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OSCILLATION OF SECOND ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. By means of Riccati transformation techniques, we establish some oscillation criteria for a second order nonlinear dynamic equation on time scales in terms of the coefficients. We give examples of dynamic equations to which previously known oscillation criteria are not applicable.

1. Introduction. Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of Hilger [**12**] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [**1**] and the references cited therein. A book on the subject of time scales by Bohner and Peterson [**5**] summarizes and organizes much of the time scale calculus. For the notions used below, we refer to the next section about the calculus on time scales and the references given therein.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of some different equations on time scales. We refer the reader to the papers [**2, 3, 6, 8**–**11**].

In [**6**], the authors consider the second order dynamic equation

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0 \quad \text{for} \quad t \in [a, b] \tag{1.1}$$

and give necessary and sufficient conditions for oscillation of all solutions on unbounded time scales. Unfortunately, the oscillation criteria are restricted in usage since additional assumptions have to be imposed on the unknown solutions. In [**9**] the authors consider the same equation and suppose that there exists some $t_0 \in \mathbf{T}$ such that p is bounded

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above on $[t_0, \infty)$, $\inf\{\mu(t) : t \in [t_0, \infty)\} > 0$ and use the Riccati equation to prove that if

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty,$$

then every solution is oscillatory on $[t_0, \infty)$. It is clear that the results given in [**9**] cannot be applied when p is unbounded, $\mu(t) = 0$ and $q(t) = t^{-\alpha}$ for $\alpha > 1$.

Following this trend, to develop the qualitative theory of dynamic equations on time scales, in this paper we shall consider the nonlinear second order dynamic equation

$$(p(t)x^{\Delta})^{\Delta} + q(t)(f \circ x^{\sigma}) = 0 \quad \text{for} \quad t \in [a, b], \tag{1.2}$$

where p and q are positive, real-valued rd-continuous functions defined on the time scales interval $[a, b]$ (throughout, $a, b \in \mathbf{T}$ with $a < b$), and we suppose that there exists a constant $K > 0$ such that $f : \mathbf{R} \to \mathbf{R}$ satisfies

$$xf(x) > 0 \quad \text{and} \quad f(x) \geq Kx \quad \text{for all} \quad x \neq 0. \tag{1.3}$$

Before doing so, let us first recall that a solution of (1.2) is a nontrivial real function x satisfying equation (1.2) for $t \geq a$. A solution x of (1.2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.2) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

In this paper we intend to use the Riccati transformation technique to obtain several oscillation criteria for (1.2) when (1.3) holds. Our results improve the results given in [**6, 9**]. Applications to equations to which previously known criteria for oscillation are not applicable are given. The paper is organized as follows. In the next section we present some basic definitions concerning the calculus on time scales. In Section 3 we develop the Riccati transformation technique for dynamic equations, and Section 4 is devoted to the proofs of our sufficient conditions for oscillation of all solutions of (1.2), subject to the condition

$$\int_{a}^{\infty} \frac{1}{p(t)}\, \Delta t = \infty. \tag{1.4}$$

In this section we also apply our results to Euler-Cauchy dynamic equations. Finally, in Section 5, we consider equations that do not satisfy (1.4). We present some conditions that ensure that all solutions are either oscillatory or convergent to zero.

2. Some preliminaries. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale $\mathbf{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. On any time scale $\mathbf{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbf{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbf{T} : s < t\}.$$

A point $t \in \mathbf{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale $\mathbf{T}$ is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbf{T} \to \mathbf{R}$ (the range $\mathbf{R}$ of f may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where} \quad s \in \mathbf{T} \setminus \{\sigma(t)\}.$$

A function $f : [a, b] \to \mathbf{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points, and f is said to be differentiable if its derivative exists. The derivative and the shift operator σ are related by the formula

$$f^{\sigma} = f + \mu f^{\Delta}, \quad \text{where} \quad f^{\sigma} := f \circ \sigma. \tag{2.1}$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$ and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}, \tag{2.2}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.3}$$

