ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 4, Winter 2004

SYMMETRIC DIOPHANTINE EQUATIONS

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ABSTRACT. In this paper we use certain properties of rational binary forms to solve several diophantine equations of the type $f(x,y) = f(u,v)$. If on applying the nonsingular linear transformation T defined by $x = \alpha u + \beta v$, $y = \gamma u + \delta v$, the binary form $\phi(x,y)$ becomes a scalar multiple of the form $\phi(u,v)$, we call $\phi(x,y)$ an eigenform of the linear transformation T. If $f(x,y) = L(x,y)\phi(x,y)$ where $\phi(x,y)$ is an eigenform of the linear transformation T and $L(x,y)$ is not an eigenform of T, the diophantine equation $f(x,y) = f(u,v)$ reduces, on making the substitution $x = m(\alpha u+\beta v)$, $y = m(\gamma u+\delta v)$, to a linear equation in the variables u and v while m is an arbitrary parameter. The solution of this linear equation readily yields a parametric solution of the original diophantine equation. We first use eigenforms to obtain parametric solutions of several general types of diophantine equations such as $L_1(x,y)Q_1^r(x,y)Q_2^s(x,y) = L_1(u,v)Q_1^r(u,v)Q_2^s(u,v)$ and $\{\Pi_{i=1}^5 L_i(x,y,z)\}Q^r(x,y,z) = \{\Pi_{i=1}^5 L_i(u,v,w)\}Q^r(u,v,w)$ where Ls and Qs denote linear and quadratic forms and r and s are arbitrary integers, and then we obtain parametric solutions of several specific diophantine equations such as the equation $f(x,y) = f(u,v)$ where $f(x,y) = x^n + x^{n-1}y + \cdots + y^n$, n being an arbitrary odd integer and the equation $x^7 + y^7 + 625z^7 = u^7 + v^7 + 625w^7$.

1. Introduction. In this paper we use certain properties of binary forms to solve several symmetric diophantine equations of the type

$$f(x,y) = f(u,v). \tag{1.1}$$

We will use Ls, Qs and Cs to denote linear, quadratic and cubic forms, respectively. All the forms considered in this paper will be assumed to be defined over the field $\mathbf{Q}$ of rational numbers. Further, reducibility of a form means reducibility over $\mathbf{Q}$.

AMS *Mathematics Subject Classification.* 11D25, 11D41.
Key words and phrases. Eigenforms of linear transformations, diophantine equations.
Received by the editors on April 3, 2002, and in revised form on January 28, 2003.

We call a binary form $\phi(x,y)$ to be an eigenform of a nonsingular linear transformation T defined by

$$x = \alpha u + \beta v,\\ y = \gamma u + \delta v, \tag{1.2}$$

where α, β, γ and δ are rational numbers if the binary form obtained from $\phi(x,y)$ by applying the linear transformation T is given by $k\phi(u,v)$, where k is a non-zero rational number. We now consider diophantine equations of the type (1.1) when $f(x,y)$ is a binary form defined by

$$f(x,y) = L(x,y)\phi(x,y) \tag{1.3}$$

where $\phi(x,y)$ is an eigenform of degree r of the linear transformation T so that $\phi(\alpha u + \beta v, \gamma u + \delta v) = k\phi(u,v)$ and $L(x,y)$ is an arbitrary linear form which is not an eigenform of T. To obtain solutions of (1.1) with $f(x,y) \neq 0$, we write

$$x = m(\alpha u + \beta v),\\ y = m(\gamma u + \delta v), \tag{1.4}$$

where m is arbitrary. We note that $\phi(x,y)$ is also an eigenform of the linear transformation defined by (1.4) and, with these substitutions, equation (1.1) reduces to the linear equation

$$m^{r+1}kL(\alpha u + \beta v, \gamma u + \delta v) = L(u,v). \tag{1.5}$$

The solution of (1.5) readily leads to a parametric solution of (1.1). We can similarly solve the equation

$$L_1(x,y)\phi(x,y) = L_2(u,v)\phi(u,v). \tag{1.6}$$

More generally, if the form obtained by applying the linear transformation T to a binary form $\phi_1(x,y)$ is $k\phi_2(u,v)$, the diophantine equation

$$L_1(x,y)\phi_1(x,y) = L_2(u,v)\phi_2(u,v) \tag{1.7}$$

reduces to a linear equation in u and v on making the substitutions (1.4), and hence we can obtain a parametric solution of (1.7).

