ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

A SYSTEM OF PELLIAN EQUATIONS AND RELATED TWO-PARAMETRIC FAMILY OF QUARTIC THUE EQUATIONS

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ABSTRACT. We show that solving of the two-parametric family of quartic Thue equations

$$x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1,$$

using the method of Tzanakis, reduces to solving the system of Pellian equations

$$V^2 - \left(m^2 + 2\right)U^2 = -2, \qquad Z^2 - \left(n^2 - 2\right)U^2 = 2,$$

where parameters m and $n \neq 0, \pm 1$ are integers. The main result in this paper can be stated as follows: If $|m|$ and $|n|$ are sufficiently large and have sufficiently large common divisor, then the system has only the trivial solutions $(V, Z, U) = (\pm m, \pm n, \pm 1)$, which implies that the original Thue equation also has only the trivial solutions $(x, y) = (\pm 1, 0), (0, \pm 1)$.

1. Introduction. Let $F \in \mathbf{Z}[X, Y]$ be a homogeneous irreducible polynomial of degree ≥ 3 and $t \neq 0$ a fixed integer. Then Diophantine equation $F(x, y) = t$ is called a Thue equation in honor of A. Thue, who proved in 1909 [**24**] that such an equation has only finitely many solutions $(x, y) \in \mathbf{Z} \times \mathbf{Z}$. Thue's proof is not effective. Using estimates for linear forms in logarithms of algebraic numbers, Baker [**1**] could give an effective upper bound for the solutions of Thue equation. Since that time, general powerful methods have been developed for the explicit solution of Thue equations, see [**21, 27, 5**], following from Baker's work. In 1990, Thomas [**23**] investigated for the first time a parametrized family of Thue equations. Since then, several families have been studied, see [**12**] for references. In particular, quartic families have been considered in [**6, 10, 12, 14, 16, 20, 22, 25, 28, 29**].

2000 AMS *Mathematics Subject Classification.* Primary 11D59, Secondary 11D25, 11B37, 11J86.
Key words and phrases. Thue equations, simultaneous Pellian equations, linear forms in logarithms.

In this paper, we consider the equation

$$x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1. \tag{1}$$

Let us note that, because of homogeneity and symmetry of equation (1), it is enough to consider the cases when m and n are nonnegative and find only all positive solutions. More precisely, $(x, y) = (a, b)$ is a solution of equation (1) if and only if $(x, y) = (b, a)$ is a solution of equation

$$x^4 + 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 - 2mnxy^3 + y^4 = 1.$$

Further, if $(x, y) = (a, b)$ is a solution of equation (1) then $(x, y) = (-a, -b)$, $(b, -a)$, $(-b, a)$ are solutions too. Thus, we will suppose, without loss of generality, that $m \geq 0$ and $n \geq 0$ are integers and consider an equation of the form (1).

Using the method of Tzanakis, given in [**26**], we will show that solving equation (1) reduces to solving the system of Pellian equations

$$V^2 - \left(m^2 + 2\right)U^2 = -2, \tag{2}$$

$$Z^2 - \left(n^2 - 2\right)U^2 = 2, \tag{3}$$

and we prove, roughly speaking, that if m and n are sufficiently large and have a sufficiently large common divisor, then the system has only the trivial solutions $(V, Z, U) = (\pm m, \pm n, \pm 1)$, which implies equation (1) has only the trivial solutions $(\pm 1, 0)$, $(0, \pm 1)$.

We will find a lower bound for solutions of this system using the "congruence method" introduced in [**11**] by Dujella and Pethő and used also in [**8, 9, 10**]. The comparison of this lower bound with an upper bound obtained from a theorem of Baker and Wüstholz [**3**] lead to the main result of this paper.

In [**26**], Tzanakis considered a certain class of quartic Thue equations whose corresponding quartic field $\mathbf{K}$ is totally real, Galois and noncyclic. Tzanakis showed that solving the equation, under the above assumptions on $\mathbf{K}$, reduces to solving a system of Pellian equations having one common unknown. Such a reduction has certain advantages. Dealing with the system of Pellian equations, all algebraic-arithmetic data we need are easily accessible. Furthermore, methods based on the theory of linear forms in logarithms lead to a linear form in three logarithms and two unknown integral coefficients instead, the known direct general methods such as [**27**], to a linear form in four logarithms and three unknown integral coefficients.

The main result of the present paper is the following theorem.

