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EXPLICIT ESTIMATES FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. We apply van der Corput's method of exponential sums to obtain explicit upper bounds for the Riemann zeta function on the line $\sigma = 1/2$. For example, we prove that if $t \geq e$, then $|\zeta(1/2+it)| \leq 3t^{1/6} \log t$. These results will be used in an application on primes to short intervals [**4**].

1. Introduction. It is well known that the distribution of prime numbers is related to the study of the Riemann zeta-function. For $\sigma > 1$, the Riemann zeta-function is defined to be the following infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where $s = \sigma + it$ with real variables σ and t.

This definition can be extended to the whole complex plane except at $s = 1$. The following definitions for $\sigma > 0$ and $s \neq 1$ can be obtained respectively by virtue of the partial and the Euler-MacLaurin summation formulae.

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{u - [u]}{u^{s+1}}\, du,$$

and

$$\zeta(s) = \frac{1}{s} + \frac{1}{2} - s \int_1^{\infty} \frac{u - [u] - 1/2}{u^{s+1}}\, du.$$

For reference, one may see [**1, 8, 13, 14**]. The following formula

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\, dx - \int_a^b f(x)\, d\bigg(x - [x] - \frac{1}{2}\bigg)$$

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is a simple form of the summation formulae (see [**3**]). For $\sigma > 1$, we apply the last formula to $\sum_{x<n\le X} n^{-s}$ for any $0 < x < 1$ and, letting $X \to \infty$, obtain

$$\zeta(s) = \frac{x^{1-s}}{1-s} + \frac{x-[x]-1/2}{x^s} - s\int_x^\infty \frac{u-[u]-1/2}{u^{s+1}}\,du, \tag{1}$$

which can also be the definition of $\zeta(s)$ for $\sigma > 0$ and $s \ne 1$.

Number theorists have devoted considerable effort to obtaining upper bounds for $\zeta(1/2+it)$. Littlewood [**11**] was the first to obtain a nontrivial bound; he used Weyl's method to prove for $t \ge e$ that

$$\zeta(1/2+it) \ll t^{1/6}\log^{3/2}(t).$$

At the time of this writing, the best known estimate is due to Huxley [**7**], who proved that

$$\zeta(1/2+it) \ll t^{89/570+\varepsilon}.$$

In this article our objective is to obtain upper bounds for $\zeta(1/2+it)$ with explicit constants. We will apply these bounds to estimates for gaps between primes [**4**]. Our approach will be to use the simplest aspects of van der Corput's method of exponential sums. This has the advantage of being simple enough to make explicit estimates possible without an extraordinary amount of effort but strong enough to make the estimates useful. We state our result in the following three theorems.

Theorem 1. *If $t \ge 0$, then*

$$|\zeta(1/2+it)| \le \frac{1}{\sqrt{4t^2+1}} + 1 + 0.267\sqrt{4t^2+1}. \tag{2a}$$

If $t \ge 2$, then

$$|\zeta(1/2+it)| \le 2t^{1/2} + 5.505. \tag{2b}$$

If $t \ge 2$ and N is a positive integer, then

$$|\zeta(1/2+it)| \le \sum_{n<N} n^{-1/2} + \frac{1}{2N^{1/2}} + \frac{2N^{1/2}}{\sqrt{4t^2+1}} + \frac{\sqrt{4t^2+1}}{24N^{3/2}} + \frac{\sqrt{4t^2+1}\sqrt{4t^2+9}}{72N^{3/2}}. \tag{2c}$$

Theorem 2. *If* $t \geq e$, *then*

$$|\zeta(1/2+it)| \leq 6t^{1/4} + 41.124.$$

Theorem 3. *If* $t \geq e$, *then*

$$|\zeta(1/2+it)| \leq 1.457t^{1/6}\log t + 40.995t^{1/6} + 1.863\log t + 123.125.$$

Theorem 1 is an easy estimate that we shall use for small values of t. The theory of exponent pairs provides a useful context for understanding the results in Theorems 2 and 3. It is well known (see, e.g., [**6**]) that if (k,l) is an exponent pair, then

$$\zeta(1/2+it) \ll t^{k/2+l/2-1/4}\log t.$$

Theorems 2 and 3 are explicit versions of this with the exponent pairs $B(0,1) = (1/2,1/2)$ and $AB(0,1) = (1/6,2/3)$, respectively.

For applications, it is convenient to combine Theorems 1 through 3 into the following result.

Corollary. *If* $0 \leq t \leq e$, *then* $|\zeta(1/2+it)| \leq 2.657$. *If* $t \geq e$, *then*

$$|\zeta(1/2+it)| \leq 3t^{1/6}\log t.$$

2. Proof of Theorem 1. The first estimate in Theorem 1 follows directly from (1). We note for any $x < X$ that

$$|\zeta(1/2)| \leq \frac{2x^{1/2}}{\sqrt{4t^2+1}} + \frac{1}{2x^{1/2}} + \frac{1}{2}\sqrt{4t^2+1} \times \left(\int_x^X \frac{|u-[u]-1/2|}{u^{3/2}}\,du + \frac{1}{2}\int_X^\infty \frac{du}{u^{3/2}}\right).$$

We choose $x = 1/4$ and $X = 28$ and get the stated result. □

Most of our other estimates will require the following lemma, which we quote from [**2**].

