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MULTILINEAR TRIF d-MAPPINGS IN BANACH MODULES OVER A C^*-ALGEBRA

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ABSTRACT. We define a multilinear Trif d-mapping, and prove the stability of multilinear Trif d-functional equations in Banach modules over a unital C^*-algebra.

1. Introduction. Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Rassias [**4**] showed that there exists a unique $\mathbf{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-2^p}\, ||x||^p$$

for all $x \in E_1$.

Recently, Trif [**6**, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$\text{(A)} \quad n_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{l=1}^{n} f(x_l) = k \sum_{1 \leq l_1 < \cdots < l_k \leq n} f\left(\frac{x_{l_1} + \cdots + x_{l_k}}{k}\right)$$

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for all $x_1, \dots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$. And he proved the stability of the functional equation (A).

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $\mathcal{U}(A)$ the unitary group of A, $A_1 = \{a \in A \mid |a| = 1\}$ and A_1^+ the set of positive elements in A_1. Let ${}_A\mathcal{B}_l$ be a left A-module with norm $||\cdot||$ for each $l = 1, \dots, d$. Let ${}_A\mathcal{D}$ be a left Banach A-module with norm $\|\cdot\|$. Let n and k be integers such that $2 \le k \le n-1$.

In [**2**, Definition 4.2.3], the authors defined a linear 2-functional. A mapping $f : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is called a *multilinear Trif d-mapping* if f satisfies the condition $D_{a_1,\dots,a_d} f(x_{11}, \dots, x_{1n}, \dots, x_{d1}, \dots, x_{dn}) = 0$, where D is defined in the beginning of the next section.

The main purpose of this paper is to prove the stability of multilinear Trif d-functional equations in Banach modules over a unital C^*-algebra.

2. Stability of multilinear Trif d-functional equations in Banach modules over a C^*-algebra. For a given mapping $f : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ and given $a_1, \dots, a_d \in A$, we set

$$\begin{aligned}
&D_{a_1,\dots,a_d} f(x_{11}, \dots, x_{1n}, \dots, x_{d1}, \dots, x_{dn}) \\
&:= n^d \ {}_{n-2}C_{k-2} f\left(\frac{a_1x_{11} + \cdots + a_1x_{1n}}{n}, \dots, \frac{a_lx_{l1} + \cdots + a_lx_{ln}}{n}, \dots, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \frac{a_dx_{d1} + \cdots + a_dx_{dn}}{n}\right) \\
&\quad + {}_{n-2}C_{k-1} \sum_{j_1,\cdots,j_d=1}^{n} f(a_1x_{1j_1}, \dots, a_lx_{lj_l}, \dots, a_dx_{dj_d}) \\
&\quad - k^d \sum_{\substack{1\le j_{11}<\cdots<j_{1k}\le n \\ \vdots \\ 1\le j_{d1}<\cdots<j_{dk}\le n}} a_1 \cdots a_d f\left(\frac{x_{1j_{11}} + \cdots + x_{1j_{1k}}}{k}, \dots, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \frac{x_{dj_{d1}} + \cdots + x_{dj_{dk}}}{k}\right)
\end{aligned}$$

for all $x_{l1}, \dots, x_{ln} \in {}_A\mathcal{B}_l$, $l = 1, \dots, d$.

Theorem 1. *Let* $q = k(n-1)/(n-k)$ *and* $r = -k/(n-k)$. *Let* $f : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *be a mapping for which there exists a function* $\varphi : \prod_{s=1}^{d} {}_A\mathcal{B}_s^n \to [0,\infty)$ *such that*

$$(\ddagger)\qquad \begin{aligned} &\psi(x_{11},\dots,x_{1n},\dots,x_{l1},\dots,x_{ln},\dots,x_{d1},\dots,x_{dn}) \\ &:= \sum_{j=0}^{\infty}\sum_{l=1}^{d} q^{1-l-jd}\varphi\Big(\underbrace{q^{j+1}x_{11},\dots,q^{j+1}x_{11}}_{n \text{ times}},\dots, \\ &\quad \underbrace{q^{j+1}x_{l-1\,1},\dots,q^{j+1}x_{l-1\,1}}_{n \text{ times}}, q^{j+1}x_{l1}, \underbrace{rq^jx_{l2},\dots,rq^jx_{ln}}_{n-1 \text{ times}}, \\ &\quad \underbrace{q^jx_{l+1\,1},\dots,q^jx_{l+1\,1}}_{n \text{ times}},\dots,\underbrace{q^jx_{d1},\dots,q^jx_{d1}}_{n \text{ times}}\Big) < \infty, \end{aligned}$$