Theorem 1. *For every* $0.5 < \varepsilon \leq 1$ *there exists an effectively computable constant* $C(\varepsilon)$ *such that if* $m \neq 0$, $\max\{m, n\} \geq C(\varepsilon)$ *and*

$$\gcd(m,n) \geq \max\{m^{\varepsilon}, n^{\varepsilon}\},$$

then the system of Pellian equations (2) *and* (3) *has only the trivial solutions* $(V,Z,U) = (\pm m, \pm n, \pm 1)$. *In particular, we may take* $C(0.999) = 10^{27}$, $C(0.99) = 10^{30}$, $C(0.9) = 10^{48}$, $C(0.8) = 10^{71}$, $C(0.7) = 10^{116}$, $C(0.6) = 10^{255}$, $C(0.51) = 10^{3138}$, $C(0.501) = 10^{36836}$.

Corollary 1. *For every* $0.5 < \varepsilon \leq 1$ *there exists an effectively computable constant* $C(\varepsilon)$ *such that if* $m \neq 0$, $\max\{m, n\} \geq C(\varepsilon)$ *and*

$$\gcd(m,n) \geq \max\{m^{\varepsilon}, n^{\varepsilon}\},$$

then Thue equation (1) *has only the trivial solutions* $(x,y) = (\pm 1, 0)$, $(0, \pm 1)$.

Remark 1. In [**13**], it is proven that for all integers m and n there are no nontrivial solutions of (1) satisfying the additional condition $\gcd(xy, mn) = 1$. The result is obtained by considering three cases: $m = n$, $m = 2n$, $n = 2m$. These cases are completely solved by applying a theorem of Bennett [**4**, Theorem 3.2] on simultaneous approximations of algebraic numbers. In all cases we obtain only trivial solutions, except for $m = 1$, $n = 2$, where there are also nontrivial solutions $(x,y) = (4,5)$, $(-4,-5)$, $(5,-4)$, $(-5,4)$. The case $m = 2n$ can be considered as a special case of the Thue equation

$$x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1,$$

which was completely solved in [**10**].

In Section 8 we will find a bound for the number of the solutions of the system (2) and (3). Using a theorem of Bennett [**4**, Theorem 3.2] we will prove the following theorem.

Theorem 2. *System of Pellian equations* (2) *and* (3) *for all* $m \geq 0$ *and* $n \geq 2$ *possess at most* 7 *solutions in positive integers* (V, Z, U).

2. Small values of parameters.

Proposition 1. *Equation* (1) *has only the trivial solutions* $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$ *in the following cases*:

i) $n \leq 1$,

ii) $m = 0$ *and* $2(n^2 - 1)$ *is not a perfect square.*

Proof. The statement of the proposition is trivially true for $n = 0$. On the other hand, for $n = 1$ we have

$$x^4 - 2mx^3y + 2m^2x^2y^2 + 2mxy^3 + y^4 = \left(x^2 - mxy - y^2\right)^2 + x^2y^2\left(m^2 + 2\right),$$

and therefore the statement is true in this case too.

For $m = 0$ we have

$$x^4 + 2(1 - n^2)x^2y^2 + y^4 = 1, \tag{4}$$

which is a special case of equation

$$x^4 - Kx^2y^2 + y^4 = 1, \tag{5}$$

where K is a positive integer. This equation was considered by Cusick in [**7**] and his result is generalized by Walsh in [**30**]. Using Ljunggren's results [**15**], it is proven that equation (5) does not have any solution in positive integers x and y except for the trivial cases where K is a square and $x = 1$ or $y = 1$. In our special case we have: If the sequence (n_k) is defined by

$$n_0 = 1, \quad n_1 = 3, \quad n_{k+2} = 6n_{k+1} - n_k, \quad k \geq 0, \tag{6}$$

then, for $n = n_k$ and $k \geq 1$, all nontrivial solutions of equation (4) are given by

$$(x, y) = \left(\pm 1, \pm\sqrt{2\left(n^2 - 1\right)}\right) \text{ and } \left(\pm\sqrt{2\left(n^2 - 1\right)}, \pm 1\right).$$

For all other values of n we have only the trivial solutions $(x, y) = (\pm 1, 0)\,,\ (0, \pm 1)\,.$ $\square$

Remark 2. For $m = 0$ system of Pellian equations (2) and (3) have nontrivial positive solution $(V, Z, U) = \left(n\sqrt{2\,(n^2-1)},\ 2n^2-3,\ 2n^2-1\right)$ if $n = n_k$, $k \geq 1$, where sequence (n_k) is given by (6).