In order to solve equations of the type (1.1) when $f(x,y)$ is given by (1.3), we need to determine a suitable linear transformation T such that the binary form $\phi(x,y)$ is an eigenform of the linear transformation T. It is seen by direct substitution that any linear form $px+qy$ is an eigenform of the linear transformation T defined by (1.4) provided α, β, γ and δ are chosen such that $\gamma \neq 0$ and

$$\beta p^2 + (\delta - \alpha)pq - \gamma q^2 = 0. \tag{1.8}$$

Similarly the arbitrary quadratic form $ax^2+bxy+cy^2$ is an eigenform of the linear transformation T if α, β, γ and δ are chosen such that $\gamma \neq 0$, $\delta = -\alpha$ and

$$b\alpha - a\beta + c\gamma = 0. \tag{1.9}$$

In fact, when α, β, γ and δ satisfy these conditions, using the substitutions (1.2), we get

$$ax^2 + bxy + cy^2 = (\alpha^2 + \beta\gamma)(au^2 + buv + cv^2). \tag{1.10}$$

It now readily follows that given a pair of forms of degrees not exceeding two, we can choose α, β, γ and δ suitably to determine a linear transformation T such that both forms are eigenforms of the linear transformation T. Further, it follows from the definition of eigenforms that the product of two or more eigenforms of a linear transformation T is also an eigenform of T. Thus, given a reducible binary cubic form $C(x,y)$ or a quartic form of the type $Q_1(x,y)Q_2(x,y)$, we can find a suitable linear transformation T such that these binary forms are eigenforms of the linear transformation T.

Using the above ideas we obtain parametric solutions of a number of diophantine equations of the type (1.1), two examples being as follows:

$$\begin{aligned} x(x+y)(x+2y)(x+3y)(x+4y) \\ = u(u+v)(u+2v)(u+3v)(u+4v) \end{aligned} \tag{1.11}$$

and

$$x^n + x^{n-1}y + \cdots + y^n = u^n + u^{n-1}v + \cdots + v^n, \tag{1.12}$$

where n is an arbitrary odd integer.

Some diophantine equations of the type $f(x, y, z) = f(u, v, w)$, may also be solved using eigenforms. As an example, we obtain a parametric solution of the following equation:

$$x^7 + y^7 + 625z^7 = u^7 + v^7 + 625w^7. \tag{1.13}$$

In Section 2 we use eigenforms to solve a number of general types of diophantine equations of the type (1.1). Generally we obtain parametric solutions but in some cases a complete solution is obtained. In Section 3 we apply these methods to obtain solutions of several specific diophantine equations and problems.

2. General diophantine equations.

2.1 The diophantine equation

$$L_1^r(x, y)L_2^s(x, y)L_3(x, y) = L_1^r(u, v)L_2^s(u, v)L_3(u, v), \tag{2.1}$$

where r and s are arbitrary integers and we assume that no two of the forms $L_i(x, y)$, $i = 1, 2, 3$, are linearly dependent as otherwise the equation reduces to a simpler equation the complete solution of which is readily determined.

First we apply the nonsingular linear transformation determined by the equations

$$\begin{aligned} L_1(x, y) &= X, \quad L_1(u, v) = U, \\ L_2(x, y) &= Y, \quad L_2(u, v) = V, \end{aligned} \tag{2.2}$$

when equation (2.1) reduces to the type

$$X^r Y^s L_4(X, Y) = U^r V^s L_4(U, V). \tag{2.3}$$

As $X^r Y^s$ is an eigenform of the linear transformation $X = \alpha U$, $Y = \beta V$, we can obtain a parametric solution of equation (2.3) by the method already described for solving the general diophantine equation (1.1). To obtain the complete solution of equation (2.3), we write

$$\frac{X}{m_1} = \frac{U}{m_2}, \quad \frac{Y}{m_3} = \frac{V}{m_4} \tag{2.4}$$

where m_i, $i = 1, 2, 3, 4$ are arbitrary integers so that equation (2.3) reduces to

$$m_1^r m_3^s L_4(m_1 m_4 U, m_2 m_3 V) = m_2^{r+1} m_4^{s+1} L_4(U, V). \tag{2.5}$$

Now (2.4) and (2.5) may be considered as three linear equations in the variables X, Y, U, V, and their complete non-zero solution may be written as follows:

$$\begin{aligned} X &= d\psi_1(m_1, m_2, m_3, m_4), \quad & Y &= d\psi_2(m_1, m_2, m_3, m_4), \\ U &= d\psi_3(m_1, m_2, m_3, m_4), \quad & V &= d\psi_4(m_1, m_2, m_3, m_4), \end{aligned} \tag{2.6}$$

where m_i, $i = 1, 2, 3, 4$ are integer parameters and d is an arbitrary rational number.