Lemma 1. *If $\sigma \geq 1/2$ and $t \geq 2$, then*

$$\zeta(s) = \sum_{n=1}^{[t]} \frac{1}{n^s} + C(s)$$

where $|C(s)| \leq 5.505$.

Now from Lemma 1, if $s = 1/2 + it$, then

$$\left| \sum_{n \leq t^\beta} \frac{1}{n^s} \right| \leq \int_0^{t^\beta} \frac{du}{u^{1/2}} \leq 2t^{\beta/2}. \tag{3}$$

This estimate in the case of $\beta = 1$ together with Lemma 1 proves estimate (2b) of Theorem 1.

To prove estimate (2c), we begin by noticing that if $\sigma > 1$ and N is a positive integer, then

$$\begin{aligned}\zeta(s) &= \sum_{n<N} n^{-s} + \int_{N^-}^{\infty} u^{-s}\, d[u] \\ &= \sum_{n<N} n^{-s} + \int_{N}^{\infty} u^{-s}\, du - \int_{N^-}^{\infty} u^{-s}\, d(u - [u] - 1/2).\end{aligned}$$

After integrating by parts, we find that

$$\zeta(s) = \sum_{n<N} n^{-s} + \frac{1}{2} N^{-s} + \frac{N^{1-s}}{s-1} - s\int_N^{\infty} (u - [u] - 1/2) u^{-s-1}\, du. \tag{4}$$

The last integral converges for $\sigma > 0$ so (4) defines an analytic continuation of $\zeta(s)$ in the half-plane $\sigma > 0$.

Now let $B_1(u) = u - 1/2$ and $B_2(u) = u^2 - u - 1/6$ be the first two Bernoulli polynomials, and set $\overline{B}_k(u) = B_k(u - [u])$. Then the integral in (4) may be written as

$$-s\int_N^{\infty} u^{-s-1}\overline{B}_1(u)\, du = -\frac{1}{2}\, s\int_N^{\infty} u^{-s-1}\, d\overline{B}_2(u).$$

After integrating this by parts and substituting into (4), we get

$$\zeta(s) = \sum_{n<N} n^{-s} + \frac{1}{2}N^{-s} + \frac{N^{1-s}}{s-1} - \frac{s}{12N^{s+1}} + \frac{1}{(s+1)N^{s+1}} - \frac{s(s+1)}{2}\int_N^\infty \frac{\overline{B}_2(u)}{u^{s+2}}\,du.$$

Estimate (2c) now follows by setting $s = 1/2 + it$ and noting that

$$\left|\int_N^\infty \frac{\overline{B}_2(u)}{u^{s+2}}\,du\right| \le \frac{1}{6}\int_N^\infty \frac{du}{u^{5/2}} = \frac{1}{9N^{3/2}}$$

when $\sigma = 1/2$.

We use Theorem 1 in the proof of the Corollary for small values of t. When $0 \le t \le e$, estimate (2a) gives $|\zeta(1/2+it)| \le 2.657$. For $e \le t \le 30$, estimate (2a) gives $|\zeta(1/2+it)| \le 3t^{1/6}\log t$. The same estimate follows from estimate (2b) for $30 \le t \le 700$. In the range $700 \le t \le 5100$, the Corollary follows by using estimate (2c) with $N = [t/4] + 1$.

3. Van der Corput's method. The following Kusmin-Landau lemma, see [**5**], gives a nontrivial estimate on the exponential sums under some conditions.

Lemma 2. *Suppose f is a continuously differentiable real-valued function with a monotonic derivative and $\|f'\| \ge U^{-1}$ for some positive real number U on the interval $(a,b]$. Then*

$$|S| = \left|\sum_{n\in(a,b]} e^{2\pi i f(n)}\right| \le \frac{1}{\pi}U + 1.$$

Proof. If we replace f by $-f$ in the last sum, then we get the conjugate for the whole sum, which has the same absolute value as the whole sum. Thus, we may assume that f' is increasing. Since f' is continuous by our assumption, there must exist an integer k such that

$$k + U^{-1} \le f'(n) \le k + 1 - U^{-1},$$

for all n's in the interval. Note that

$$\left|\sum_{n=L}^{M} e^{2\pi i f(n)}\right| = \left|e^{-kn}\sum_{n=L}^{M} e^{2\pi i f(n)}\right| = \left|\sum_{n=L}^{M} e^{2\pi i f(n)-kn}\right|.$$

We may further assume that $U^{-1} \le f'(n) \le 1 - U^{-1}$.

Define $g(n) = f(n+1) - f(n)$. We have $g'(n) = f'(n+1) - f'(n)$. The assumption that f' is increasing implies that $g' > 0$, so that g is increasing. Furthermore, using the mean-value theorem, we know that $g(n) = f'(m_n)$ for some $n \le m_n \le n+1$. It follows that

$$U^{-1} \le f'(n) \le g(n) \le f'(n+1) \le 1 - U^{-1}. \tag{5}$$

Denote

$$G(n) = \frac{1}{1 - e^{2\pi i g(n)}} = \frac{1}{2} + \frac{1}{2}\, i\cot(\pi g(n)).$$