$$\text{(i)}\quad \widetilde{\varphi}(x_1,\dots,x_l,\dots,x_d) := \psi(\underbrace{x_1,\dots,x_1}_{n \text{ times}},\dots,\underbrace{x_l,\dots,x_l}_{n \text{ times}},\dots,\underbrace{x_d,\dots,x_d}_{n \text{ times}}),$$

$$\text{(ii)}\quad \|D_{u_1,\dots,u_d} f(x_{11},\dots,x_{1n},\dots,x_{d1},\dots,x_{dn})\| \le \varphi(x_{11},\dots,x_{1n},\dots,x_{d1},\dots,x_{dn})$$

for all $u_1,\dots,u_d \in \mathcal{U}(A)$, *all* $(x_1,\dots,x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$, *and all* $x_{l1},\dots,x_{ln} \in {}_A\mathcal{B}_l$, $l = 1,\dots,d$. *Assume that* $f(x_1,\dots,x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1,\dots,d$. *Then there exists a unique A-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *such that*

$$\text{(iii)}\quad \|f(x_1,\dots,x_d) - M(x_1,\dots,x_d)\| \le \frac{1}{k\ {}_{n-1}C_{k-1}}\widetilde{\varphi}(x_1,\dots,x_d)$$

for all $(x_1,\dots,x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

Proof. Put $u_1 = \cdots = u_d = 1 \in \mathcal{U}(A)$. For each fixed l, let $x_{11} = \cdots = x_{1n} = x_1, \dots, x_{l-1\,1} = \cdots = x_{l-1\,n} = x_{l-1}, x_{l+1\,1} = \cdots = x_{l+1\,n} = x_{l+1}, \dots, x_{d1} = \cdots = x_{dn} = x_d$ and $x_{l1} = qx_l$, $x_{l2} = \cdots = x_{ln} = rx_l$ in (ii). Then we get

$$\begin{aligned} &\|{}_{n-2}C_{k-1} f(x_1,\dots,x_{l-1},qx_l,x_{l+1},\dots,x_d) \\ &\qquad - k\ {}_{n-1}C_{k-1} f(x_1,\dots,x_{l-1},x_l,x_{l+1},\dots,x_d)\| \\ &\quad \le \varphi(x_1,\dots,x_1,\dots,x_{l-1},\dots,x_{l-1},qx_l,rx_l,\dots,rx_l, \\ &\qquad\qquad x_{l+1},\dots,x_{l+1},\dots,x_d,\dots,x_d), \end{aligned}$$

and hence

$$\begin{aligned}&\|f(x_1,\dots,x_{l-1},x_l,x_{l+1},\dots,x_d)-q^{-1}\\&\qquad\times f(x_1,\dots,x_{l-1},qx_l,x_{l+1},\dots,x_d)\|\\&\quad\le\frac{1}{k\ {}_{n-1}C_{k-1}}\,\varphi(x_1,\dots,x_1,\dots,x_{l-1},\dots,x_{l-1},qx_l,rx_l,\dots,rx_l,\\&\qquad\qquad x_{l+1},\dots,x_{l+1},\dots,x_d,\dots,x_d)\end{aligned}$$

for all $(x_1,\dots,x_d)\in\prod_{s=1}^d {}_A\mathcal{B}_s$. So one can obtain

$$\begin{aligned}&\|q^{1-l}f(qx_1,\dots qx_{l-1},x_l,\dots,x_d)-q^{-l}f(qx_1,\dots,qx_l,x_{l+1},\dots,x_d)\|\\&\quad\le\frac{q^{1-l}}{k\ {}_{n-1}C_{k-1}}\,\varphi(qx_1,\dots,qx_1,\dots,qx_{l-1},\dots,qx_{l-1},qx_l,rx_l,\dots,rx_l,\\&\qquad\qquad x_{l+1},\dots,x_{l+1},\dots,x_d,\dots,x_d)\end{aligned}$$

for all $(x_1,\dots,x_d)\in\prod_{s=1}^d {}_A\mathcal{B}_s$. Thus

$$\begin{aligned}&\|f(x_1,\dots,x_d)-q^{-d}f(qx_1,\dots,qx_d)\|\\&\quad\le\sum_{l=1}^d\frac{q^{1-l}}{k\ {}_{n-1}C_{k-1}}\varphi(qx_1,\dots,qx_1,\dots,qx_{l-1},\dots,\\&\qquad\qquad qx_{l-1},qx_l,rx_l,\dots,rx_l,x_{l+1},\dots,x_{l+1},\dots,x_d,\dots,x_d)\end{aligned}$$