To show that this solution of equation (2.3) is complete, let X_1, Y_1, U_1 and V_1 be any given solution of (2.3) so that

$$X_1^r Y_1^s L_4(X_1, Y_1) = U_1^r V_1^s L_4(U_1, V_1). \tag{2.7}$$

If we choose $m_1 = X_1$, $m_2 = U_1$, $m_3 = Y_1$, $m_4 = V_1$, it is easy to see that X_1, Y_1, U_1 and V_1 satisfy the linear equations (2.4) and (2.5). Since (2.6) gives the complete non-zero solution of these equations, we must have

$$\begin{aligned} X_1 &= d\psi_1(m_1, m_2, m_3, m_4), \quad & Y_1 &= d\psi_2(m_1, m_2, m_3, m_4), \\ U_1 &= d\psi_3(m_1, m_2, m_3, m_4), \quad & V_1 &= d\psi_4(m_1, m_2, m_3, m_4), \end{aligned} \tag{2.8}$$

for the values of m_i, $i = 1, 2, 3, 4$ already chosen and a suitable value of d. This shows that the solution (2.6) of equation (2.3) is complete and hence it leads to a complete solution of equation (2.1).

We also note that we can suitably modify the above method to obtain the complete solution of equations of the type

$$L_1^r(x, y) L_2^s(x, y) L_3(x, y) = L_4^r(u, v) L_5^s(u, v) L_6(u, v). \tag{2.9}$$

2.2 The diophantine equation

$$\begin{aligned} L_1(x, y) L_2(x, y) L_3(x, y) L_4(x, y) \\ = L_5(u, v) L_6(u, v) L_7(u, v) L_8(u, v), \end{aligned} \tag{2.10}$$

where we assume that no two of the four forms on each side of the equation (2.10) are linearly dependent.

There exist non-zero integers a_i, b_i, $i = 1, 2, 3$ such that both $a_1L_1(x, y) + a_2L_2(x, y) + a_3L_3(x, y)$ and $b_1L_4(u, v) + b_2L_5(u, v) + b_3L_6(u, v)$ vanish identically. The nonsingular linear transformations defined by

$$\begin{aligned} a_1L_1(x, y) &= X, \quad a_2L_2(x, y) = Y, \\ b_1L_4(u, v) &= U, \quad b_2L_5(u, v) = V, \end{aligned} \tag{2.11}$$

reduce equation (2.10) to the type

$$XY(X + Y)L_9(X, Y) = UV(U + V)L_{10}(U, V). \tag{2.12}$$

It is easily seen that $XY(X+Y)$ is an eigenform of each of the following six linear transformations:

(i) $X = U$, $Y = V$,

(ii) $X = V$, $Y = U$,

(iii) $X = U$, $Y = -(U + V)$,

(iv) $X = V$, $Y = -(U + V)$,

(v) $X = -(U + V)$, $Y = U$,

(vi) $X = -(U + V)$, $Y = V$.

We can thus get six parametric solutions of equation (2.12) by the method already described and these, in turn, lead to six parametric solutions of (2.10). For determining the integers a_i we can choose any three linear forms out of the four linear forms on the lefthand side of equation (2.10), and this can be done in four different ways. Similarly for determining b_i we can choose three linear forms on the righthand side of (2.10) in four different ways. Thus, we could get a maximum of $4 \times 4 \times 6 = 96$ parametric solutions of (2.10) in this manner.

As a special case of (2.10), we note that we can solve the equation

$$\begin{aligned} &L_1(x, y)L_2(x, y)L_3(x, y)L_4(x, y) \\ &\qquad = L_1(u, v)L_2(u, v)L_3(u, v)L_4(u, v). \end{aligned} \tag{2.13}$$

2.3 The diophantine equation

$$\lambda L(x, y)Q(x, y) = \mu L(u, v)Q(u, v). \tag{2.14}$$

We assume that $L(x,y)$ is not a factor of $Q(x,y)$ as otherwise (2.14) reduces to a simpler equation of which the complete solution is readily obtained. Let $Q(x,y) = ax^2 + bxy + cy^2$ and we choose α, β and γ to be rational numbers satisfying the relation $b\alpha - a\beta + c\gamma = 0$. Then $Q(x,y)$ is an eigenform of the linear transformation defined by

$$\begin{aligned} x &= \alpha u + \beta v, \\ y &= \gamma u - \alpha v, \end{aligned} \tag{2.15}$$

and with these substitutions,

$$Q(x,y) = (\alpha^2 + \beta\gamma)Q(u,v). \tag{2.16}$$

On substituting the values of x and y given by (2.15) in (2.14), we get

$$\lambda(\alpha^2 + \beta\gamma)L(\alpha u + \beta v, \gamma u - \alpha v) = \mu L(u,v). \tag{2.17}$$