Then

$$G(n)(e^{2\pi i f(n)} - e^{2\pi i f(n+1)}) = \frac{e^{2\pi i f(n)} - e^{2\pi i f(n+1)}}{1 - e^{2\pi i g(n)}} = e^{2\pi i f(n)},$$

and

$$G(n) - G(n-1) = \frac{1}{2i}\{\cot(g(n-1)) - \cot(g(n))\}.$$

It follows that

$$\begin{aligned}
\left|\sum_{n=L}^{M} e^{2\pi i f(n)}\right| &= \left|\sum_{n=L}^{M-1} (e^{2\pi i f(n)} - e^{2\pi i f(n+1)})G(n) + e^{2\pi i f(M)}\right| \\
&= \Bigg|e^{2\pi i f(L)}G(L) + \sum_{n=L+1}^{M-1} e^{2\pi i f(n)}(G(n) - G(n-1)) \\
&\qquad + e^{2\pi i f(M)}(1 - G(M-1))\Bigg| \\
&\le \left|\sum_{n=L+1}^{M-1} (G(n) - G(n-1))e^{2\pi i f(n)}\right| \\
&\qquad + |G(L)e^{2\pi i f(L)}| + |(1 - G(M-1))e^{2\pi i f(M)}| \\
&\le \sum_{n=L+1}^{M-1} |G(n) - G(n-1)| + |G(L)| + |1 - G(M-1)|.
\end{aligned}$$

Recalling (5) and noting that $\cot(x)$ is a decreasing function for $0 < x < \pi$ and the values of $G(n) - G(n-1)$ have the same sign for all n's, we get

$$\begin{aligned}\Big|\sum_{n=L}^{M} e^{2\pi i f(n)}\Big| &\le \frac{1}{2}\{\cot(\pi g(L)) - \cot(\pi g(M-1))\}\\ &\quad + \frac{1}{2} + \frac{1}{2}\cot(\pi g(L)) + \frac{1}{2} + \frac{1}{2}\cot(g(M-1))\\ &= \cot(\pi g(L)) + 1.\end{aligned}$$

We use the fact that $\cot(x) < 1/x$ for $0 < x \le \pi/2$, getting

$$\Big|\sum_{n=L}^{M} e^{2\pi i f(n)}\Big| \le \frac{1}{\pi\|g(L)\|} + 1 \le \frac{1}{\pi U^{-1}} + 1,$$

which proves the lemma. □

The next lemma shall be used to estimate the "zeta sums," defined as those sums on the right sides of (8) and (17).

Lemma 3. *Assume that f is a real-valued function with two continuous derivatives on $[N+1, N+L]$. If there exist two real numbers $V < W$ with $W > 1$ such that*

$$\frac{1}{W} \le |f''(x)| \le \frac{1}{V} \tag{6}$$

for x on $[N+1, N+L]$, then

$$\Big|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\Big| \le \frac{1}{5}\Big(\frac{L}{V} + 1\Big)(8W^{1/2} + 15).$$

Proof. The condition (6) implies that either $f''(x) > 0$ or $f''(x) < 0$ for $N+1 \le x \le N+L$. Without loss of generality, we may assume that $f''(x) > 0$ for $N+1 \le x \le N+L$. Under this assumption we know that $f'(x)$ is increasing and by the mean-value theorem we have, for some

$N+1 \le x_0 \le N+L$, that $f'(N+L) - f'(N+1) = f''(x_0)L \le L/V$. Denote $C_1 = [f'(N+1)]$, $C_j = C_{j-1}+1$ for $j = 2,3,\dots,k-1$, and $C_k = \lceil f'(N+L)\rceil$. Then $k \le L/V + 2$.

Let Δ, $0 < \Delta < 1/2$, be a parameter to be chosen later, and let

$$\begin{aligned} x_j &= (f')^{-1}(C_j - \Delta), \quad \text{for } j = 2,3,\dots,k \\ y_j &= (f')^{-1}(C_j + \Delta), \quad \text{for } j = 1,2,\dots,k-1, \end{aligned}$$

and

$$z_j = (f')^{-1}(C_j), \quad \text{for } j = 1,2,\dots,k.$$

On each interval $[y_j, x_{j+1}]$, we have $\|f'(x)\| \ge \Delta$. We have $k-1$ such intervals; the sub-sum corresponding to each of them, by Lemma 2, is bounded by $1/(\pi\Delta)+1$.

On all other $2(k-1)$ intervals, we use the trivial estimate

$$\left| \sum_{a \le n \le b} e^{2\pi i f(n)} \right| \le b - a + 1.$$

By the mean-value theorem, we have

$$f^{-1}(C_j + \Delta) - f^{-1}(C_j) = \frac{1}{f'(x_0)}(C_j + \Delta - C_j) \le W\Delta,$$

and

$$f^{-1}(C_j) - f^{-1}(C_j - \Delta) = \frac{1}{f'(x_0)}(C_j - C_j + \Delta) \le W\Delta.$$

Thus, we get

$$\begin{aligned} \left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right| &\le (L/V+1)(1/(\pi\Delta)+1) + 2(L/V+1)(W\Delta+1) \\ &= (L/V+1)(1/(\pi\Delta) + 2W\Delta + 3). \end{aligned}$$

Taking $\Delta = 1/\sqrt{2\pi W}$ and noting that $2\sqrt{2}/\sqrt{\pi} < 8/5$, we obtain the claimed result. $\square$

4. Estimate for sums over those n's between t^α and t. In this section we will use Lemma 3 to obtain the following estimates.