for all $(x_1,\dots,x_d)\in\prod_{s=1}^d {}_A\mathcal{B}_s$. Hence we get

$$\begin{aligned}&\|q^{-jd}f(q^jx_1,\dots,q^jx_d)-q^{-(j+1)d}f(q^{j+1}x_1,\dots,q^{j+1}x_d)\|\\&\quad\le\sum_{l=1}^d\frac{q^{1-l}}{k\ {}_{n-1}C_{k-1}}\,q^{-jd}\varphi(q^{j+1}x_1,\dots,q^{j+1}x_1,\dots,q^{j+1}x_{l-1},\dots,\\&\qquad\qquad q^{j+1}x_{l-1},q^{j+1}x_l,rq^jx_l,\dots,rq^jx_l,q^jx_{l+1},\dots,\\&\qquad\qquad q^jx_{l+1},\dots,q^jx_d,\dots,q^jx_d)\end{aligned}$$

for all $(x_1,\dots,x_d)\in\prod_{s=1}^d {}_A\mathcal{B}_s$. So

(1)

$$\begin{aligned}&\|f(x_1,\dots,x_d)-q^{-pd}f(q^px_1,\dots,q^px_d)\|\\&\quad\le\sum_{j=0}^{p-1}\sum_{l=1}^d\frac{q^{1-l}}{k\ {}_{n-1}C_{k-1}}q^{-jd}\varphi(q^{j+1}x_1,\dots,q^{j+1}x_1,\dots,q^{j+1}x_{l-1},\\&\qquad\qquad \dots,q^{j+1}x_{l-1},q^{j+1}x_l,rq^jx_l,\dots,rq^jx_l,\\&\qquad\qquad q^jx_{l+1},\dots,q^jx_{l+1},\dots,q^jx_d,\dots,q^jx_d)\end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$.

For each $l = 1, \dots, d$, let x_l be an element in ${}_A\mathcal{B}_l$. For positive integers p and m with $p > m$,

$$\begin{aligned}
&\|q^{-md} f(q^m x_1, \dots, q^m x_d) - q^{-pd} f(q^p x_1, \dots, q^p x_d)\| \\
&\quad \le \sum_{j=m}^{p-1} \sum_{l=1}^{d} \frac{q^{1-l}}{k\ {}_{n-1}C_{k-1}}\, q^{-jd} \varphi(q^{j+1} x_1, \dots, q^{j+1} x_1, \dots, \\
&\qquad\qquad q^{j+1} x_{l-1}, \dots, q^{j+1} x_{l-1}, q^{j+1} x_l, \\
&\qquad\qquad rq^j x_l, \dots, rq^j x_l, q^j x_{l+1}, \dots, q^j x_{l+1}, \dots, q^j x_d, \dots, q^j x_d),
\end{aligned}$$

which tends to zero as $m \to \infty$ by (‡) and (i). So $\{q^{-jd} f(q^j x_1, \dots, q^j x_d)\}$ is a Cauchy sequence for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. Since ${}_A\mathcal{D}$ is complete, the sequence $\{q^{-jd} f(q^j x_1, \dots, q^j x_d)\}$ converges for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. We can define a mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ by

$$M(x_1, \dots, x_d) = \lim_{j\to\infty} q^{-jd} f(q^j x_1, \dots, q^j x_d) \tag{2}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$.

By (‡) and (2), we get

$$\begin{aligned}
&\|D_{1,\dots,1} M(x_1, \dots, x_1, \dots, x_{l-1}, \dots, x_{l-1}, x_{l1}, \dots, x_{ln}, \\
&\qquad\qquad x_{l+1}, \dots, x_{l+1}, \dots, x_d, \dots, x_d)\| \\
&\quad = \lim_{j\to\infty} q^{-jd} \|D_{1,\dots,1} f(q^j x_1, \dots, q^j x_1, \dots, q^j x_{l-1}, \dots, q^j x_{l-1}, \\
&\qquad\qquad q^j x_{l1}, \dots, q^j x_{ln}, q^j x_{l+1}, \dots, q^j x_{l+1}, \dots, q^j x_d, \dots, q^j x_d)\| \\
&\quad \le \lim_{j\to\infty} q^{-jd} \varphi(q^j x_1, \dots, q^j x_1, \dots, q^j x_{l-1}, \dots, q^j x_{l-1}, \\
&\qquad\qquad q^j x_{l1}, \dots, q^j x_{ln}, q^j x_{l+1}, \dots, q^j x_{l+1}, \dots, q^j x_d, \dots, q^j x_d) = 0,
\end{aligned}$$