Now (2.15) and (2.17) are three linear equations in the variables x, y, u and v and their complete solution may be written, on substituting $\gamma = (a\beta - b\alpha)/c$, as follows:

$$\begin{aligned} x &= d\psi_1(\alpha,\beta), \quad & y &= d\psi_2(\alpha,\beta), \\ u &= d\psi_3(\alpha,\beta), \quad & v &= d\psi_4(\alpha,\beta), \end{aligned} \tag{2.18}$$

where α, β and d are arbitrary parameters. We will now show that this solution is complete. Let X, Y, U and V be any given solution of equation (2.14) so that

$$\lambda L(X,Y)Q(X,Y) = \mu L(U,V)Q(U,V). \tag{2.19}$$

We will show that there exist suitable rational values of α and β such that the solution (2.18) generates the given solution X, Y, U and V. We will choose α, β and γ so as to satisfy the equations

$$\begin{aligned} X &= \alpha U + \beta V, \\ Y &= \gamma U - \alpha V, \\ b\alpha - a\beta + c\gamma &= 0. \end{aligned} \tag{2.20}$$

With these values of α, β and γ, it is easily seen from (2.16) and (2.20) that

$$Q(X,Y) = (\alpha^2 + \beta\gamma)Q(U,V), \tag{2.21}$$

and hence it follows from (2.19), (2.20) and (2.21) that

$$\lambda(\alpha^2+\beta\gamma)L(\alpha U+\beta V,\gamma U-\alpha V)=\mu L(U,V). \tag{2.22}$$

Thus, X, Y, U and V satisfy the equations (2.15) and (2.17) where α and β have been chosen as mentioned above. Since (2.18) gives the complete non-zero solution of these equations, we must have

$$\begin{aligned} X&=d\psi_1(\alpha,\beta), \quad Y=d\psi_2(\alpha,\beta),\\ U&=d\psi_3(\alpha,\beta), \quad V=d\psi_4(\alpha,\beta) \end{aligned} \tag{2.23}$$

for the values of α and β already chosen and a suitable value of d. This shows that (2.18) is a complete solution of equation (2.14). We may present this complete solution in terms of integer parameters p, q and r by writing $\alpha=p/r$, $\beta=q/r$ and clearing denominators which is possible since equation (2.14) is a homogeneous equation.

2.4 The diophantine equation

$$L^r(x,y)Q^s(x,y)=L^r(u,v)Q^s(u,v). \tag{2.24}$$

We assume that the exponents r and s are relatively prime. Using the substitutions defined by (2.15) where, as before, we take α and β to be arbitrary and $\gamma=(a\beta-b\alpha)/c$, equation (2.24) reduces to

$$(\alpha^2+\beta\gamma)^sL^r(\alpha u+\beta v,\gamma u-\alpha v)=L^r(u,v) \tag{2.25}$$

which may be written as

$$\left(\alpha^2-\frac{b\alpha\beta}{c}+\frac{a\beta^2}{c}\right)^s=\left\{\frac{L(u,v)}{L(\alpha u+\beta v,\gamma u-\alpha v)}\right\}^r=t^{rs} \tag{2.26}$$

where t is some rational number. To solve (2.26), we write

$$\alpha^2-(b\alpha\beta)/c+(a\beta^2)/c=t^r, \tag{2.27}$$

$$L(u,v)=t^s\{L(\alpha u+\beta v,\gamma u-\alpha v)\}. \tag{2.28}$$

When r is even, one solution of equation (2.27) is $\alpha=t^{r/2}$, $\beta=0$ and hence the complete solution of this quadratic equation in α and

β is readily found. With these values of α and β, we get a solution for u and v from the linear equation (2.28) and thereafter x and y are determined by (2.15) where $\gamma = (a\beta - b\alpha)/c$. This gives a complete solution of equation (2.24).

When r is odd, a parametric solution of equation (2.27) may be found by writing $t = t_1^2$ when one solution of equation (2.27) is $\alpha = t_1^r$, $\beta = 0$. By solving the three linear equations (2.15) and (2.28) for the variables x, y, u and v we get a parametric solution of equation (2.24).

2.5 The diophantine equation

$$L_1^r(x,y)L_2(x,y)Q^s(x,y) = L_1^r(u,v)L_2(u,v)Q^s(u,v). \tag{2.29}$$

To solve this equation, we choose a linear transformation T such that both $L_1(x,y)$ and $Q(x,y)$ are eigenforms of this linear transformation. Then the form $L_1^r(x,y)Q^s(x,y)$ is also an eigenform of the linear transformation T and we now obtain, in general, a parametric solution of equation (2.29) by the method already described for solving the general equation (1.1). As a special case when $r = s = 1$, we can obtain, in general, a parametric solution of the equation

$$L_1(x,y)L_2(x,y)Q(x,y) = L_1(u,v)L_2(u,v)Q(u,v). \tag{2.30}$$

2.6 The diophantine equation

$$L_1(x,y)Q_1^r(x,y)Q_2^s(x,y) = L_1(u,v)Q_1^r(u,v)Q_2^s(u,v). \tag{2.31}$$

To solve this equation we choose a linear transformation T so that both $Q_1(x,y)$ and $Q_2(x,y)$ are eigenforms of the linear transformation T. Then the form $Q_1^r(x,y)Q_2^s(x,y)$ is also an eigenform of the linear transformation T and we obtain, in general, a parametric solution of equation (2.31) by the method already described for solving the general equation (1.1).