Lemma 4. *If $t \ge e$, then*

$$\Big|\sum_{t^{2/3}<n\le t} n^{-1/2-it}\Big| \le 1.5t^{1/6} + 101.006,$$

and

$$\Big|\sum_{t^{1/2}<n\le t} n^{-1/2-it}\Big| \le 4t^{1/4} + 35.619.$$

Proof. For this proof, the number α is either $2/3$ or $1/2$. We let τ (> 1) and t_1 ($> e$) be positive constants whose values will be determined later. We shall apply Lemma 3 when $t \ge t_1$ and use a trivial estimate for $e \le t \le t_1$.

We let $X_j = \tau^j t^\alpha$ and $N_j = [X_j]$ for $j = 0, 1, \dots, J$ with $j \le [(1-\alpha)\log t/\log\tau] + 1$. It follows that

$$\sum_{t^\alpha<n\le t} \frac{1}{n^s} = \sum_{j=1}^{J} \sum_{n=N_{j-1}+1}^{\min\{N_j,t\}} \frac{1}{n^s}. \tag{7}$$

Using the partial summation formula for each inner sum in the last expression, we get

$$\Big|\sum_{n=N_{j-1}+1}^{\min\{N_j,t\}} \frac{1}{n^s}\Big| \le \frac{1}{(N_{j-1}+1)^{1/2}} \max_{L\le N_j-N_{j-1}} \Big|\sum_{n=N_{j-1}+1}^{N_{j-1}+L} e^{-it\log n}\Big|. \tag{8}$$

Let $f(x) = -t\log x/2\pi$. For $X_{j-1} < N_{j-1}+1 \le x \le N_j \le X_j$, we have

$$\frac{t}{2\pi\tau^2 X_{j-1}^2} = \frac{t}{2\pi X_j^2} \le |f''(x)| = \left|\frac{t}{2\pi x^2}\right| < \frac{t}{2\pi X_{j-1}^2}.$$

Applying Lemma 3 by letting $V = 2\pi X_{j-1}^2/t$ and $W = 2\pi\tau^2 X_{j-1}^2/t$ to the zeta sums on the right side of (8) and noting that

$L \le N_j - N_{j-1} < X_j - X_{j-1} + 1 \le (\tau - 1)X_{j-1} + 1$, we get

$$\begin{aligned} \Bigg|\sum_{n=N_{j-1}+1}^{N_{j-1}+L} e^{-it\log n}\Bigg| &\le \frac{1}{5}\bigg(\frac{(\tau-1)t}{2\pi X_{j-1}} + \frac{t}{2\pi X_{j-1}^2} + 1\bigg) \\ &\qquad \times \bigg(\frac{2^{7/2}\pi^{1/2}\tau X_{j-1}}{t^{1/2}} + 15\bigg) \\ &= \frac{1}{5}(K_1 + K_2 + K_3 + K_4 + K_5 + K_6), \end{aligned} \tag{9}$$

where

$$K_1 = \frac{2^{5/2}\tau(\tau-1)t^{1/2}}{\pi^{1/2}}, \quad K_2 = \frac{2^{5/2}\tau t^{1/2}}{\pi^{1/2}X_{j-1}}, \quad K_3 = \frac{2^{7/2}\pi^{1/2}\tau X_{j-1}}{t^{1/2}},$$
$$K_4 = \frac{15(\tau-1)t}{2\pi X_{j-1}}, \qquad K_5 = \frac{15t}{2\pi X_{j-1}^2}, \qquad K_6 = 15.$$

Noting $N_{j-1} + 1 > X_{j-1}$ and combining (7), (8) and (9), we obtain

$$\Bigg|\sum_{t^\alpha < n \le t} \frac{1}{n^s}\Bigg| \le \frac{1}{5}\sum_{j=1}^{J} \frac{K_1 + \cdots + K_6}{X_{j-1}^{1/2}}.$$

In the term K_3 we have to sum $X_{j-1}^{1/2}$. In this case

$$\sum_{j=1}^{J} X_{j-1}^{1/2} = t^{\alpha/2}\sum_{j=0}^{J-1}\tau^{j/2} \le \frac{\tau^{J/2}t^{\alpha/2}}{\tau^{1/2}-1} \le \frac{\tau^{1/2}t^{1/2}}{\tau^{1/2}-1}. \tag{10}$$

In other cases we note that, for any $0 < \delta < 2$,

$$\sum_{j=1}^{J} X_{j-1}^{-\delta/2} \le \sum_{j=0}^{\infty} X_j^{-\delta/2} = \sum_{j=0}^{\infty}(\tau^j t^\alpha)^{-\delta/2} = \frac{\tau^{\delta/2}}{(\tau^{\delta/2}-1)}t^{-\delta\alpha/2}. \tag{11}$$

We wind up with the estimate

$$\Bigg|\sum_{t^\alpha < n \le t} n^{-s}\Bigg| \le \frac{1}{5}(L_1 + \cdots + L_6), \tag{12}$$

where

$$L_1 = \frac{2^{5/2}\tau^{3/2}(\tau^{1/2}+1)}{\pi^{1/2}} t^{(1-\alpha)/2}, \quad L_2 = \frac{2^{5/2}\tau^{5/2}}{\pi^{1/2}(\tau^{3/2}-1)} t^{(1-3\alpha)/2},$$
$$L_3 = \frac{2^{7/2}\pi^{1/2}\tau^{3/2}}{\tau^{1/2}-1}, \qquad L_4 = \frac{15\tau^{1/2}(\tau^{1/2}+1)}{2\pi} t^{(2-3\alpha)/2},$$
$$L_5 = \frac{15\tau^{5/2}}{2\pi(\tau^{5/2}-1)} t^{(2-5\alpha)/2}, \qquad L_6 = \frac{15\tau^{1/2}}{(\tau^{1/2}-1)} t^{\alpha/2}.$$