hence

$$\begin{aligned}
&D_{1,\dots,1} M(x_1, \dots, x_1, \dots, x_{l-1}, \dots, x_{l-1}, x_{l1}, \dots, x_{ln}, \\
&\qquad\qquad x_{l+1}, \dots, x_{l+1}, \dots, x_d, \dots, x_d) = 0
\end{aligned}$$

for all $(x_1, \dots, x_{l-1}, x_{l1}, x_{l+1}, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$ and all $x_{l2}, \dots, x_{ln} \in {}_A\mathcal{B}_l$, which implies that the mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$

satisfies the functional equation (A) in the lth variable for each $l = 1, \dots, d$. So M is additive in the lth variable for each $l = 1, \dots, d$. Moreover, by passing to the limit in (1) as $p \to \infty$, we get the inequality (iii).

Now let $L : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ be another multi-additive mapping satisfying

$$\|f(x_1, \dots, x_d) - L(x_1, \dots, x_d)\| \le \frac{1}{k \ {}_{n-1}C_{k-1}} \widetilde{\varphi}(x_1, \dots, x_d)$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Then

$$\begin{aligned}
&\|M(x_1, \dots, x_d) - L(x_1, \dots, x_d)\| \\
&\quad = q^{-jd}\|M(q^j x_1, \dots, q^j x_d) - L(q^j, \dots, q^j x_d)\| \\
&\quad \le q^{-jd}\|M(q^j x_1, \dots, q^j x_d) - f(q^j x_1, \dots, q^j x_d)\| \\
&\qquad + q^{-jd}\|f(q^j x_1, \dots, q^j x_d) - L(q^j x_1, \dots, q^j x_d)\| \\
&\quad \le 2 \cdot q^{-jd}\widetilde{\varphi}(q^j x_1, \dots, q^j x_d),
\end{aligned}$$

which tends to zero as $j \to \infty$ by (‡) and (i). Thus $M(x_1, \dots, x_d) = L(x_1, \dots, x_d)$ for all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. This proves the uniqueness of M.

By the assumption, for each $u_l \in \mathcal{U}(A)$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$,

$$\begin{aligned}
&q^{-jd}\|D_{1,\dots,1,u_l,1,\dots,1}f(q^j x_1, \dots, q^j x_1, \dots, q^j x_l, \dots, \\
&\qquad\qquad q^j x_l, \dots, q^j x_d, \dots, q^j x_d)\| \\
&\quad \le q^{-jd}\varphi(q^j x_1, \dots, q^j x_1, \dots, q^j x_l, \dots, q^j x_l, \dots, q^j x_d, \dots, q^j x_d),
\end{aligned}$$

which tends to zero as $j \to \infty$ by (‡). So

$$\begin{aligned}
&D_{1,\dots,1,u_l,1,\dots,1}M(x_1, \dots, x_1, \dots, x_l, \dots, x_l, x_d, \dots, x_d) \\
&\quad = \lim_{j\to\infty} q^{-jd} D_{1,\dots,1,u_l,1,\dots,1}f(q^j x_1, \dots, q^j x_1, \dots, q^j x_l, \dots, \\
&\qquad\qquad q^j x_l, \dots, q^j x_d, \dots, q^j x_d) = 0
\end{aligned}$$

for all $u_l \in \mathcal{U}(A)$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So we get

$$\begin{aligned}
&M(x_1, \dots, x_{l-1}, u_l x_l, x_{l+1}, \dots, x_d) \\
&\qquad = u_l M(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_d)
\end{aligned}$$

for all $u_l \in \mathcal{U}(A)$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$.

Now let $a \in A$, $a \neq 0$, and let K be an integer greater than $4|a|$. Then

$$\left|\frac{a}{K}\right| = \frac{1}{K}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [**3**, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3a/K = u_1 + u_2 + u_3$. And

$$\begin{aligned} M(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_d) \\ &= M\Big(x_1, \dots, x_{l-1}, 3 \cdot \frac{1}{3}x_l, x_{l+1}, \dots, x_d\Big) \\ &= 3M\Big(x_1, \dots, x_{l-1}, \frac{1}{3}x_l, x_{l+1}, \dots, x_d\Big) \end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. So

$$\begin{aligned} M\Big(x_1, \dots, x_{l-1}, \frac{1}{3}x_l, x_{l+1}, \dots, x_d\Big) \\ &= \frac{1}{3}M(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_d) \end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. Thus