As special cases of (2.31), we can obtain, in general, parametric solutions of the following fifth degree equations:

$$L_1(x,y)Q_1(x,y)Q_2(x,y) = L_1(u,v)Q_1(u,v)Q_2(u,v), \tag{2.32}$$

$$L_1(x,y)L_2(x,y)L_3(x,y)Q(x,y) = L_1(u,v)L_2(u,v)L_3(u,v)Q(u,v), \tag{2.33}$$

$$L_1(x,y)L_2(x,y)L_3(x,y)L_4(x,y)L_5(x,y) = L_1(u,v)L_2(u,v)L_3(u,v)L_4(u,v)L_5(u,v). \tag{2.34}$$

2.7 The diophantine equation

$$L(x,y)C(x,y) = L(u,v)C(u,v). \tag{2.35}$$

If $C(x,y)$ is a reducible cubic form, equation (2.35) reduces to an equation of type (2.13), (2.24) or (2.30). If $C(x,y)$ is irreducible and its discriminant is a perfect square, then also (2.35) can be solved using eigenforms. For example, if $C(x,y) = x^3 + axy^2 + by^2$ and its discriminant $-4a^3 - 27b^2$ is a perfect square, say t^2, it can be verified by direct computation that the cubic form $C(x,y)$ is an eigenform of the linear transformation T defined by (1.2) where $\alpha = 9b + t$, $\beta = -2a^2$, $\gamma = -6a$ and $\delta = t - 9b$, and equation (2.35) can accordingly be solved. If, however, $C(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, we first apply the nonsingular linear transformation $x = X - bY/(3a)$, $y = Y$, $u = U - bV/(3a)$, $v = V$ to equation (2.35) which now reduces to the type

$$L_1(X,Y)C_1(X,Y) = L_1(U,V)C_1(U,V). \tag{2.36}$$

where $C_1(X,Y) = X^3 + AXY^2 + BY^3$. Since the discriminant is an invariant of the linear transformation applied to the cubic form $C(x,y)$, the discriminant of the cubic form $C_1(X,Y)$ is also a perfect square and we can therefore obtain a parametric solution of (2.36) as already described. This solution, in turn, leads to a parametric solution of the equation (2.35).

2.8 The diophantine equation

$$\{\Pi_{i=1}^5 L_i(x,y,z)\}Q^r(x,y,z) = \{\Pi_{i=1}^5 L_i(u,v,w)\}Q^r(u,v,w). \tag{2.37}$$

To solve equation (2.37), we first choose integers p, q, r and s, not all zero, such that we have identically

$$pL_2(x,y,z) + qL_3(x,y,z) + rL_4(x,y,z) + sL_5(x,y,z) = 0. \tag{2.38}$$

We next solve for z the equation

$$pL_2(x,y,z) + qL_3(x,y,z) = 0$$

to get $z = L(x,y)$. We also write $w = L(u,v)$. If we now write $L'_i(x,y)$, $i = 1,2,3,4,5$ and $Q'(x,y)$ to denote the forms obtained by substituting $z = L(x,y)$ in $L_i(x,y,z)$, $i = 1,2,3,4,5$ and $Q(x,y,z)$, respectively, equation (2.37) reduces to

$$\begin{aligned} &L'_1(x,y)\{L'_2(x,y)L'_4(x,y)\}^2\{Q'(x,y)\}^r \\ &\qquad\qquad = L'_1(u,v)\{L'_2(u,v)L'_4(u,v)\}^2\{Q'(u,v)\}^r \end{aligned}$$

which is a special case of equation (2.31) and a parametric solution may accordingly be obtained.