3. The method of Tzanakis. In this section we will describe the method of Tzanakis [**26**] for solving quartic Thue equations whose corresponding quartic field $\mathbf{K}$ has the properties stated in Section 1.

Consider the quartic Thue equation

$$
\begin{aligned}
&f(x, y) = t \\
&f(x, y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 \\
&\qquad\qquad + 4a_3xy^3 + a_4y^4 \in \mathbf{Z}\,[x, y]\,, \quad a_0 > 0,
\end{aligned}
\tag{7}
$$

whose corresponding quartic field $\mathbf{K}$ is Galois and non-cyclic. By [**18**], this condition on $\mathbf{K}$ is equivalent with $\mathbf{K}$ having three quadratic subfields, which happens exactly when the cubic resolvent of the quartic Thue equation has three distinct rational roots.

It is more convenient to consider the cubic equation

$$
4\rho^3 - g_2\rho - g_3 = 0 \tag{8}
$$

with roots opposite to those of the cubic resolvent of the quartic equation $f(x, 1) = 0$. Here g_2 and g_3 are invariants of the form:

$$
g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\,\mathbf{Z},
$$
$$
g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\,\mathbf{Z}\,.
$$

Therefore, the above conditions on $\mathbf{K}$ are equivalent to the fact that the cubic equation (8) has three rational roots ρ_1, ρ_2, ρ_3. Assuming that $\mathbf{K}$ is not totally complex, then by classification of Nagell (Table on p. 351 of [**18**]), $\mathbf{K}$ is totally real, in fact, it is a compositum of two real quadratic fields. This happens exactly when

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}. \tag{9}$$

Let $H(x, y)$ and $G(x, y)$ be the quartic and sextic covariants of $f(x, y)$, respectively, see [**17**, Chapter 25], i.e.,

$$H(x, y) = -\frac{1}{144}\begin{vmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{vmatrix} \in \frac{1}{48}\,\mathbf{Z}\,[x, y]\,,$$

$$G(x, y) = -\frac{1}{8}\begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial H/\partial x & \partial H/\partial y \end{vmatrix} \in \frac{1}{96}\,\mathbf{Z}\,[x, y]\,.$$

Then

$$4H^3 - g_2 H f^2 - g_3 f^3 = G^2. \tag{10}$$

If we put $H = (1/48)H_0$, $G = (1/96)G_0$, $\rho_i = (1/12)r_i$, $i = 1, 2, 3$, then $H_0, G_0 \in \mathbf{Z}\,[x, y]$, $r_i \in \mathbf{Z}$, $i = 1, 2, 3$. Since f and H are relatively prime in $\mathbf{Q}\,[x, y]$, then, in view of (10),

$$(H_0 - 4r_1 f)(H_0 - 4r_2 f)(H_0 - 4r_3 f) = 3G_0^2$$

and the tree factors on the lefthand side are pairwise relatively prime in $\mathbf{Z}\,[x, y]$. So there exist positive square-free integers k_1, k_2, k_3 and quadratic forms G_1, G_2, $G_3 \in \mathbf{Z}\,[x, y]$ such that

$$H_0 - 4r_i f = k_i G_i^2, \quad i = 1, 2, 3$$

and $k_1 k_2 k_3 (G_1 G_2 G_3)^2 = 3G_0^2$. If $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ is a solution of (7), then

$$k_2 G_2^2 - k_1 G_1^2 = 4(r_1 - r_2)t, \tag{11}$$

$$k_3 G_3^2 - k_1 G_1^2 = 4(r_1 - r_3)t. \tag{12}$$

In this way, solving the Thue equation (7) reduces to solving the system of Pellian equations (11) and (12) with one common unknown.

4. Binary recursive sequences. Let us apply the method from Section 3 to the equation

$$f(x,y) = x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1\,.$$

We have

$$\begin{aligned}
g_2 &= 1 + m^2n^2 + \frac{1}{3}\left(m^2 - n^2 + 1\right)^2,\\
g_3 &= -\frac{1}{54}\left(m^2 + 2n^2 - 2\right)\left(2m^2 + n^2 + 2\right)\left(m^2 - n^2 + 4\right),\\
\rho_1 &= \frac{1}{6}\,m^2 - \frac{1}{6}\,n^2 + \frac{2}{3},\\
\rho_2 &= \frac{1}{6}\,m^2 + \frac{1}{3}\,n^2 - \frac{1}{3},\\
\rho_3 &= -\frac{1}{3}\,m^2 - \frac{1}{6}\,n^2 - \frac{1}{3}.
\end{aligned}$$