We have used the fact $X_{j-1} \geq t^{\alpha}$. When $\alpha = 2/3$ and $t \geq t_1$, we get

$$\left| \sum_{t^{2/3} < n \leq t} n^{-s} \right| \leq c_1 t^{1/6} + c_2, \tag{13}$$

where

$$c_1 = k_1 = \frac{2^{5/2}\tau^{3/2}(\tau^{1/2}+1)}{5\pi^{1/2}}, \quad c_2 = k_3 + k_4 + \frac{k_2}{t_1^{1/2}} + \frac{k_5}{t_1^{2/3}} + \frac{k_6}{t_1^{1/3}}$$

with

$$k_2 = \frac{2^{5/2}\tau^{5/2}}{5\pi^{1/2}(\tau^{3/2}-1)}, \quad k_3 = \frac{2^{7/2}\pi^{1/2}\tau^{3/2}}{5(\tau^{1/2}-1)}, \quad k_4 = \frac{3\tau^{1/2}(\tau^{1/2}+1)}{2\pi},$$
$$k_5 = \frac{3\tau^{5/2}}{2\pi(\tau^{5/2}-1)}, \qquad k_6 = \frac{3\tau^{1/2}}{\tau^{1/2}-1}.$$

Taking $\tau = 1.096$ and $t_1 = 48449$ gives (13) with $c_1 = 1.5$ and $c_2 = 101.006$, provided $t \geq t_1$. When $e < t \leq t_1$, we use the estimate

$$\left| \sum_{t^{\alpha} < n \leq t} \frac{1}{n^s} \right| \leq \sum_{t^{\alpha} < n \leq t} \frac{1}{n^{1/2}} \leq \int_1^t \frac{du}{u^{1/2}} \leq 2t_1^{1/2}. \tag{14}$$

This completes the proof of the first part of Lemma 4.

Similarly, in the case of $\alpha = 1/2$, we use (12) when $t \geq t_1'$ to obtain

$$\left| \sum_{t^{2/3} < n \leq t} n^{-s} \right| \leq d_1 t^{1/4} + d_2, \tag{15}$$

with

$$d_1 = k_1 + k_4 \quad \text{and} \quad d_2 = k_3 + \frac{k_2 + k_4 + k_6}{t_1'^{1/4}}.$$

Taking $\tau' = 1.088$ and $t_1' = 4773$, we have $d_1 = 2.5$ and $d_2 = 115.417$. When $e \le t < t_1'$, the same upper bound follows from (14). We may also take $\tau' = 1.523$ and $t_1' = 882$ to get $d_1 = 4$ and $d_2 = 35.619$.

5. Weyl–van der Corput lemma. To deal with zeta sums over those n's between $t^{1/3}$ and $t^{2/3}$ by applying van der Corput's method, we first use Corput's version of "Weyl differencing." For references, one may see [**5, 6, 8, 12, 15**].

Let M be a positive integer. We have

$$M \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} = \sum_{m=1}^{M} \sum_{n=N+1-m}^{N+L-m} e^{2\pi i f(n+m)}.$$

Interchanging the order of the summation, we get

$$M \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} = \sum_{n=N+1-M}^{N+L-1} \sum_{m=\max\{N+1-n,1\}}^{\min\{M,N+L-n\}} e^{2\pi i f(n+m)}.$$

Thus we have

$$M^2 \left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 = \left| \sum_{n=N+1-M}^{N+L-1} \sum_{m=\max\{N+1-n,1\}}^{\min\{M,N+1-n\}} e^{2\pi i f(n+m)} \right|^2.$$

Regarding the inner sum over m for each fixed n as b_n and $a_n = 1$ and using Cauchy's inequality $|\sum_n a_n b_n|^2 \le |\sum_n |a_n|^2 \sum_n |b_n|^2$, we obtain

$$M^2 \left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 < (L+M) \sum_{n=N+1-M}^{N+L-1} \left| \sum_{m=\max\{N+1-n,1\}}^{\min\{M,N+L-n\}} e^{2\pi i f(n+m)} \right|^2.$$

We then use the inequalities that $\overline{e^{2\pi i f}} = e^{-2\pi i f}$ and $|z|^2 = z\bar{z}$ for the last sum over m in the above inequality and interchange the order of the summation, getting

$$\begin{aligned}
&M^2\bigg|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\bigg|^2 \\
&\le (L+M)\sum_{n=N+1-M}^{N+L-1}\ \sum_{m_1=\max\{N+1-n,1\}}^{\min\{M,N+L-n\}} \\
&\quad\times \sum_{m_2=\max\{N+1-n,1\}}^{\min\{M,N+L-n\}} e^{2\pi i[f(n+m_1)-f(n+m_2)]} \\
&= (L+M)\sum_{m_1=1}^{M}\sum_{m_2=1}^{M}\ \sum_{n=\max\{N+1-m_1,N+1-m_2\}}^{\min\{N+L-m_1,N+L-m_2\}} e^{2\pi i[f(n+m_1)-f(n+m_2)]}.
\end{aligned}$$