By using the product rule, the derivative of $f(t) = (t-\alpha)^m$ for $m \in \mathbf{N}$ and $\alpha \in \mathbf{T}$ can be calculated as

$$f^{\Delta}(t) = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^{\nu}(t-\alpha)^{m-\nu-1}. \tag{2.4}$$

For $a, b \in \mathbf{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{2.5}$$

and infinite integrals are defined by

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

Example 2.1. In case $\mathbf{T} = \mathbf{R}$ we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) \equiv 0, \quad f^{\Delta} = f' \text{ and } \int_a^b f(t)\Delta t = \int_a^b f(t)\, dt,$$

and in case $\mathbf{T} = \mathbf{Z}$ we have

$$\sigma(t) = t+1, \quad \rho(t) = t-1, \quad \mu(t) \equiv 1, \quad f^{\Delta} = \Delta f$$

and (if $a < b$)

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

3. Riccati transformation technique. The classical Riccati transformation technique for differential equations consists essentially in "completing the square." For dynamic equations we will need to "complete the circle square." Recall [**5**] that the set $\mathcal{R}$ of regressive (i.e., the function f satisfies $1+\mu(t)f(t) \neq 0$ for all $t \in \mathbf{T}$) and rd-continuous functions is an Abelian group under the addition

$$f \oplus g = f + g + \mu f g$$

and that the inverse with respect to $\oplus$ is

$$\ominus f = -\frac{f}{1+\mu f}.$$

We define $f \ominus g = f \oplus (\ominus g)$. The circle square of an $f \in \mathcal{R}$ is defined as

$$f^{②} = (-f)(\ominus f).$$

Note also the formula [**5**]

$$f + (\ominus f) = \mu f^{②}. \tag{3.1}$$

Lemma 3.1. *For $f, g \in \mathcal{R}$, we have*

$$(f \ominus g)^{②} = f^{②} + f(\ominus g) + (\ominus f) g + g^{②}. \tag{3.2}$$

Proof. We have

$$(f \ominus g)^{②} = \frac{(f\ominus g)^2}{1+\mu(f \ominus g)} = \frac{(f-g)^2/(1+\mu g)^2}{1+\mu(f-g)/(1+\mu g)} = \frac{(f-g)^2}{(1+\mu f)(1+\mu g)},$$

and hence

$$\begin{aligned}(f \ominus g)^{②} - f^{②} - g^{②} &= \frac{(f-g)^2}{(1+\mu f)(1+\mu g)} - \frac{f^2}{1+\mu f} - \frac{g^2}{1+\mu g} \\ &= \frac{f^2 - 2fg + g^2 - f^2 - \mu f^2 g - g^2 - \mu g^2 f}{(1+\mu f)(1+\mu g)} \\ &= -fg \frac{1+\mu f + 1 + \mu g}{(1+\mu f)(1+\mu g)} \\ &= f(\ominus g) + (\ominus f) g\end{aligned}$$

implies (3.2). □

In the next lemma we collect some identities that are needed in the proof of our Riccati transformation result. These identities follow easily from (2.1) and hence we omit a proof.

Lemma 3.2. *Suppose f is differentiable with $ff^\sigma \neq 0$ and define $g = f^\Delta/f$. Then*

$$1 + \mu g = \frac{f^\sigma}{f}, \quad \ominus g = -\frac{f^\Delta}{f^\sigma}, \quad \textit{and} \quad g^{②} = \frac{(f^\Delta)^2}{ff^\sigma}. \tag{3.3}$$