If $n \geq 2$, then

$$\begin{aligned}
\frac{a_1^2}{a_0} - a_2 - \rho_1 &= \frac{1}{4}\left(m^2 + 2\right)\left(n^2 - 2\right) \geq 0,\\
\frac{a_1^2}{a_0} - a_2 - \rho_2 &= \frac{1}{4}\,m^2\left(n^2 - 2\right) \geq 0,\\
\frac{a_1^2}{a_0} - a_2 - \rho_3 &= \frac{1}{4}\,n^2\left(m^2 + 2\right) \geq 0.
\end{aligned}$$

Thus, condition (9) is clearly satisfied for $n \geq 2$.

Furthermore, we obtain

$$\begin{aligned}
H_0 - 4r_1f &= 12\left(n^2 - 2\right)\left(m^2 + 2\right)\left(x^2 + y^2\right)^2,\\
H_0 - 4r_2f &= 12\left(n^2 - 2\right)\left(mx^2 + 2nxy - my^2\right)^2,\\
H_0 - 4r_3f &= 12\left(m^2 + 2\right)\left(-nx^2 + 2mxy + ny^2\right)^2.
\end{aligned}$$

By putting $k_1 = 3\left(n^2 - 2\right)\left(m^2 + 2\right)$, $k_2 = 3\left(n^2 - 2\right)$, $k_3 = 3\left(m^2 + 2\right)$,

$$\begin{aligned}
U &= \frac{G_1}{2} = x^2 + y^2,\\
V &= \frac{G_2}{2} = mx^2 + 2nxy - my^2,\\
Z &= \frac{G_3}{2} = -nx^2 + 2mxy + ny^2,
\end{aligned}$$

we get, from (11) and (12), the system of Pellian equations (2) and (3). Consider the system of Pellian equations (2) and (3). Neither m^2+2 nor n^2-2 is a square and both $\mathbf{Q}\left(\sqrt{m^2+2}\right)$ and $\mathbf{Q}\left(\sqrt{n^2-2}\right)$ are real quadratic real number fields. Moreover, by [**19**, Theorem 105], $m^2+1+m\sqrt{m^2+2}$ and $n^2-1+n\sqrt{n^2-2}$ are nontrivial units of norm 1 in number rings $\mathbf{Z}\left[\sqrt{m^2+2}\right]$ and $\mathbf{Z}\left[\sqrt{n^2-2}\right]$, respectively. By [**19**, Theorem 108a], we find that

$$v_1+u_1\sqrt{m^2+2}=m+\sqrt{m^2+2},$$
$$v_2+u_2\sqrt{m^2+2}=-m+\sqrt{m^2+2}$$

are the possible fundamental solutions of equation (2). By [**19**, p. 58], these solutions belong to the same class, so we have only one fundamental solution $m+\sqrt{m^2+2}$.

Similarly, by [**19**, Theorem 108], we find that $n+\sqrt{n^2-2}$ is the only fundamental solution of equation (3).

Hence, all solutions of equation (2) in positive integers are given by

$$v+u\sqrt{m^2+2}=\left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k, \tag{13}$$

where $k\in\mathbf{Z}$ and $k\geq 0$ or by $u=U_k$ and $v=V_k$, where the sequences (U_k) and (V_k) are defined by the recurrences

$$\begin{gathered} U_0=1,\ U_1=2m^2+1,\ U_{k+2}=2\left(m^2+1\right)U_{k+1}-U_k,\ k\geq 0;\\ V_0=m,\ V_1=m\left(2m^2+3\right),\ V_{k+2}=2\left(m^2+1\right)V_{k+1}-V_k,\\ k\geq 0. \end{gathered} \tag{14}$$

All solutions of equation (3) in positive integers are given by

$$z+t\sqrt{n^2-2}=\left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l \tag{15}$$

where $l\in\mathbf{Z}$ and $l\geq 0$ or by $t=T_l$ and $z=Z_l$, where the sequences (T_l) and (Z_l) are defined by the recurrences

$$\begin{gathered} T_0=1,\quad T_1=2n^2-1,\quad T_{l+2}=2\left(n^2-1\right)T_{l+1}-T_l,\quad l\geq 0;\\ Z_0=n,\quad Z_1=n\left(2n^2-3\right),\\ Z_{l+2}=2\left(n^2-1\right)Z_{l+1}-Z_l,\quad l\geq 0. \end{gathered} \tag{16}$$

In this way we reformulated the system of Pellian equations (2) and (3) to the Diophantine equation of the form

$$U_k = T_l$$

in integers $k, l \geq 0$. In order to prove Theorem 1, it suffices to show that $U_k = T_l$ implies $k = l = 0$ in all these special cases.