We calculate the inner sum for $m_1 = m_2$ and $m_1 \neq m_2$ separately. If $m_1 = m_2$, then the involved terms contribute to

$$\sum_{m=1}^{M}\sum_{n=N+1-m}^{N+L-m} 1 = ML.$$

If $m_1 \neq m_2$, then the sum over the involved terms is twice that of the corresponding one under the condition $m_1 > m_2$. We denote $m = m_1 - m_2$. For each fixed m, the equation $m_1 - m_2 = m$ has $M - m$ solutions for the ordered pair $\langle m_1, m_2\rangle$ under the condition $1 \le m_1 \le M$ and $1 \le m_2 \le M$, which are $\langle M, m-m\rangle, \dots, \langle m+1, 1\rangle$. Let us change the variable $n+m_2$ to n. Thus the involved terms under the condition $m_1 \neq m_2$ contribute to

$$2\sum_{m=1}^{M-1}(M-m)\sum_{n=N+1}^{N+L-m} e^{2\pi i[f(n+m)-f(n)]}.$$

It follows that

$$\begin{aligned}
&M^2\bigg|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\bigg|^2 \\
&\quad\le (L+M)\bigg\{ML + 2\sum_{m=1}^{M-1}(M-m)\bigg|\sum_{n=N+1}^{N+L-m} e^{2\pi i[f(n+m)-f(n)]}\bigg|\bigg\}.
\end{aligned}$$

Dividing the factor M^2 on both sides and denoting the last sum in the last expression by S', we get the following lemma.

Lemma 5. *Let $f(n)$ be a real-valued function and M a positive integer. Then*

$$\Bigl|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\Bigr|^2 \le \frac{(L+M)L}{M}+\frac{2(L+M)}{M}\sum_{m=1}^{M-1}\Bigl(1-\frac{m}{M}\Bigr)\max_{K\le L}|S'_m(K)|,$$

where

$$S'_m(K)=\sum_{n=N+1}^{N+K} e^{2\pi i[f(n+m)-f(n)]}.$$

6. Estimate for those n's between $t^{1/3}$ and $t^{2/3}$.

Lemma 6. *If $t \ge e$, then*

$$\Bigl|\sum_{t^{1/3}<n\le t^{2/3}} n^{-1/2-it}\Bigr| \le 1.457t^{1/6}\log t+37.495t^{1/6}+1.863\log t+16.614.$$

Note that $t^{1/3}\ge 1$. For small t, we use the trivial estimate

$$\Bigl|\sum_{t^{1/3}<n\le t^{2/3}}\frac{1}{n^s}\Bigr| \le \sum_{1<n\le t^{2/3}}\frac{1}{n^{1/2}} \le \int_1^{t^{2/3}}\frac{du}{u^{1/2}} \le 2t^{1/3}.$$

This suffices to prove the lemma when $e \le t \le t_2$, where $t_2 = 2.028\times 10^{13}$. Henceforth, we assume that $t \ge t_2$. As in Section 3 we let κ, $\kappa>1$, be a positive constant to be determined later. We let $X_j=\kappa^j t^{1/3}$ for $j=0,1,\dots,J$ and $N_j=[X_j]$ so that $X_J\ge t^{2/3}$ where $J<(\log t/3\log\kappa)+1$. Thus,

$$\sum_{t^{1/3}<n\le t^{2/3}}\frac{1}{n^s}=\sum_{j=1}^{J}\sum_{n=N_{j-1}+1}^{\min\{N_j,t^{2/3}\}}\frac{1}{n^s}, \tag{16}$$

and

$$\left|\sum_{n=N_{j-1}+1}^{\min\{N_j,t^{2/3}\}} \frac{1}{n^s}\right| \le \frac{1}{X_{j-1}^{1/2}} \max_{L\le N_j-N_{j-1}} \left|\sum_{n=N_{j-1}+1}^{N_{j-1}+L} e^{-it\log n}\right|. \tag{17}$$

Instead of using Lemma 3 only, we need to use Lemma 5 first in this case. We denote the last sum by S_j, let $f(x) = -(t/2\pi)\log x$ and note that $L \le N_j - N_{j-1} \le (\kappa-1)X_{j-1}+1$. We note here that the choice of M will be subject to $2 \le \gamma \le M := \gamma X_{j-1}/t^{1/3}$, where γ is a constant whose value will be determined later. Applying Lemma 5 for each j, we get

$$\begin{aligned}|S_j| \le{}& \frac{((\kappa-1)X_{j-1}+1+\gamma t_2^{-1/3}X_{j-1})^{1/2}((\kappa-1)X_{j-1}+1)^{1/2}}{M^{1/2}}\\ &+\frac{2^{1/2}(\kappa X_{j-1}+1)^{1/2}}{M^{1/2}}\left(\sum_{m=1}^{M-1}\left(1-\frac{m}{M}\right)\max_{K\le L}|S'_m(K)|\right)^{1/2},\end{aligned} \tag{18}$$

where

$$S'_m(K) = \sum_{n=N_{j-1}+1}^{N_{j-1}+K} e^{-it[\log(n+m)-\log(n)]};$$

here we have used the inequality

$$(a+b)^{1/2} \le a^{1/2}+b^{1/2} \tag{19}$$

for any $a \ge 0$ and $b \ge 0$.

