &= H^x \bar{H}^y H^z + H^z \bar{H}^y H^x \\ &\quad - \frac{2}{cn^3 - (n-2)r}\{cn(n+1) - r\}(A^x_y H^z + A^z_y H^x). \end{aligned}$$

By the direct and complicated calculation, it follows from (5.2), (6.3) and (6.4) that we obtain

$$\begin{aligned} \sum_{x,y,z} \varepsilon_x \varepsilon_y \varepsilon_z G^{xyz} \bar{G}^{xyz} &= \frac{1}{8n^3\{cn^3 - (n-2)r\}}(n+2)\{cn(3n+2) - 4r\} \\ &\quad \times (cn^2 - r)\{cn(n+1) - r\}\{cn(n+2) - r\}I. \end{aligned} \tag{6.5}$$

Since M is not totally geodesic, we have by (3.23)

$$cn(n+1) - r < 0.$$

And taking account of this inequality, we get

$$\begin{gathered} cn(n+2) - r < cn(n+2) - cn(n+1) = cn < 0, \\ cn^3 - (n-2)r < cn^3 - c(n-2)n(n+1)c = cn(n+2) < 0. \end{gathered}$$

Using the above three equations, we have by (6.5)

$$\{cn(3n+2) - 4r\}(cn^2 - r) \geqq 0$$

because the lefthand side of the equation (6.5) is nonpositive.

This completes the proof. $\square$

Moreover, we obtain by (5.3)

$$r = \frac{c}{n+2q} n^2 (n+q+1). \tag{6.6}$$

If $r \leqq cn^2$, then the above equation implies that $q \leqq 1$, namely, we have $q = 1$. And if $r \geqq c/4n(3n+2)$, then we have $q \geqq \frac{n}{2}$. Thus we obtain

$$q = 1 \quad \text{or} \quad q \geqq \frac{n}{2}. \tag{6.7}$$

Lemma 6.2. *Let M be an $n(\geqq 2)$-dimensional space-like Einstein complex submanifold of $M_p^{n+p}(c)$, $c < 0$. If it is not totally geodesic, then there exists an $(n+q)$-dimensional totally geodesic submanifold M' in $M_p^{n+p}(c)$ in which the given submanifold M is immersed, where $q = \operatorname{rank} A > 0$.*

Proof. For the unitary frame $\{E_\alpha\} = \{E_j, E_y\}$ at any point x, we define the normal space to M at x, which is denoted by N_x

$$N_x = \Big\{ \sum_y \xi^y E_y : \xi^y \in \mathbf{C} \Big\},$$

where $\mathbf{C}$ is the complex field. We define a mapping f of $N_x \times N_x$ into $\mathbf{C}$ by

$$f(Y, Z) = \sum_{y,z} A_z^y \bar{\xi}^y \eta^z, \quad Y = \sum_y \xi^y E_y, \; Z = \sum_z \eta^z E_z.$$

Let H_p be a set of all Hermitian matrices of order p, which is considered as a complex vector space. Then the unitary group $U(p)$ operates H_p as follows:

For any Hermitian matrix H in H_p and any unitary matrix U in $U(p)$,

$$U(H) = U^* H U,$$

where * denotes the complex conjugate and transpose operator. Since the matrix A is invariant under $U(p)$, the mapping f is well-defined and it is a positive semi-definite Hermitian form of order q, so that it can be normalized. This means that we can new unitary frame $\{E_i, E_\alpha, E_\lambda\}$ such that

$$\omega_{\alpha i} \neq 0, \qquad \omega_{\lambda i} = 0, \tag{6.8}$$

where the range of indices is as follows:

$$\begin{aligned} i, j, \ldots &= 1, \ldots, n, \\ \alpha, \beta, \ldots &= n+1, \ldots, n+q, \\ \lambda, \mu, \ldots &= n+q+1, \ldots, n+p. \end{aligned}$$

By definition of h^λ_{ijk}, we have $\sum_\alpha h^\alpha_{ij} \omega_{\lambda\alpha} = 0$. It implies that

$$\omega_{\lambda\alpha} = 0 \tag{6.9}$$

for any indices α and λ. From (6.8) and (6.9), we can define a distribution DM defined by

$$\omega_\lambda = 0, \qquad \omega_{\lambda i} = 0, \qquad \omega_{\lambda\alpha} = 0.$$

Then it follows from the structure equations on $M^{n+p}_p(c)$ that we obtain

$$\begin{aligned} d\omega_\lambda &= -\sum_j \omega_{\lambda j} \wedge \omega_j - \sum_\alpha \omega_{\lambda\alpha} \wedge \omega_\alpha - \sum_\mu \omega_{\lambda\mu} \wedge \omega_\mu \\ &\equiv \ 0 \ (\mathrm{mod}\ \omega_\lambda,\ \omega_{\lambda i},\ \omega_{\lambda\alpha}), \\ d\omega_{\lambda i} &= -\sum_j \omega_{\lambda j} \wedge \omega_{ji} - \sum_\alpha \omega_{\lambda\alpha} \wedge \omega_{\alpha i} - \sum_\mu \omega_{\lambda\mu} \wedge \omega_{\mu i} + \ \Omega_{\lambda i} \\ &\equiv \ 0 \ (\mathrm{mod}\ \omega_\lambda,\ \omega_{\lambda i},\ \omega_{\lambda\alpha}), \\ d\omega_{\lambda\alpha} &= -\sum_j \omega_{\lambda j} \wedge \omega_{j\alpha} - \sum_\beta \omega_{\lambda\beta} \wedge \omega_{\beta\alpha} - \sum_\mu \omega_{\lambda\mu} \wedge \omega_{\mu\alpha} + \ \Omega_{\lambda\alpha} \\ &\equiv \ 0 \ (\mathrm{mod}\ \omega_\lambda,\ \omega_{\lambda i},\ \omega_{\lambda\alpha}). \end{aligned}$$