Solving recurrences (14) and (16) we find

$$\begin{aligned}(17)\quad U_k = \frac{1}{2\sqrt{m^2+2}}\Big[&\left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k \\ &-\left(m-\sqrt{m^2+2}\right)\left(m^2+1-m\sqrt{m^2+2}\right)^k\Big],\end{aligned}$$

$$\begin{aligned}(18)\quad T_l = \frac{1}{2\sqrt{n^2-2}}\Big[&\left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l \\ &-\left(n-\sqrt{n^2-2}\right)\left(n^2-1-n\sqrt{n^2-2}\right)^k\Big].\end{aligned}$$

5. Linear form in logarithms of algebraic numbers.

Lemma 1. *If $m \geq n \geq 2$, $U_k = T_l$ and $k \neq 0$, then the following hold*:

$\mathbf{1}^0$

$$\begin{aligned}0 < l\log\left(n^2-1+n\sqrt{n^2-2}\right) - k\log\left(m^2+1+m\sqrt{m^2+2}\right)& \\ + \log\frac{\sqrt{m^2+2}\,\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\,\left(m+\sqrt{m^2+2}\right)} < 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}&.\end{aligned}$$

$\mathbf{2}^0$ $k < l < 0.5674\cdot t(m)\cdot k$, *where* $t(m) = \log\left(m^2+1+m\sqrt{m^2+2}\right)$.

Proof. $\mathbf{1}^0$ If we put

$$P = \left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k,$$
$$Q = \left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l,$$

then

$$P^{-1} = \frac{-1}{2}\left(m - \sqrt{m^2+2}\right)\left(m^2+1-m\sqrt{m^2+2}\right)^k,$$
$$Q^{-1} = \frac{1}{2}\left(n - \sqrt{n^2-2}\right)\left(n^2-1-n\sqrt{n^2-2}\right)^l.$$

Now, from (17) and (18) it follows that the relation $U_k = T_l$ implies

$$P + 2P^{-1} = A\left(Q - 2Q^{-1}\right),$$

where

$$A = \sqrt{\frac{m^2+2}{n^2-2}} > 1.$$

It is clear that $P > 1$, $Q > 1$, and from

$$AQ - P = 2\left(AQ^{-1} + P^{-1}\right) > \frac{1}{A}Q^{-1} - P^{-1} = \frac{1}{A}(P - AQ)\,Q^{-1}P^{-1},$$

it follows that $AQ > P$, which implies $Q^{-1} < AP^{-1}$. Thus, we have

$$AQ - P = 2AQ^{-1} + 2P^{-1} < 2A^2P^{-1} + 2P^{-1} = 2\left(A^2+1\right)P^{-1}.$$

Since $k \geq 1$, we have

$$\begin{aligned} 2\left(A^2+1\right)P^{-2} &\leq \frac{2\left((m^2+2)/(n^2-2)+1\right)}{\left(\left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)\right)^2} \\ &= f(m,n). \end{aligned}$$

For $m \geq n \geq 2$, function $f(m,n)$ is decreasing in the both variables, which implies

$$2\left(A^2+1\right)P^{-2} \leq f(2,2) < 4.2 \times 10^{-3}.$$

Thus, we have

$$\frac{AQ-P}{AQ} < \frac{2\left(A^2+1\right)}{A}P^{-1}Q^{-1} < 2\left(A^2+1\right)P^{-2} < 4.2 \times 10^{-3}.$$

Hence,

$$\begin{aligned}
0 < \log\frac{AQ}{P} &= -\log\left(1-\frac{AQ-P}{AQ}\right)\\
&< \left(\frac{AQ-P}{AQ}\right)+\left(\frac{AQ-P}{AQ}\right)^2\\
&< \left[1+2\left(A^2+1\right)P^{-2}\right]\cdot 2\left(A^2+1\right)P^{-2}\\
&< \left(1+4.2\times 10^{-3}\right)\cdot\frac{2\left((m^2+2)/(n^2-2)+1\right)}{\left(m+\sqrt{m^2+2}\right)^2}\\
&\quad\cdot\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}.
\end{aligned}$$

The function

$$f_1(m,n)=\frac{2\left((m^2+2)/(n^2-2)+1\right)}{\left(m+\sqrt{m^2+2}\right)^2}$$

is decreasing in both variables, so $f_1(m,n)\le f_1(2,2)<0.4041$. Thus we have

$$0<\log\frac{AQ}{P}<0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k},$$

which implies the assertion.