$$\begin{aligned} M(x_1, \dots, ax_l, \dots, x_d) &= M\Big(x_1, \dots, \frac{K}{3} \cdot 3\frac{a}{K}\, x_l, \dots, x_d\Big) \\ &= \frac{K}{3}\, M\Big(x_1, \dots, 3\frac{a}{K}\, x_l, \dots, x_d\Big) \\ &= \frac{K}{3}\, M(x_1, \dots, u_1x_l + u_2x_l + u_3x_l, \dots, x_d) \\ &= \frac{K}{3}\, (u_1 + u_2 + u_3)M(x_1, \dots, x_l, \dots, x_d) \\ &= \frac{K}{3} \cdot 3\frac{a}{K}\, M(x_1, \dots, x_l, \dots, x_d) \\ &= aM(x_1, \dots, x_l, \dots, x_d) \end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. Obviously,

$$M(x_1, \dots, 0x_l, \dots, x_d) = 0M(x_1, \dots, x_l, \dots, x_d)$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. Hence

$$\begin{aligned} M(x_1, \dots, ax_l + by_l, \dots, x_d) &= M(x_1, \dots, ax_l, \dots, x_d) \\ &\quad + M(x_1, \dots, x_{l-1}, by_l, x_{l+1}, \dots, x_d) \\ &= aM(x_1, \dots, x_l, \dots, x_d) \\ &\quad + bM(x_1, \dots, x_{l-1}, y_l, x_{l+1}, \dots, x_d) \end{aligned}$$

for all $a, b \in A$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$ and all $y_l \in {}_A\mathcal{B}_l$. So the unique multi-additive mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is an A-multilinear mapping. □

Corollary 1. *Let $q = k(n-1)/n-k$ and $r = -k/(n-k)$. Let $\theta \geq 0$, and let $\eta : [0, \infty) \to [0, \infty)$ be a function such that*

$$\begin{aligned} \eta(\alpha\beta) &\leq \eta(\alpha)\eta(\beta), \\ \eta(q) &< q^d \end{aligned}$$

for all $\alpha, \beta \in [0, \infty)$. Assume that a mapping $f : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ satisfies

$$\|D_{u_1, \dots, u_d} f(x_{11}, \dots, x_{1n}, \dots, x_{d1}, \dots, x_{dn})\| \leq \theta \sum_{l=1}^{d} \sum_{\nu=1}^{n} \eta(||x_{l\nu}||)$$

for all $u_1, \dots, u_d \in \mathcal{U}(A)$, and all $x_{l1}, \dots, x_{ln} \in {}_A\mathcal{B}_l$, $l = 1, \dots, d$, and that $f(x_1, \dots, x_d) = 0$ if $x_l = 0$ for any $l = 1, \dots, d$. Then there exists a unique A-multilinear mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ such that

(iv)

$$\begin{aligned} &\|f(x_1, \dots, x_d) - M(x_1, \dots, x_d)\| \\ &\quad \leq \frac{\theta}{k\ {}_{n-1}C_{k-1}} \sum_{l=1}^{d} \sum_{j=0}^{\infty} q^{1-l-jd} \big(n\ \eta(q^j||qx_1||) + \cdots \\ &\qquad\qquad + n\ \eta(q^j||qx_{l-1}||) + \eta(q^j||qx_l||) \\ &\qquad + (n-1)\eta(q^j||rx_l||) + n\ \eta(q^j||x_{l+1}||) + \cdots + n\ \eta(q^j||x_d||)\big) \end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$.

Proof. It follows from Theorem 1. Indeed, for all $x_{l1}, \ldots, x_{ln} \in {}_A\mathcal{B}_l$, $l = 1, \ldots, d$, we have