Theorem 3.3. *Suppose that x solves (1.2) with $x(t) \neq 0$ for all $t \geq t_0$. Let z be a differentiable function and define w on $[t_0, \infty)$ by*

$$w = \frac{z^2 p x^\Delta}{x}. \tag{3.4}$$

Then we have on $[t_0, \infty)$

$$-w^\Delta = (z^\sigma)^2 q \, \frac{f \circ x^\sigma}{x^\sigma} - p(z^\Delta)^2 + pzz^\sigma (r \ominus s)^{②}, \tag{3.5}$$

where

$$r = \frac{x^\Delta}{x} \quad \textit{and} \quad s = \frac{z^\Delta}{z}.$$

If additionally (1.3) holds and $x(t)x^\sigma(t) > 0$ for all $t \geq t_0$, then on $[t_0, \infty)$

$$-w^\Delta \geq qK(z^\sigma)^2 - p(z^\Delta)^2. \tag{3.6}$$

Proof. We calculate

$$\begin{aligned} -w^\Delta &\overset{(2.2)}{=} -\left[z^\Delta \left(\frac{zpx^\Delta}{x} \right) + z^\sigma \left(\frac{zpx^\Delta}{x} \right)^\Delta \right] \\ &\overset{(2.2)}{=} -z^\Delta zpr - z^\sigma \left[z^\sigma \left(\frac{px^\Delta}{x} \right)^\Delta + z^\Delta \, \frac{px^\Delta}{x} \right] \end{aligned}$$

$$
\begin{aligned}
&\overset{(2.3)}{=} -z^{\Delta} zpr - (z^{\sigma})^2 \left[\frac{(px^{\Delta})^{\Delta} x - x^{\Delta} p x^{\Delta}}{x x^{\sigma}} \right] - z^{\sigma} z^{\Delta} pr \\
&\overset{(3.3)}{=} -(z^{\sigma})^2 \frac{(px^{\Delta})^{\Delta}}{x^{\sigma}} + (z^{\sigma})^2 p r^{\textcircled{2}} - z^{\Delta} zpr - z^{\sigma} z^{\Delta} pr \\
&\overset{(1.2)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} \left[\frac{z^{\sigma}}{z} r^{\textcircled{2}} - \frac{z^{\Delta}}{z^{\sigma}} r - \frac{z^{\Delta}}{z} r \right] \\
&\overset{(2.1)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} \left[\frac{z + \mu z^{\Delta}}{z} r^{\textcircled{2}} - \frac{z^{\Delta}}{z} r - \frac{z^{\Delta}}{z^{\sigma}} r \right] \\
&\overset{(3.3)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} [r^{\textcircled{2}} + \mu s r^{\textcircled{2}} - sr + (\ominus s) r] \\
&\overset{(3.1)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} [r^{\textcircled{2}} + s(\ominus r) + (\ominus s) r] \\
&\overset{(3.2)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} [(r \ominus s)^{\textcircled{2}} - s^{\textcircled{2}}] \\
&\overset{(3.3)}{=} (z^{\sigma})^2 q \frac{f \circ x^{\sigma}}{x^{\sigma}} + pzz^{\sigma} (r \ominus s)^{\textcircled{2}} - p(z^{\Delta})^2,
\end{aligned}
$$

where we simply "completed the square." This proves (3.5). To obtain (3.6), note that

$$zz^{\sigma} p (r \ominus s)^{\textcircled{2}} = \frac{z^2 x (r - s)^2}{x^{\sigma}}$$

holds (apply the formula in the proof of Lemma 3.1 and the identities (3.3) from Lemma 3.2). This and (1.3) imply (3.6). $\square$

4. Oscillation criteria. In this section we assume (1.4) and present some oscillation criteria for (1.2). We start with the following auxiliary result.