3. Solution of specific diophantine equations and problems.

3.1 The diophantine equation

$$a(x^3 + y^3) = b(u^3 + v^3). \tag{3.1}$$

This equation where a and b are non-zero integers such that gcd $(a,b) = 1$ has been considered by Oppenheim [**5**] who gave a method of obtaining rational solutions and by Choudhry [**2**] who has given a complete solution in parametric form. Solving equation (3.1) by the method described in Section 2.3, we get the complete primitive solution given below:

$$\begin{aligned} dx &= a(p^4 + 2p^3q + 3p^2q^2 + 2pq^3 + q^4) + b(p-q)r^3, \\ dy &= -a(p^4 + 2p^3q + 3p^2q^2 + 2pq^3 + q^4) + b(2p+q)r^3, \\ du &= r\{a(p^3 - q^3) + br^3\}, \\ dv &= r\{a(2p^3 + 3p^2q + 3pq^2 + q^3) - br^3\}, \end{aligned} \tag{3.2}$$

where p, q, r are arbitrary integer parameters and d is an integer so chosen that $\gcd(x, y, u, v) = 1$. This solution of (3.1) has not been published earlier.

3.2 Pythagorean triangles with areas in a given ratio. We now consider the problem of finding two Pythagorean triangles with areas in a given ratio. Dickson [**3**, pp. 174–175] has outlined the solutions to this problem given by Fermat, Euler and others, and has further stated that Euler found ten pairs of such triangles with areas in a given ratio. If we take two Pythagorean triangles with sides $2xy$, $x^2 - y^2$, $x^2 + y^2$ and $2uv$, $u^2 - v^2$, $u^2 + v^2$, their areas will be in any given ratio $a : b$ if

$$bxy(x^2 - y^2) = auv(u^2 - v^2). \tag{3.3}$$

This equation can be solved by the method indicated in Section 2.2 and we get the following solution of (3.3):

$$\begin{aligned} x &= 2ap^4q + bq^5, \quad & u &= ap^5 + 2bpq^4, \\ y &= ap^4q - bq^5, \quad & v &= ap^5 - bpq^4, \end{aligned}$$

where p and q are arbitrary parameters. We thus obtain infinitely many pairs of Pythagorean triangles with areas in a given ratio $a : b$. When $a = b$, the above solution provides examples of equiareal Pythagorean triangles. More parametric solutions of (3.3) can be obtained as indicated in Section 2.2.

3.3 The diophantine equation

$$x^4 + y^4 + z^4 = u^4 + v^4 + w^4. \tag{3.4}$$

While Dickson [**3**, pp. 653–655] mentions several parametric solutions of this equation, its complete solution is not yet known. To solve this equation, we write $z = u + v$, $w = x + y$ when (3.4) reduces to

$$xy(2x^2 + 3xy + 2y^2) = uv(2u^2 + 3uv + v^2).$$

This is an equation of type (2.30) and noting that $y(2x^2 + 3xy + 2y^2)$ is an eigenform of the linear transformation $x = m(2u + 3v)$, $y = -2mv$, we may solve this equation and hence obtain a parametric solution of

(3.4) which, on substituting $m = p/q$ and clearing denominators, may be written as follows:

$$\begin{aligned} x &= 3pq^4, & u &= 24p^4q,\\ y &= 2p(16p^4+q^4), & v &= q(16p^4+q^4),\\ z &= q(8p^4-q^4), & w &= p(32p^4-q^4). \end{aligned}$$

3.4 Arithmetic progressions of equal lengths and equal products of terms. We now consider the problem of finding two arithmetic progressions of n terms each and with equal products of terms. Partial solutions for $n = 3$ and $n = 4$ have been given by Gabovich [**4**], for $n = 4$ and $n = 5$ by Choudhry [**1**], and for arbitrary n by Mirkowski and Makowski [**6**] as well as by Choudhry [**1**].

To obtain such arithmetic progressions when $n = 4$, we have to solve the diophantine equation

$$x(x+y)(x+2y)(x+3y) = u(u+v)(u+2v)(u+3v). \tag{3.5}$$

This is a special case of equation (2.13) and, following the procedure described for solving this equation, we apply the linear transformation given by $x = X$, $y = -X - Y/2$, $u = 2U + V/2$, $v = -U - V/2$, which reduces (3.5) to the type (2.12) and we thus get six parametric solutions of equation (3.5). These solutions, written in terms of arbitrary integers p and q are given in Table 1.

TABLE 1. Solutions of equation (3.5).

x	y	u	v
$p(3p^4+q^4)$	$-p(p^4-q^4)$	$4p^4q$	$-q(p^4-q^4)$
$2p(3p^4-q^4)$	$-p(2p^4-5q^4)$	$13p^4q$	$-q(7p^4+2q^4)$
$2p(3p^4+4q^4)$	$-p(2p^4+7q^4)$	$-13p^4q$	$q(5p^4-2q^4)$
$2p(3p^4-4q^4)$	$-p(2p^4-5q^4)$	$7p^4q$	$-q(5p^4-2q^4)$
$6p(p^4+q^4)$	$-p(2p^4+7q^4)$	$-15p^4q$	$q(7p^4+2q^4)$
$p+3q$	$-q$	p	q

The last solution does not lead to any solution of our diophantine problem even though it is a nontrivial solution of equation (3.5).