$$\begin{aligned}
&\psi(x_{11}, \ldots, x_{1n}, \ldots, x_{l1}, \ldots, x_{ln}, \ldots, x_{d1}, \ldots, x_{dn}) \\
&\quad = \theta \sum_{l=1}^{d} \sum_{j=0}^{\infty} q^{1-l-jd} \big(n\ \eta(q^j ||qx_{11}||) + \cdots + n\ \eta(q^j ||qx_{l-1\ 1}||) \\
&\qquad + \eta(q^j||qx_{l1}||) + \eta(q^j||rx_{l2}||) + \cdots + \eta(q^j||rx_{ln}||) \\
&\qquad\qquad + n\ \eta(q^j||x_{l+1\ 1}||) + \cdots + n\ \eta(q^j||x_{d1}||) \big) \\
&\quad \le \theta \sum_{l=1}^{d} \sum_{j=0}^{\infty} q^{1-l-jd} \eta(q)^j \big(n\ \eta(||qx_{11}||) + \cdots + n\ \eta(||qx_{l-1\ 1}||) \\
&\qquad + \eta(||qx_{l1}||) + \eta(||rx_{l2}||) + \cdots + \eta(||rx_{ln}||) \\
&\qquad\qquad + n\ \eta(||x_{l+1\ 1}||) + \cdots + n\ \eta(||x_{d1}||) \big) \\
&\quad = \theta \sum_{j=0}^{\infty} (\frac{\eta(q)}{q^d})^j \sum_{l=1}^{d} q^{1-l} \big(n\ \eta(||qx_{11}||) + \cdots + n\ \eta(||qx_{l-1\ 1}||) \\
&\qquad + \eta(||qx_{l1}||) + \eta(||rx_{l2}||) + \cdots + \eta(||rx_{ln}||) \\
&\qquad\qquad + n\ \eta(||x_{l+1\ 1}||) + \cdots + n\ \eta(||x_{d1}||) \big) \\
&\quad = \frac{q^d \theta}{q^d - \eta(q)} \sum_{l=1}^{d} q^{1-l} \big(n\ \eta(||qx_{11}||) + \cdots + n\ \eta(||qx_{l-1\ 1}||) \\
&\qquad + \eta(||qx_{l1}||) + \eta(||rx_{l2}||) + \cdots + \eta(||rx_{ln}||) \\
&\qquad\qquad + n\ \eta(||x_{l+1\ 1}||) + \cdots + n\ \eta(||x_{d1}||) \big) < \infty.
\end{aligned}$$

Now

$$\begin{aligned}
&\psi(\underbrace{x_1, \ldots, x_1}_{n \text{ times}}, \ldots, \underbrace{x_l, \ldots, x_l}_{n \text{ times}}, \ldots, \underbrace{x_d, \ldots, x_d}_{n \text{ times}}) \\
&\quad = \theta \sum_{l=1}^{d} \sum_{j=0}^{\infty} q^{1-l-jd} (n\ \eta(q^j||qx_1||) + \cdots \\
&\qquad + n\ \eta(q^j||qx_{l-1}||) + \eta(q^j||qx_l||) \\
&\qquad + (n-1)\eta(q^j||rx_l||) + n\ \eta(q^j||x_{l+1}||) + \cdots + n\ \eta(q^j||x_d||))
\end{aligned}$$

for all $(x_1, \ldots x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So one can obtain (iv). $\square$

Corollary 2. *Let* $q = k(n-1)/n-k$ *and* $r = -k/(n-k)$. *Let* $\theta \geq 0$, $0 < p < d$, *and let* $\mu : [0,\infty)^{nd} \to [0,\infty)$ *be a function such that*

$$\mu(\lambda\beta_{11}, \ldots, \lambda\beta_{1n}, \ldots, \lambda\beta_{d1}, \ldots, \lambda\beta_{dn}) = \lambda^p \ \mu(\beta_{11}, \ldots, \beta_{1n}, \ldots, \beta_{d1}, \ldots, \beta_{dn})$$

for all $\lambda, \beta_{11}, \ldots, \beta_{1n}, \ldots, \beta_{d1}, \ldots, \beta_{dn} \in [0,\infty)$. *Assume that a mapping* $f : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *satisfies*

$$\|D_{u_1,\ldots,u_d} f(x_{11}, \ldots, x_{1n}, \ldots, x_{d1}, \ldots, x_{dn})\| \leq \theta\mu(||x_{11}||, \ldots, ||x_{1n}||, \ldots, ||x_{d1}||, \ldots, ||x_{dn}||)$$

for all $u_1, \ldots, u_d \in \mathcal{U}(A)$, *and all* $x_{l1}, \ldots, x_{ln} \in {}_A\mathcal{B}_l$, $l = 1, \ldots, d$, *and that* $f(x_1, \ldots, x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, \ldots, d$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathcal{B}_\int \to {}_\mathcal{A}\mathcal{D}$ *such that*

$$\begin{aligned}
&\|f(x_1, \ldots, x_d) - M(x_1, \ldots, x_d)\| \\
&\leq \frac{\theta}{k\ {}_{n-1}C_{k-1}} \cdot \frac{q^d}{q^d - q^p} \sum_{l=1}^{d} q^{1-l}\, \mu(\underbrace{||qx_1||, \ldots, ||qx_1||}_{n \text{ times}}, \ldots, \\
&\underbrace{||qx_{l-1}||, \ldots, ||qx_{l-1}||}_{n \text{ times}}, ||qx_l||, \underbrace{||rx_l||, \ldots, ||rx_l||}_{n-1 \text{ times}}, \underbrace{||x_{l+1}||, \ldots, ||x_{l+1}||}_{n \text{ times}}, \\
&\ldots, \underbrace{||x_d||, \ldots, ||x_d||}_{n \text{ times}})
\end{aligned}$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