Lemma 4.1. *Assume* (1.4). *Suppose that* x *is a nonoscillatory solution of* (1.2). *Then there exists* $t_0 \in \mathbf{T}$ *such that*

$$x(t) x^{\Delta}(t) > 0 \quad \textit{for all} \quad t \geq t_0. \tag{4.1}$$

Proof. Since x is nonoscillatory, it is either eventually positive or eventually negative. We only prove the lemma for the first case as the second case is similar and hence omitted. Assume that there exists $t_0 \in \mathbf{T}$ such that

$$x(t) > 0 \quad \text{for all} \quad t \geq t_0.$$

Define $y = px^\Delta$. Let $t \geq t_0$. Then $x(\sigma(t)) > 0$ and hence

$$y^\Delta(t) = -q(t)f(x^\sigma(t)) < 0$$

so that y is decreasing. Assume that there exists $t_1 \geq t_0$ with $y(t_1) =: c < 0$. Then

$$p(s)x^\Delta(s) = y(s) \leq y(t_1) = c \quad \text{for all} \quad s \geq t_1$$

and therefore

$$x^\Delta(s) \leq \frac{c}{p(s)} \quad \text{for all} \quad s \geq t_1.$$

Let $t \geq t_1$. Then

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^t x^\Delta(s)\Delta s \\ &\leq x(t_1) + \int_{t_1}^t \frac{c}{p(s)}\,\Delta s \\ &= x(t_1) + c\left\{\int_{t_1}^t \frac{\Delta s}{p(s)}\right\} \\ &\overset{(1.4)}{\longrightarrow} -\infty \quad \text{as} \quad t \longrightarrow \infty, \end{aligned}$$

a contradiction. Hence, $y(t) > 0$ for all $t \geq t_0$ and thus $x^\Delta(t) > 0$ for all $t \geq t_0$, i.e., (4.1) holds. $\square$

Now we are ready to present the main results of this paper.

Theorem 4.2. *Assume that* (1.3) *and* (1.4) *hold. Furthermore, assume that there exists a differentiable function z with*

$$\limsup_{t\to\infty} \int_a^t \Big[Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\Big]\Delta s = \infty. \tag{4.2}$$

Then every solution of (1.2) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.2). Then (4.1) from Lemma 4.1 implies that there exists $t_0 \in \mathbf{T}$ such that

$$w(t) > 0 \quad \text{for all} \quad t \geq t_0,$$

where w is defined by (3.4). All assumptions from Theorem 3.3 are satisfied, and hence we may integrate (3.6) from t_0 to $t \geq t_0$ to obtain

$$\begin{aligned} w(t_0) &\geq w(t_0) - w(t) \\ &= -\int_{t_0}^{t} w^{\Delta}(s)\Delta s \\ &\geq \int_{t_0}^{t} \Big[Kq(s)(z^{\sigma}(s))^2 - p(s)(z^{\Delta}(s))^2\Big]\Delta s \\ &\overset{(4.2)}{\longrightarrow} \infty, \end{aligned}$$

which is not possible. The proof is complete. $\square$

Corollary 4.3. *Assume that* (1.3) *and* (1.4) *hold. Furthermore, assume that there exists a positive function* δ *with*

$$\text{(4.3)} \quad \limsup_{t\to\infty} \int_a^t \left[Kq(s)\delta^{\sigma}(s) - p(s)\left(\frac{\delta^{\Delta}(s)}{\sqrt{\delta(s)} + \sqrt{\delta^{\sigma}(s)}}\right)^2\right]\Delta s = \infty.$$

Then every solution of (1.2) *is oscillatory on* $[a, \infty)$.

Proof. Define $z = \sqrt{\delta}$ and note that [**5**]

$$z^{\Delta} = \frac{\delta^{\Delta}}{\sqrt{\delta} + \sqrt{\delta^{\sigma}}}.$$

Since (4.3) holds for δ, we see that (4.2) holds for $z = \sqrt{\delta}$. Hence the claim follows from Theorem 4.2. $\square$

From Theorem 4.2 and Corollary 4.3 we can obtain different conditions for oscillation of all solutions of (1.2) by different choices of δ. For instance, if $z(t) = \delta(t) \equiv 1$, then oscillation criteria, including those known from [**9**] for (1.1), follow.

Corollary 4.4. *Assume that* (1.3) *and* (1.4) *hold. If*

$$\limsup_{t\to\infty}\int_a^t q(s)\Delta s=\infty,$$

then every solution of (1.2) *is oscillatory on* $[a,\infty)$.