Many more parametric solutions of equation (3.5) may be obtained as indicated in Section 2.2.

When $n = 5$, we have to solve the equation

$$x(x+y)(x+2y)(x+3y)(x+4y) = u(u+v)(u+2v)(u+3v)(u+4v) \tag{3.6}$$

This equation is a special case of equation (2.32). We can choose the forms $L_1(x,y)$, $Q_1(x,y)$ and $Q_2(x,y)$ in a variety of ways but these lead to only seven parametric solutions of (3.6) that generate different solutions of our diophantine problem. These solutions, written in terms of integer parameters p and q are given in Table 2.

TABLE 2. Solutions of equation (3.6).

x	y	u	v
$11pq^5$	$-p(27p^5+5q^5)$	$99p^5q$	$-q(45p^5+q^5)$
$14pq^5$	$p(27p^5-5q^5)$	$126p^5q$	$-q(45p^5+q^5)$
$5pq^5$	$p(p^5-q^5)$	$5p^5q$	$-q(p^5-q^5)$
$12p(36p^5+q^5)$	$-p(432p^5+7q^5)$	$2q(216p^5+q^5)$	$-2q(126p^5+q^5)$
$-16p(p^5-q^5)$	$p(16p^5-5q^5)$	$2q(32p^5+q^5)$	$-2q(10p^5+q^5)$
$18p(3p^5+q^5)$	$-p(54p^5+5q^5)$	$2q(81p^5+q^5)$	$-q(45p^5+2q^5)$
$-2p(8p^5-q^5)$	$8p^6$	$2q(4p^5-q^5)$	q^6

When n is arbitrary, we take the two arithmetic progressions as $x, x+y, \ldots, x+(n-1)y$, and $u+v, u+2v, \ldots, u+ny$ so that we have to solve the diophantine equation

$$\begin{aligned} x(x+y)(x+2y)\cdots\{x+(n-1)y\} \\ = (u+v)(u+2v)\cdots\{u+(n-1)v\}(u+ny) \end{aligned} \tag{3.7}$$

which is of type (1.6). As the form $[(x+y)(x+2y)\cdots\{x+(n-1)y\}]$ is an eigenform of the linear transformation $x = mu$, $y = mv$, we can obtain a parametric solution of (3.7). This parametric solution has already been given earlier by Choudhry [**1**].

3.5 Diophantine equations involving fifth powers. We will now obtain parametric solutions of diophantine equations of the type

$$X^5 + Y^5 + \lambda Z^5 = U^5 + V^5 + \lambda W^5 \tag{3.8}$$

for several integer values of λ, for example, when $\lambda = 2, 6, 8, 16, 19, 38, 44, 59, 61, 88$, etc. We will also obtain parametric solutions of the diophantine equations

$$2X^5 + 2Y^5 + Z^5 = 2U^5 + 2V^5 + W^5 \tag{3.9}$$

and

$$4X^5 + 4Y^5 + Z^5 = 4U^5 + 4V^5 + W^5. \tag{3.10}$$

To solve (3.8) when $\lambda = 2$, we write

$$\begin{aligned} X &= x + y, & U &= u + v, \\ Y &= x - y, & V &= u - v, \\ Z &= -x, & W &= -u, \end{aligned}$$

when (3.8) reduces to

$$xy^2(2x^2 + y^2) = uv^2(2u^2 + v^2)$$

which is of type (2.32) and can accordingly be solved. This leads to the following solution of (3.8) with $\lambda = 2$:

$$\begin{aligned} X &= 2p(4p^5 + q^5), & U &= q(8p^5 + q^5), \\ Y &= 2p(4p^5 - q^5), & V &= -q(8p^5 - q^5), \\ Z &= -8p^6, & W &= -q^6. \end{aligned}$$

To solve (3.8) for other values of λ, we write

$$\begin{aligned} X &= ax + y, & U &= au + v, \\ Y &= ax - y, & V &= au - v, \\ Z &= x, & W &= u, \end{aligned} \tag{3.11}$$

and choose $\lambda = \lambda(a, r) = 2a(5r^2 - a^4)$ when (3.8) reduces to

$$x(r^2x^4 + 2a^2x^2y^2 + y^4) = u(r^2u^4 + 2a^2u^2v^2 + v^4). \tag{3.12}$$

It is easily verified that the quartic form $r^2x^4 + 2a^2x^2y^2 + y^4$ is an eigenform of the linear transformation $x = mv$, $y = mru$, and with these substitutions, we get a solution of (3.12) and hence also of (3.8) in terms of the parameter m. On substituting $m = p/q$ and clearing denominators, we get the following parametric solution of (3.8):

$$\begin{aligned} X &= p(aq^5 + p^5r^3), & U &= q(ap^5r^2 + q^5), \\ Y &= p(aq^5 - p^5r^3), & V &= q(ap^5r^2 - q^5), \\ Z &= pq^5, & W &= p^5qr^2. \end{aligned}$$