Proof. It follows from Theorem 1. Indeed, for all $x_{l1}, \dots , x_{ln} \in {}_A\mathcal{B}_l$, $l = 1, \dots , d$, we have

$$\begin{aligned}
&\psi(x_{11}, \dots , x_{1n}, \dots , x_{l1}, \dots , x_{ln}, \dots , x_{d1}, \dots , x_{dn}) \\
&\quad = \theta \sum_{l=1}^{d} \sum_{j=0}^{\infty} q^{1-l-jd} \mu\big(q^j||qx_{11}||, \dots , q^j||qx_{11}||, \dots , q^j||qx_{l-1\ 1}||, \dots , \\
&\qquad q^j||qx_{l-1\ 1}||, q^j||qx_{l1}||, q^j||rx_{l2}||, \dots , q^j||rx_{ln}||, q^j||x_{l+1\ 1}||, \dots , \\
&\qquad q^j||x_{l+1\ 1}||, \dots , q^j||x_{d1}||, \dots , q^j||x_{d1}||\big) \\
&\quad = \theta \sum_{l=1}^{d} q^{1-l} \sum_{j=0}^{\infty} q^{-jd} q^{jp} \mu\big(||qx_{11}||, \dots , ||qx_{11}||, \dots , ||qx_{l-1\ 1}||, \dots , \\
&\qquad ||qx_{l-1\ 1}||, ||qx_{l1}||, ||rx_{l2}||, \dots , ||rx_{ln}||, ||x_{l+1\ 1}||, \dots , \\
&\qquad ||x_{l+1\ 1}||, \dots , ||x_{d1}||, \dots , ||x_{d1}||\big) \\
&\quad = \frac{q^d\theta}{q^d - q^p} \sum_{l=1}^{d} q^{1-l} \mu\big(||qx_{11}||, \dots , ||qx_{11}||, \dots , ||qx_{l-1\ 1}||, \dots , \\
&\qquad ||qx_{l-1\ 1}||, ||qx_{l1}||, ||rx_{l2}||, \dots , ||rx_{ln}||, ||x_{l+1\ 1}||, \dots , \\
&\qquad ||x_{l+1\ 1}||, \dots , ||x_{d1}||, \dots , ||x_{d1}||\big) < \infty.
\end{aligned}$$

Now

$$\begin{aligned}
&\psi(\underbrace{x_1, \dots , x_1}_{n\text{times}}, \dots , \underbrace{x_l, \dots , x_l}_{n \text{ times}}, \dots , \underbrace{x_d, \dots , x_d}_{n \text{ times}}) \\
&\quad = \frac{q^d\theta}{q^d - q^p} \sum_{l=1}^{d} q^{1-l}\, \mu\big(||qx_1||, \dots , ||qx_1||, \dots , ||qx_{l-1}||, \dots , \\
&\qquad ||qx_{l-1}||, ||qx_l||, ||rx_l||, \dots , ||rx_l||, ||x_{l+1}||, \dots , \\
&\qquad ||x_{l+1}||, \dots , ||x_d||, \dots , ||x_d||\big)
\end{aligned}$$

for all $(x_1, \dots , x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So one can obtain (v). □

Corollary 3. *Let $q = k(n-1)/(n-k)$ and $r = -k/(n-k)$. Let $\theta \geq 0$, and let $0 < p < d$. Assume that a mapping $f : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ satisfies*

$$\|D_{u_1,\dots,u_d} f(x_{11}, \dots , x_{1n}, \dots , x_{d1}, \dots , x_{dn})\| \leq \theta \sum_{l=1}^{d} \sum_{\nu=1}^{n} ||x_{l\nu}||^p$$

for all $u_1, \dots, u_d \in \mathcal{U}(A)$*, and all* $x_{l1}, \dots, x_{ln} \in {}_A\mathcal{B}_l$*,* $l = 1, \dots, d$*, and that* $f(x_1, \dots, x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, \dots, d$*. Then there exists a unique A-multilinear mapping* $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *such that*

$$\begin{aligned}&\|f(x_1, \dots, x_d) - M(x_1, \dots, x_d)\| \\ &\quad\le \frac{\theta}{k\ {}_{n-1}C_{k-1}} \cdot \frac{q^d}{q^d - q^p} \sum_{l=1}^d q^{1-l} \big(n||qx_1||^p + \dots + n||qx_{l-1}||^p \\ &\qquad + ||qx_l||^p + (n-1)||rx_l||^p + n||x_{l+1}||^p + \dots + n||x_d||^p\big)\end{aligned}$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$*.*