If $\delta(t)=t$, then Corollary 4.3 yields the following result.

Corollary 4.5. *Assume that* (1.3) *and* (1.4) *hold. If*

$$\limsup_{t\to\infty}\int_a^t\left[K\sigma(s)q(s)-\frac{p(s)}{(\sqrt{s}+\sqrt{\sigma(s)}\,)^2}\right]\Delta s=\infty, \tag{4.4}$$

then every solution of (1.2) *is oscillatory on* $[a,\infty)$.

Example 4.6. Consider the Euler dynamic equation

$$x^{\Delta\Delta}+\frac{\gamma}{t\sigma(t)}\,x^{\sigma}=0 \quad \text{for} \quad t\in[1,\infty). \tag{4.5}$$

Here $p(t)\equiv 1$, $K=1$ and $q(t)=\gamma/(t\sigma(t))$. Then (4.4) from Corollary 4.5 reads

$$\limsup_{t\to\infty}\int_1^t\left[\frac{\gamma}{s}-\frac{1}{(\sqrt{s}+\sqrt{\sigma(s)}\,)^2}\right]\Delta s=\infty.$$

Note that the estimate

$$\begin{aligned}\frac{\gamma}{s}-\frac{1}{(\sqrt{s}+\sqrt{\sigma(s)}\,)^2}&\geq\frac{\gamma}{s}-\frac{1}{(\sqrt{s}+\sqrt{s})^2}\\&=\frac{\gamma}{s}-\frac{1}{(2\sqrt{s})^2}\\&=\frac{\gamma-(1/4)}{s}\end{aligned}$$

implies the following result. If $\mathbf{T}$ is a time scale that satisfies

$$\int_1^\infty\frac{\Delta t}{t}=\infty, \tag{4.6}$$

and if $\gamma > 1/4$, then (4.5) is oscillatory. Note that (4.6) holds for the time scales $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$ as

$$\lim_{t\to\infty} \log t = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

It also holds for the time scale $\mathbf{T} = q^{\mathbf{N}_0} := \{q^n : n \in \mathbf{N}_0\}$ where $q > 1$, since for this time scale,

$$\int_1^\infty \frac{\Delta t}{t} = \sum_{k=0}^{\infty} \int_{q^k}^{q^{k+1}} \frac{\Delta t}{t} = \sum_{k=0}^{\infty} \frac{\mu(q^k)}{q^k} = \sum_{k=0}^{\infty} \frac{(q-1)q^k}{q^k} = \sum_{k=0}^{\infty} (q-1) = \infty.$$

In fact, (4.6) holds for every time scale that is unbounded above, and this is shown in [**4**].

Note that our result is compatible with the well-known oscillatory behavior of (4.5) when $\mathbf{T} = \mathbf{R}$ (see [**14**]) and when $\mathbf{T} = \mathbf{Z}$ (see [**15**]). For the case $\mathbf{T} = \mathbf{Z}$, it is also known from [**15**] that, for $\gamma \le 1/4$, (4.5) has a nonoscillatory solution. Hence, Theorem 4.2 and Corollary 4.5 are sharp. Finally note that the results in [**6, 9**], i.e., Corollary 4.4, cannot be applied to (4.5) as

$$\int_1^\infty q(t)\Delta t = \int_1^\infty \frac{\gamma \Delta t}{t\sigma(t)} = \gamma \int_1^\infty \left(-\frac{1}{t}\right)^\Delta \Delta t = \gamma \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \gamma.$$

Example 4.7. Let $0 < p(t) \le 1$ for all t, e.g., $p(t) = t/(t+1)$, and consider the nonlinear dynamic equation

$$(p(t)x^\Delta)^\Delta + \frac{\gamma}{t\sigma(t)} x^\sigma (1 + (x^\sigma)^2) = 0 \quad \text{for} \quad t \ge 1. \tag{4.7}$$