Thus we have obtained a parametric solution of (3.8) whenever λ is of the type $2a(5r^2 - a^4)$. We observe that $\lambda(-3,4) = 6$, $\lambda(1,1) = 8$, $\lambda(2,2) = 16$, $\lambda(-4,16) = 608 = 32 \cdot 19$, $\lambda(1,2) = 38$, $\lambda(-4,4) = 44$, $\lambda(-4,2) = 1888 = 32 \cdot 59$, $\lambda(4,10) = 1952 = 32 \cdot 61$, $\lambda(1,3) = 88$. As any fifth power, for example, 32, which is a factor of λ may be merged with Z and W on the two sides, we get solutions of (3.8) for λ =6, 8, 16, 19, 38, 44, 59, 61, 88, etc.

Finally we note that any integer solution of

$$X^5 + Y^5 + 16Z^5 = U^5 + V^5 + 16W^5$$

leads, on multiplying by 2, to the relation

$$2X^5 + 2Y^5 + (2Z)^5 = 2U^5 + 2V^5 + (2W)^5$$

which gives a solution of (3.9). Similarly any solution of (3.8) with $\lambda = 8$, leads, on multiplying by 4, to a solution of (3.10).

3.6 The diophantine equation

$$X^7 + Y^7 + 625Z^7 = U^7 + V^7 + 625W^7. \tag{3.13}$$

We write

$$\begin{aligned} X &= 5x + 2y, & U &= 5u + 2v, \\ Y &= 5x - 2y, & V &= 5u - 2v, \\ Z &= 2x, & W &= 2u, \end{aligned} \tag{3.14}$$

when equation (3.13) reduces to

$$\begin{aligned} x(3375x^6 + 7500x^4y^2 + 2000x^2y^4 + 64y^6) \\ = u(3375u^6 + 7500u^4v^2 + 2000u^2v^4 + 64v^6). \end{aligned} \tag{3.15}$$

We note that $(3375x^6 + 7500x^4y^2 + 2000x^2y^4 + 64y^6)$ is an eigenform of the linear transformation $x = 4mv$, $y = 15mu$, and with these substitutions, we get a solution of (3.15) which, on writing $m = p/q$, leads to the following solution of (3.13):

$$\begin{aligned} X &= 5p(10125p^7 + q^7), & U &= q(16875p^7 + q^7),\\ Y &= -5p(10125p^7 - q^7), & V &= q(16875p^7 + q^7),\\ Z &= 2pq^7, & W &= 6750p^7q. \end{aligned}$$

3.7 The diophantine equation

$$x^n + x^{n-1}y + \cdots + y^n = u^n + u^{n-1}v + \cdots + v^n. \tag{3.16}$$

We will obtain a parametric solution of equation (3.16) when n is an arbitrary odd integer. We may write equation (3.16) as

$$(x^{n+1} - y^{n+1})(u - v) = (u^{n+1} - v^{n+1})(x - y). \tag{3.17}$$

When n is odd, it is easily seen that the form $(x^{n+1} - y^{n+1})$ is an eigenform of the linear transformation $x = mu$, $y = -mv$, and with these substitutions, (3.17) reduces to a linear equation in u and v which is readily solved and this leads, on writing $m = p/q$, to the following parametric solution of (3.16):

$$\begin{aligned} x &= p^{n+1} + pq^n, & u &= p^nq + q^{n+1},\\ y &= -p^{n+1} + pq^n, & v &= p^nq - q^{n+1}. \end{aligned}$$

Acknowledgments. I am grateful to the referee for his comments which have led to improvements in the paper.

REFERENCES

1. A. Choudhry, *On arithmetic progressions of equal lengths and equal products of terms*, Acta Arith. **82** (1997), 95–97.

2. ———, *The general solution of a cubic diophantine equation*, Ganita **50** (1999), 1–4.

3. L.E. Dickson, *History of theory of numbers*, Vol. 2, Chelsea Publ. Co., New York, 1952, reprint.

4. Y. Gabovich, *On arithmetic progressions with equal products of terms*, Colloq. Math. **15** (1966), 45–48.

5. A. Oppenheim, *The rational integral solution of the equation* $a(x^3+y^3) = b(u^3+v^3)$ *and allied diophantine equations*, Acta Arith. **9** (1964), 221–226.

6. Problèmes P543 et 545, R1, Colloq. Math. **19** (1968), 179–180.

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