Proof. It follows either from Corollary 1 for $\eta(t) = t^p$, or from Corollary 2 for $\mu(\beta_{11}, \dots, \beta_{1n}, \dots, \beta_{d1}, \dots, \beta_{dn}) = \beta_{11}^p + \dots + \beta_{1n}^p + \dots + \beta_{d1}^p + \dots + \beta_{dn}^p$. □

Theorem 2. *Let* $q = k(n-1)/(n-k)$ *and* $r = -k/(n-k)$*. Let* $f : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *be a mapping for which there exists a function* $\varphi : \prod_{s=1}^d {}_A\mathcal{B}_s^n \to [0, \infty)$ *satisfying* (‡) *such that*

$$\begin{aligned}&\|D_{a_1,\dots,a_d} f(x_{11}, \dots, x_{1n}, \dots, x_{d1}, \dots, x_{dn})\| \\ &\qquad\qquad \le \varphi(x_{11}, \dots, x_{1n}, \dots, x_{d1}, \dots, x_{dn})\end{aligned}$$

for all $a_1, \dots, a_d \in A_1^+ \cup \{i\}$ *and all* $x_{l1}, \dots, x_{ln} \in {}_A\mathcal{B}_l$*,* $l = 1, \dots, d$*. Assume that* $f(x_1, \dots, x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, \dots, d$*, and that for each* $l = 1, \dots, d$*,* $f(x_1, \dots, x_{l-1}, \lambda x_l, x_{l+1}, \dots, x_d)$ *is continuous in* $\lambda \in \mathbf{R}$ *for each fixed* $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$*. Then there exists a unique A-multilinear mapping* $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *satisfying* (iii)*.*

Proof. Put $a_1 = \dots = a_d = 1 \in A_1^+$. By the same reasoning as in the proof of Theorem 1, there exists a unique multi-additive mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ satisfying (iii).

For each fixed $l = 1, \dots, d$, since $f(x_1, \dots, \lambda x_l, \dots, x_d)$ is continuous in $\lambda \in \mathbf{R}$ for each fixed $(x_1, \dots, x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$, by the same reasoning as in the proof of [**4**, Theorem], the multi-additive mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is $\mathbf{R}$-linear in the lth variable. So the multi-additive mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is $\mathbf{R}$-multilinear.

By the same reasoning as in the proof of Theorem 1,

$$\begin{aligned}M(x_1, \dots , x_{l-1}, ax_l, x_{l+1}, \dots , x_d)\\ = aM(x_1, \dots , x_{l-1}, x_l, x_{l+1}, \dots , x_d)\end{aligned} \tag{3}$$

for all $a \in A_1^+ \cup \{i\}$ and $(x_1, \dots , x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$.

For any element $a \in A$, $a = (a + a^*)/2 + i(a - a^*)/2i$, and $(a + a^*)/2$ and $(a - a^*)/2i$ are self-adjoint elements; furthermore, $a = ((a+a^*)/2)^+ - ((a+a^*)/2)^- + i((a-a^*)/2i)^+ - i((a-a^*)/2i)^-$, where $((a+a^*)/2)^+$, $((a+a^*)/2)^-$, $i((a-a^*)/2i)^+$, and $i((a-a^*)/2i)^-$ are positive elements, see [**1**, Lemma 38.8]. Using the **R**-multilinearity and (3), one can easily show that

$$\begin{aligned}M(x_1, \dots , x_{l-1}, ax_l, x_{l+1}, \dots , x_d)\\ = aM(x_1, \dots , x_{l-1}, x_l, x_{l+1}, \dots , x_d)\end{aligned}$$

for all $a \in A$ and all $(x_1, \dots , x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$. Hence

$$\begin{aligned}M(x_1, \dots , x_{l-1}, ax_l + by_l, x_{l+1}, \dots , x_d)\\ &= M(x_1, \dots , x_{l-1}, ax_l, x_{l+1}, \dots , x_d)\\ &\quad + M(x_1, \dots , x_{l-1}, by_l, x_{l+1}, \dots , x_d)\\ &= aM(x_1, \dots , x_{l-1}, x_l, x_{l+1}, \dots , x_d)\\ &\quad + bM(x_1, \dots , x_{l-1}, y_l, x_{l+1}, \dots , x_d)\end{aligned}$$

for all $a, b \in A$, all $(x_1, \dots , x_d) \in \prod_{s=1}^d {}_A\mathcal{B}_s$ and $y_l \in {}_A\mathcal{B}_l$. So the unique multi-additive mapping $M : \prod_{s=1}^d {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is an A-multilinear mapping. □

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