Here $K = 1$ and $q(t) = \gamma/(t\sigma(t))$. Then (4.4) from Corollary 4.5 reads

$$\limsup_{t\to\infty} \int_1^t \left[\frac{\gamma}{s} - \frac{p(s)}{(\sqrt{s} + \sqrt{\sigma(s)}\,)^2}\right] \Delta s = \infty.$$

Note that the estimate

$$\frac{\gamma}{s} - \frac{p(s)}{(\sqrt{s} + \sqrt{\sigma(s)}\,)^2} \ge \frac{\gamma}{s} - \frac{1}{(\sqrt{s} + \sqrt{s})^2} = \frac{\gamma - (1/4)}{s}$$

implies that every solution of (4.7) is oscillatory when $\gamma > 1/4$. Note also that the results in [**6, 9**] cannot be applied to (4.7).

Theorem 4.8. *Assume that* (1.3) *and* (1.4) *hold. Furthermore, assume that there exists a differentiable function z and an odd $m \in \mathbf{N}$ with*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m \Big[Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\Big] \Delta s = \infty. \tag{4.8}$$

Then every solution of (1.2) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.2). Then (4.1) from Lemma 4.1 implies that there exists $t_0 \in \mathbf{T}$ such that

$$w(t) > 0 \quad \text{for all} \quad t \geq t_0,$$

where w is defined by (3.4). All assumptions from Theorem 3.3 are satisfied, and hence we may multiply (3.6) by $(t-s)^m$ for $t \geq s$ and integrate the resulting inequality from t_0 to $t \geq t_0$ to obtain

$$\begin{aligned}
&\int_{t_0}^t (t-s)^m \Big[Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\Big] \Delta s \\
&\leq -\int_{t_0}^t (t-s)^m w^\Delta(s) \Delta s \\
&= -\Big\{ -(t-t_0)^m w(t_0) - (-1)^m \int_{t_0}^t \sum_{\nu=0}^{m-1} (\sigma(t)-s)^\nu (t-s)^{m-\nu-1} w(\sigma(s)) \Delta s \Big\} \\
&\leq (t-t_0)^m w(t_0),
\end{aligned}$$

where we have used the integration by parts formula from (2.5), (2.4) and the fact that $m \in \mathbf{N}$ is odd. Therefore,

$$\frac{1}{t^m} \int_{t_0}^t (t-s)^m \Big[Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\Big] \Delta s \leq \left(1 - \frac{t_0}{t}\right)^m w(t_0),$$

which is a contradiction to (4.8). The proof is complete. $\square$

Remark 4.9. Note that when $z(t) \equiv 1$, then (4.8) reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m\, q(s)\Delta s = \infty, \tag{4.9}$$

which can be considered as an extension of the Kamenev-type oscillation criteria for second order differential equations given in [**13**]. When $\mathbf{T} = \mathbf{R}$, then (4.9) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m\, q(s)\, ds = \infty,$$

and when $\mathbf{T} = \mathbf{Z}$, then (4.9) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \sum_{s=T}^{t-1} (t-s)^m\, q(s) = \infty.$$

5. Other criteria. In this section we consider (1.2), where p does not satisfy (1.4), i.e.,

$$\int_a^\infty \frac{1}{p(t)}\, \Delta t < \infty. \tag{5.1}$$

In addition to (1.3), we impose the additional assumption

$$f \quad \text{is nondecreasing.} \tag{5.2}$$

We start with the following auxiliary result.