2^0 From relations (17) and (18) it is clear that $k<l$. We have, by **1**^{0}, that

$$\begin{aligned}
l\log\left(n^2-1+n\sqrt{n^2-2}\right)-k\log\left(m^2+1+m\sqrt{m^2+2}\right)\\
< 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}\\
-\log\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}\\
\le 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2}\\
-\log\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)} = g\left(m,n\right).
\end{aligned}$$

Since $g(m,n) = g_1(m) + g_2(n)$, where

$$g_1(m) = 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2} + \log\frac{m+\sqrt{m^2+2}}{\sqrt{m^2+2}},$$
$$g_2(n) = \log\frac{\sqrt{n^2-2}}{n+\sqrt{n^2-2}},$$

then $g(m,n) \le g_1(m) + g_2(m) = g_3(m)$. Function $g_3(m)$ is continuous, strictly increasing and $\lim_{m\to\infty} g_3(m) = 0$ and we conclude that $g_3(m) < 0$ for $m \ge 2$. Thus, we have

$$l\log\left(n^2-1+n\sqrt{n^2-2}\right) - k\log\left(m^2+1+m\sqrt{m^2+2}\right) < 0,$$

which implies

$$\frac{l}{k} < \frac{\log\left(m^2+1+m\sqrt{m^2+2}\right)}{\log\left(n^2-1+n\sqrt{n^2-2}\right)} = T(m,n).$$

Since function $T(m,n)$ attains its greatest value at $n=2$ for every m, we have

$$\frac{l}{k} < T(m,2) < 0.5674\cdot\log\left(m^2+1+m\sqrt{m^2+2}\right),$$

which implies the assertion. □

Lemma 2. *If $1 \le m < n$, $n \ge 3$, $U_k = T_l$ and $l \ne 0$, then the following hold*:

$\mathbf{1}^0$

$$\begin{aligned}0 &< l\log\left(n^2-1+n\sqrt{n^2-2}\right) - k\log\left(m^2+1+m\sqrt{m^2+2}\right)\\ &\quad - \log\frac{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}\\ &< 0.211\left(n^2-1+n\sqrt{n^2-2}\right)^{-2l},\end{aligned}$$

$\mathbf{2}^0$ $l < k < 0.76\cdot t(n)\cdot l$, *where* $t(n) = \log\left(1.354\left(n^2-1+n\sqrt{n^2-2}\right)\right)$.

Proof. Similarly as in the proof of Lemma 2. For details see [**13**, Lemma 5.14]. □

Note that we have to consider the cases $m = 1$ and $n = 2$ separately. Here systems (2) and (3) have the form

$$V^2 - 3U^2 = -2,$$
$$Z^2 - 2U^2 = 2.$$

We can show, by using a slight modification of [**11**, Lemma 5], which is a variant of the Baker-Davenport reduction procedure [**2**], that this system has two positive solutions $(V, Z, U) = (1, 2, 1), (41, 71, 58)$, so the corresponding Thue equation has solutions given by $(x, y) = (\pm 1, 0), (0, \pm 1), (4, 5), (-4, -5), (5, -4), (-5, 4)$. For details, see [**13**, Lemma 5.10].

6. The congruence method. Let $g = \gcd(m, n)$.

Lemma 3. *Let the sequences* (U_k) *and* (T_l) *be defined by* (14) *and* (16)*. Then for all* $k, l \geq 0$ *we have*

$$U_k \equiv \left[k\,(k+1)\,m^2 + 1\right] \pmod{2g^4}, \tag{19}$$
$$T_l \equiv (-1)^{l+1}\left[l\,(l+1)\,n^2 - 1\right] \pmod{2g^4}. \tag{20}$$

Proof. Both relations are obviously true for $k, l \in \{0, 1\}$.