To estimate $S'_m(K)$ we apply Lemma 3 by letting $f(x) = -(t/2\pi)[\log(x+m)-\log(x)]$. Note that

$$f'(x) = \frac{t}{2\pi}\left(\frac{1}{x}-\frac{1}{x+m}\right),$$

and

$$f''(x) = -\frac{t}{2\pi}\left(\frac{1}{x^2}-\frac{1}{(x+m)^2}\right) = -\frac{mt}{\pi(x+m_0)^3}$$

for some m_0, $0 \le m_0 \le m \le M$. Putting $V = \pi X_{j-1}^3/mt$ and $W = \pi(\kappa+1)^3 X_{j-1}^3/mt$, we have $1/W \le |f''(x)| \le 1/V$ for $X_{j-1} < N_{j-1}+1 \le x \le N_{j-1}+K \le X_j$. It follows that

$$\max_{K\le L} |S'_m(K)| \le \frac{1}{5}\bigg(m^{1/2}(H_1+H_2) + \frac{H_3}{m^{1/2}} + m(H_4+H_5)+H_6\bigg),$$

where

$$H_1 = \frac{8(\kappa-1)(\kappa+1)^{3/2}t^{1/2}}{\pi^{1/2}X_{j-1}^{1/2}}, \quad H_2 = \frac{8(\kappa+1)^{3/2}t^{1/2}}{\pi^{1/2}X_{j-1}^{3/2}},$$
$$H_3 = \frac{8\pi^{1/2}(\kappa+1)^{3/2}X_{j-1}^{3/2}}{t^{1/2}}, \quad H_4 = \frac{15(\kappa-1)t}{\pi X_{j-1}^2},$$
$$H_5 = \frac{15t}{\pi X_{j-1}^3}, \quad H_6 = 15.$$

Next we consider

$$(20) \quad \sum_{m=1}^{M-1}\bigg(1-\frac{m}{M}\bigg)\max_{K\le L}|S'_m(K)| \le \frac{1}{5}\sum_{m=1}^{M-1}\bigg(\bigg(1-\frac{m}{M}\bigg)m^{1/2}(H_1+H_2) + \frac{H_3}{m^{1/2}} + m(H_4+H_5)+H_6\bigg).$$

We recall a standard result in numerical analysis from [**11**].

Lemma 7. *Let $f : C[a,b] \to R$ be twice continuously differentiable. Then the error for the trapezoidal rule can be represented in the form*

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\,[f(a)+f(b)] - \frac{(b-a)^3}{12}\,f''(\xi)$$

with some $\xi \in [a,b]$.

We are going to use the last formula in the form of the inequality

$$\int_{k-1}^{k} f(x)\,dx \ge \frac{1}{2}[f(k-1)+f(k)],$$

with $k-1$ and k in places of a and b, respectively, and the condition that $f''(x) < 0$.

For $f(x) = [1-(x/M)]x^{1/2}$, we sum from $k=2$ to $k=M$ to avoid the problem of f not being differentiable at 0. We get

$$\int_1^M f(x)\,dx \geq \sum_{1\leq m\leq M-1} f(m) - \frac{1}{2}\,[f(1)+f(M)],$$

or

$$\frac{4}{15}\,M^{3/2} - \frac{2}{3} + \frac{2}{5M} \geq \sum_{1\leq m\leq M-1} f(m) - \frac{1}{2} + \frac{1}{2M}.$$

With a little algebra, we then get

$$\sum_{1\leq m\leq M-1}\left(1-\frac{m}{M}\right)m^{1/2} \leq \frac{4}{15}\,M^{3/2} - \frac{1}{6} - \frac{1}{10M} \leq \frac{4}{15}\,M^{3/2}$$

for the sum of the first two terms on the right side of (20). We have

$$\sum_{m=1}^{M-1}\left(1-\frac{m}{M}\right)m^{-1/2} \leq \int_0^M\left(1-\frac{u}{M}\right)\frac{1}{u^{1/2}}\,du \leq \frac{4}{3}\,M^{1/2}$$

for the sum of the third one in that. We then utilize

$$\sum_{m=1}^{M-1}\left(1-\frac{m^{1/2}}{M}\right)m = \frac{1}{6}\,M^2 - \frac{1}{6} \leq \frac{1}{6}\,M^2$$

for the sum of the fourth and fifth terms. Corresponding to the term H_6, it is easy to have

$$\sum_{m=1}^{M-1}\left(1-\frac{m}{M}\right) = \frac{1}{2}\,M.$$

It follows, if we denote the sum on the left of the inequality (20) by S_0, that

(21)

$$S_0 \leq \frac{1}{5}\left(\frac{4}{15}\,M^{3/2}(H_1+H_2) + \frac{4}{3}\,M^{1/2}H_3 + \frac{1}{6}\,M^2(H_4+H_5) + \frac{1}{2}\,MH_6\right).$$

Recalling (18) and (20), we obtain

$$|S_j| \le \frac{(\kappa X_{j-1}+1)^{1/2}((\kappa-1)X_{j-1}+1)^{1/2}}{\gamma^{1/2}X_{j-1}^{1/2}}t^{1/6} + \frac{2^{1/2}(\kappa X_{j-1}+1)^{1/2}}{\gamma^{1/2}X_{j-1}^{1/2}}t^{1/6}S_0^{1/2}.$$