Therefore, the distribution DM is of dimensional $(n+q)$ and it becomes completely integrable. For any point x, we consider the maximal integral submanifold $M'(x)$ of M through x. Then $M'(x)$ is of $(n+q)$-dimensional and it is totally geodesic in $M^{n+p}_p(c)$ by the construction. Moreover, M is immersed in $M'(x)$.

This completes the proof. $\square$

Now we are in a position to prove the main theorem stated in the Introduction.

The immersion of M into $M_p^{n+p}(c)$ is said to be *full* if M cannot be immersed in $(n+p')$-dimensional totally geodesic submanifold in $M_p^{n+p}(c)$, where $p > p' > 0$. The first assertion of the theorem follows immediately from (6.7) and Lemma 6.2 and the rigidity theorem due to Montiel and Romero [**8**] for an Einstein semi-Kähler hypersurface in an indefinite hyperbolic space.

Next, we shall prove another case. In this case, we may suppose $p = q$ because of the full immersion. By the assumption of the theorem, we have $p = q \geqq \frac{1}{2}n(n+1)$. We denote by $r(q)$ the right hand side of (6.6), namely, we see $r = r(q)$. We can regard $r(q)$ as the function with one variable q and then it is easily seen that it is monotonic increasing with respect to q because c is negative, and hence

$$r = r(q) \geqq \frac{c}{2}n(n+1),$$

from which together with (5.11), it follows that

$$r = \frac{c}{2}n(n+1).$$

By taking account of Lemma 5.3 and (6.6), $p = \frac{1}{2}n(n+1)$ and M is a complex space form $M^n(\frac{c}{2})$ of constant holomorphic sectional curvature $\frac{c}{2}$.

This completes the proof. $\square$

Problem 6.1. Does there exist an $n(\geqq 2)$-dimensional space-like Einstein Kähler submanifold of an indefinite complex space form $M_p^{n+p}(c)$, $c < 0$, $\frac{n}{2} < p < \frac{1}{2}n(n+1)$?

Remark 6.1. Is the estimate of the codimension in the assertion (1) of the theorem best possible? As seen in Example 4.6, the product manifold $CH^{\frac{n}{2}}(c) \times CH^{\frac{n}{2}}(c)$ is a space-like Einstein complex submanifold $CH_p^{n+p}(c)$, where $p = 1/4n^2$. So, if $n = 2$, then $p = 1$. However, it means essentially that it is complex quadric.

Remark 6.2. Let $M = M_s^n(c)$ be a complex n-dimensional semi-definite Kähler manifold of constant holomorphic sectional curvature c and of index $2s$, and let $M_S^N(c')$ be a complex N-dimensional semi-definite complete simply connected complex space form of constant holomorphic sectional curvature c' and of index $2S$. Then a holomorphic isometric immersion $f : M_s^n(c) \to M_S^N(c')$ is said to be *full* if $f(M_s^n(c))$ is not contained in a totally geodesic submanifold of $M_S^N(c')$. It is seen in [**13**] that $M_s^n(c)$, $c > 0$, admits a full holomorphic isometric immersion into $M_S^N(c')$, $c' > 0$, if and only if $c' = kc$ for some positive integer k,

$$N = {}_{n+k}C_k - 1 =: N(n,k)$$

and

$$S = \sum_{j=0}^{[\frac{k+1}{2}]-1} {}_{s+2j}C_{2j+1}\ {}_{n-s+k-2j-1}C_{k-2j-1} =: S(n,s,k) \quad \text{if } s > 0,$$

where $[\frac{k+1}{2}]$ denotes the greatest integer less than or equal to $\frac{1}{2}(k+1)$, and $S = 0$ and if $s = 0$. The local version is true.

Changing the Kähler metric of $M_n^n(c)$, $c > 0$, by its opposite, we have that there exists a full holomorphic isometric immersion of $M^n(-c)$, $c > 0$, into $M_{S'(n,k)}^{N(n,k)}(-kc)$, $c > 0$, where $S'(n,k) = N(n,k) - S(n,n,k)$. It is seen that

$$N(n,2) - n = S'(n,2) = \frac{1}{2}n(n+1)$$

and

$$N(n,k) - n > S'(n,k) \quad \text{if } k > 2.$$

This means that there exists only one full holomorphic isometric immersion of $M^n(c)$, $c < 0$, into $M_p^{n+p}(c')$, $c' < 0$, as space-like submanifolds except for the trivial immersion as a totally geodesic one. In this case, we see $k = 2$ and $p = \frac{1}{2}n(n+1)$.

Remark 6.3. In their paper [**3**], Aiyama, Nakagawa and Suh proved the following fact. Let M be a space-like complex submanifold with constant scalar curvature r of $M_p^{n+p}(c)$, $c < 0$. If $r > \frac{c}{2}n(n+1)$, then M is a complex space form $M^n(\frac{c}{2})$ and $p \geqq \frac{1}{2}n(n+1)$.

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