Assume that (19) is valid for $k-1$ and k. Then

$$\begin{aligned} U_{k+1} &= 2\left(m^2+1\right)U_k - U_{k-1} \\ &\equiv 2\left(m^2+1\right)\left(k^2m^2+km^2+1\right) - \left((k-1)^2\,m^2 + (k-1)\,m^2+1\right) \\ &= \left(2k\,(k+1)\,m^4 + \left(k^2+3k+2\right)m^2+1\right) \\ &\equiv \left[(k+1)\,((k+1)+1)\,m^2+1\right] \pmod{2g^4}. \end{aligned}$$

Assume that (20) is valid for $l-1$ and l. Then

$$\begin{aligned}
T_l &= 2\left(n^2-1\right)T_l - T_{l-1}\\
&\equiv 2\left(n^2-1\right)(-1)^{l+1}\left(l^2n^2+ln^2-1\right) - (-1)^l\left((l-1)^2n^2+(l-1)\,n^2-1\right)\\
&= 2\,(-1)^{l+1}\left(l^2+l\right)n^4 + (-1)^l\left(l^2+3l+2\right)n^2 - (-1)^l\\
&\equiv (-1)^{l+2}\left[(l+1)\,((l+1)+1)\,n^2-1\right] \pmod{2g^4}. \qquad \square
\end{aligned}$$

Suppose that k and l are positive integers such that $U_k = T_l$. Then, of course, $U_k \equiv T_l \pmod{2g^4}$. Furthermore suppose that $m = m_1 g$, $n = n_1 g$ and $g > 1$. By Lemma 3, we have $1 \equiv (-1)^l \pmod{2g^2}$ and therefore l is even, say $l = 2l_1$. Further, Lemma 3 implies

$$k\,(k+1)\,m^2 + 1 \equiv 1 - 2l_1\,(2l_1+1)\,n^2 \pmod{2g^4},$$

and

$$k\,(k+1)\,m_1^2 \equiv -2l_1\,(2l_1+1)\,n_1^2 \pmod{2g^2},$$

which implies

$$\frac{k\,(k+1)}{2}\,m_1^2 \equiv -l_1\,(2l_1+1)\,n_1^2 = -\frac{l\,(l+1)}{2}\,n_1^2 \pmod{2g^2}. \tag{21}$$

Consider the positive integer

$$Y = \frac{k\,(k+1)}{2}\,m_1^2 + \frac{l\,(l+1)}{2}\,n_1^2.$$

1) Let $m \geq n$. Assume that $l \leq g/m_1$. We have by Lemma 1, 2^0, that $k < l$, and since $n_1 \leq m_1$ we also have

$$\frac{l\,(l+1)}{2}\,n_1^2 \leq \frac{l\,(l+1)}{2}\,m_1^2 \leq g^2$$

and

$$\frac{k\,(k+1)}{2}\,m_1^2 < \frac{l\,(l+1)}{2}\,m_1^2 \leq g^2.$$

So we have $0 < Y < 2g^2$ and, by (21), $Y \equiv 0 \pmod{2g^2}$, a contradiction.

2) Let $m < n$. Assume that $k \leq g/n_1$. We have, by Lemma 2, 2^0, that $l < k$, and since $m_1 < n_1$ we also have

$$\frac{k\,(k+1)}{2}\,m_1^2 < \frac{k\,(k+1)}{2}\,n_1^2 \leq g^2$$

and

$$\frac{l\,(l+1)}{2}\,n_1^2 < \frac{k\,(k+1)}{2}\,n_1^2 \leq g^2.$$

So, we have $0 < Y < 2g^2$, and by (21), we obtain a contradiction as before.

Therefore we proved

Proposition 2. *If $U_k = T_l$, $(k,l) \neq (0,0)$, $g = \gcd(m,n) > 1$, $m = m_1 g$, $n = n_1 g$, then we have*:

1) *if $m \geq n$ then $l > g/m_1$,*

2) *if $m < n$ then $k > g/n_1$.*

Corollary 2. *If $U_k = T_l$, $(k,l) \neq (0,0)$, $g = \gcd(m,n) > 1$, $m = m_1 g$, $n = n_1 g$, then we have*:

1) *if $m \geq n$ then $U > \left(2n^2 - 3\right)^{g/m_1}$,*

2) *if $m < n$ then $U > \left(1.8m^2 + 0.9\right)^{g/n_1}$.*

Proof. From relations (18) and (17) we have

$$\begin{aligned}
U &> \left(n^2 - 1 + n\sqrt{n^2-2}\right)^l > \left(2n^2-3\right)^l,\\
U &> \frac{m + \sqrt{m^2+2}}{2\sqrt{m^2+2}}\left(m^2 + 1 + m\sqrt{m^2+2}\right)^k\\
&> \frac{9}{10}\left(2m^2+1\right)^k > \left(1.8m^2+0.9\right)^k,
\end{aligned}$$

respectively, so assertions **1)** and **2)** follow directly from Proposition 2. $\square$