Also recall (16), (17) and (21). For the factor S_0 we shall use an inequality similar to (19) in six components instead of two. We also need several similar inequalities with the same or different summands to (10) and (11) and recall that $X_{j-1} \ge t^{1/3}$. For $t \ge t_2$, we wind up with

$$\left|\sum_{t^{1/3}<n\le t^{2/3}} \frac{1}{n^s}\right| \le U_0+U_1+\cdots+U_6,$$

where

$$U_0 = \frac{(\kappa+(\gamma+1)t_2^{-1/3}-1)^{1/2}(\kappa-1+t_2^{-1/3})^{1/2}}{\gamma^{1/2}}\left(\frac{\log t}{3\log\kappa}+1\right)t^{1/6},$$

$$U_1 = \frac{8\gamma^{1/4}(\kappa-1)^{1/2}(\kappa+1)^{3/4}(\kappa+t_2^{-1/3})^{1/2}}{3^{1/2}5\pi^{1/4}}\left(\frac{\log t}{3\log\kappa}+1\right)t^{1/6},$$

$$U_2 = \frac{8\gamma^{1/4}\kappa^{1/2}(\kappa+1)^{3/4}(\kappa+t_2^{-1/3})^{1/2}}{3^{1/2}5\pi^{1/4}(\kappa^{1/2}-1)},$$

$$U_3 = \frac{8\pi^{1/4}\kappa^{1/2}(\kappa+1)^{3/4}(\kappa+t_2^{-1/3})^{1/2}}{3^{1/2}5^{1/2}\gamma^{1/4}(\kappa^{1/2}-1)}t^{1/6},$$

$$U_4 = \frac{\gamma^{1/2}\kappa^{1/2}(\kappa-1)^{1/2}(\kappa+t_2^{-1/3})^{1/2}}{\pi^{1/2}(\kappa^{1/2}-1)}t^{1/6},$$

$$U_5 = \frac{\gamma^{1/2}\kappa(\kappa+t_2^{-1/3})^{1/2}}{\pi^{1/2}(\kappa-1)},$$

and

$$U_6 = 3^{1/2}(\kappa+t_2^{-1/3})^{1/2}\left(\frac{\log t}{3\log\kappa}+1\right).$$

We conclude that, for $t \ge t_2$,

$$\left|\sum_{t^{1/3}<n\le t^{2/3}} \frac{1}{n^s}\right| \le c_3t^{1/6}\log t + c_4t^{1/6} + c_5\log t + c_6, \tag{22}$$

where $c_3 = c_{30} + c_{31}$, $c_4 = c_{40} + c_{41} + c_{43} + c_{44}$, $c_5 = c_{56}$ and $c_6 = c_{62} + c_{65} + c_{66}$, with

$$c_{30} = \frac{(\kappa + (\gamma+1)t_2^{-1/3} - 1)^{1/2}(\kappa - 1 + t_2^{-1/3})^{1/2}}{3\gamma^{1/2}\log\kappa},$$

$$c_{31} = \frac{8\gamma^{1/4}(\kappa-1)^{1/2}(\kappa+1)^{3/4}(\kappa + t_2^{-1/3})^{1/2}}{3^{3/2}5\pi^{1/4}\log\kappa},$$

$$c_{40} = \frac{(\kappa + (\gamma+1)t_2^{-1/3} - 1)^{1/2}(\kappa - 1 + t_2^{-1/3})^{1/2}}{\gamma^{1/2}},$$

$$c_{41} = \frac{8\gamma^{1/4}(\kappa-1)^{1/2}(\kappa+1)^{3/4}(\kappa + t_2^{-1/3})^{1/2}}{3^{1/2}5\pi^{1/4}},$$

$$c_{43} = \frac{8\pi^{1/4}\kappa^{1/2}(\kappa+1)^{3/4}(\kappa + t_2^{-1/3})^{1/2}}{3^{1/2}5^{1/2}\gamma^{1/4}(\kappa^{1/2}-1)},$$

$$c_{44} = \frac{\gamma^{1/2}\kappa^{1/2}(\kappa-1)^{1/2}(\kappa + t_2^{-1/3})^{1/2}}{\pi^{1/2}(\kappa^{1/2}-1)},$$

$$c_{56} = \frac{(\kappa + t_2^{-1/3})^{1/2}}{3^{1/2}\log\kappa}, \quad c_{62} = \frac{8\gamma^{1/4}\kappa^{1/2}(\kappa+1)^{3/4}(\kappa + t_2^{-1/3})^{1/2}}{3^{1/2}5\pi^{1/4}(\kappa^{1/2}-1)},$$

$$c_{65} = \frac{\gamma^{1/2}\kappa(\kappa + t_2^{-1/3})^{1/2}}{\pi^{1/2}(\kappa-1)},$$

and

$$c_{66} = 3^{1/2}(\kappa + t_2^{-1/3})^{1/2}.$$

We choose $\kappa = 1.453$ and $\gamma = 2$ to obtain the lemma for $t \geq t_2$.

7. Conclusion. Combining Lemma 1, the second part of Lemma 4 and (3) with $\beta = 1/2$ gives Theorem 2. Similarly, we combine Lemma 1, the first part of Lemma 4, Lemma 6 and (3) with $\beta = 1/3$ to get Theorem 3. We have already proved in Section 2 that the corollary is valid for $t \leq 5100$. We finish the proof of the corollary by applying Theorem 2 for $5100 \leq t \leq 4 \times 10^{14}$ and utilizing Theorem 3 for $t \geq 4 \times 10^{14}$.

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