Lemma 5.1. *Assume* (5.1), (1.3), (5.2) *and*

$$\int_a^\infty \frac{1}{p(t)} \int_a^t q(s)\Delta s \Delta t = \infty. \tag{5.3}$$

Suppose that x is a nonoscillatory solution of (1.2) *such that there exists $t_1 \in \mathbf{T}$ with*

$$x(t)x^\Delta(t) < 0 \quad \textit{for all} \quad t \geq t_1. \tag{5.4}$$

Then

$$\lim_{t\to\infty} x(t) \quad \textit{exists and is zero.}$$

Proof. Since x is nonoscillatory, it is either eventually positive or eventually negative. We only prove the lemma for the first case as the second case is similar and hence omitted. Assume there exists $t_1 \in \mathbf{T}$ such that

$$x(t) > 0 \quad \text{and} \quad x^{\Delta}(t) < 0 \quad \text{for all} \quad t \geq t_1. \tag{5.5}$$

Hence x is positive and decreasing, and therefore $\lim_{t\to\infty} x(t) =: b$ clearly exists. We have to show $b = 0$. Let us assume the opposite, i.e., $b > 0$. By (1.3), $f(b) > 0$. Hence

$$x(\sigma(t)) \geq b > 0 \quad \text{for all} \quad t \geq t_1$$

implies by (5.2)

$$f\big(x(\sigma(t))\big) \geq f(b) > 0 \quad \text{for all} \quad t \geq t_1.$$

Define $y = px^{\Delta}$ and integrate the inequality

$$y^{\Delta}(s) \leq -q(s)f\big(x(\sigma(s))\big) \leq -q(s)f(b)$$

from t_1 to $t \geq t_1$ to find

$$y(t) = y(t_1) + \int_{t_1}^{t} y^{\Delta}(s)\Delta s \leq \int_{t_1}^{t} y^{\Delta}(s)\Delta s \leq -\int_{t_1}^{t} q(s)f(b)\Delta s$$

so that

$$x^{\Delta}(t) \leq -f(b)\,\frac{1}{p(t)}\int_{t_1}^{t} q(s)\Delta s.$$

Now we integrate this inequality from t_1 to $T \geq t_1$ to obtain

$$\begin{aligned} x(T) &= x(t_1) + \int_{t_1}^{T} x^{\Delta}(t)\Delta t \\ &\leq -f(b)\int_{t_1}^{T} \frac{1}{p(t)}\int_{t_1}^{t} q(s)\Delta s\Delta t \\ &\xrightarrow{(5.3)} -\infty \quad \text{as} \quad T \longrightarrow \infty. \end{aligned}$$

This is contradictory to (5.5), and the proof is complete. □

Using Lemma 5.1, we can now derive the following criteria.

Theorem 5.2. *Assume* (5.1)*,* (1.3)*,* (5.2) *and* (5.3)*. If there exists a differentiable function* z *satisfying* (4.2)*, then every solution of* (1.2) *is either oscillatory or converges to zero.*

Proof. We assume that x is a nonoscillatory solution of (1.2). Hence x is either eventually positive or eventually negative; i.e., there exists $t_0 \in \mathbf{T}$ with $x(t) > 0$ for all $t \geq t_0$ or $x(t) < 0$ for all $t \geq t_0$. Let $y = px^\Delta$. If there exists $t_1 \geq t_0$ with $y(t_1) < 0$, then

$$y(t) \leq y(t_1) < 0 \quad \text{for all} \quad t \geq t_1$$

since y is decreasing, and hence $x^\Delta(t) < 0$ for all $t \geq t_1$. If, however, $y(t) > 0$ for all $t \geq t_0$, then $x^\Delta(t) > 0$ for all $t \geq t_0$. Therefore, either

$$x(t)x^\Delta(t) > 0 \quad \text{for all} \quad t \geq t_1,$$

in which case we can use Theorem 3.3 to derive a contradiction as in the proof of Theorem 4.2, or

$$x(t)x^\Delta(t) < 0 \quad \text{for all} \quad t \geq t_1,$$

in which case we see from Lemma 5.1 that $x(t)$ converges to zero as t tends to infinity. This completes the proof. □

Similarly we can prove the following theorem.

Theorem 5.3. *Assume* (5.1)*,* (1.3)*,* (5.2) *and* (5.3)*. If there exists a differentiable function* z *satisfying* (4.8)*, then every solution of* (1.2) *is either oscillatory or converges to zero.*

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