7. The proof of Theorem 1. Now we will apply the following famous theorem of Baker and Wüstholz [**3**]:

Theorem 3. *For a linear form $\Lambda \neq 0$ in logarithms of s algebraic numbers $\alpha_1, \dots, \alpha_s$ with rational integer coefficients $b_1, \dots, b_s$ we have*

$$\log \Lambda \geq -18(s+1)!\, s^{s+1}(32d)^{s+2}h'(\alpha_1)\cdots h'(\alpha_s)\log(2sd)\log G\,,$$

where $G = \max\{|b_1|, \dots, |b_s|\}$, and where d is the degree of the number field generated by $\alpha_1, \dots, \alpha_s$.

Here

$$h'(\alpha) = \frac{1}{d}\max\{h(\alpha), |\log\,\alpha|, 1\}\,,$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

1) Let $m \geq n$, $g = \gcd(m,n)$, $n = n_1 g$, $m = m_1 g = Bn_1 g$. We will apply Theorem 3 to the form from Lemma 1. We have $s = 3$, $d = 4$, $G = l$,

$$\alpha_1 = n^2 - 1 + n\sqrt{n^2-2}, \quad \alpha_2 = m^2 + 1 + m\sqrt{m^2+2},$$
$$\alpha_3 = \frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}.$$

Under the assumption $g > 1$, we have

$$\begin{aligned}
h'(\alpha_1) &= \frac{1}{2}\log\left(n^2-1+n\sqrt{n^2-2}\right) < \frac{1}{2}\log\left(2n^2\right) \\
&= \log\left(\sqrt{2}n_1 g\right) < 4.329\cdot\log\left(\sqrt{2}n_1\right)\cdot\log g \\
h'(\alpha_2) &= \frac{1}{2}\log\left(m^2+1+m\sqrt{m^2+2}\right) \\
&< \frac{1}{2}\log\left(\left(2+\sqrt{2}\right)m^2\right) \\
&< \log\left(2Bn_1 g\right) < 2.886\cdot\log\left(2Bn_1\right)\cdot\log g.
\end{aligned}$$

Furthermore, we find that

$$\alpha_3 = |\alpha_3| = \frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)} < \frac{n\sqrt{m^2+2}}{m\sqrt{n^2-2}},$$

and the conjugates of α_3 satisfy

$$\alpha_3' = |\alpha_3'| = \frac{\sqrt{m^2+2}\left(n-\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)} < 1,$$
$$|\alpha_3''| = \frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(\sqrt{m^2+2}-m\right)} < \frac{2n\left(m^2+2\right)}{\sqrt{n^2-2}},$$
$$|\alpha_3'''| = \frac{\sqrt{m^2+2}\left(n-\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(\sqrt{m^2+2}-m\right)} < \frac{m^2+2}{n^2-2}.$$

Therefore,

$$\begin{aligned} h'\left(\alpha_3\right) &= h\left(\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}\right) \\ &= \frac{1}{4}\log\left[\left(n^2-2\right)^2\prod_{i=1}^{4}\max_j\left(1,\left|\alpha^{(j)}\right|\right)\right] \\ &< \frac{1}{4}\log\left(\frac{2n^2}{m}\left(m^2+2\right)^{5/2}\right) \\ &< 7.576\cdot\log\left(\sqrt{2}n_1B^{5/7}\right)\cdot\log g. \end{aligned}$$

Hence, Theorem 3 implies

$$\begin{aligned} \log\Lambda > &-3.617\times10^{17}\cdot\log\left(\sqrt{2}n_1\right)\cdot\log\left(2Bn_1\right) \\ &\cdot\log\left(\sqrt{2}n_1B^{5/7}\right)\cdot\left(\log g\right)^3\cdot\log l. \end{aligned} \tag{22}$$

On the other hand, Lemma 1 implies

$$\begin{aligned} \log\Lambda &< \log\left[0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}\right] \\ &< -2k\log\left(m^2+1+m\sqrt{m^2+2}\right) < -2k\cdot\log g^2 \\ &< 7.04\cdot\frac{l}{t\left(m\right)}\cdot\log g, \end{aligned} \tag{23}$$

where

$$\begin{aligned} t\left(m\right) &= \log\left(m^2+1+m\sqrt{m^2+2}\right) < 2\log\left(2Bn_1g\right) \\ &< 5.773\cdot\log\left(2Bn_1\right)\cdot\log g \end{aligned} \tag{24}$$

Combining (22), (23) and (24) we obtain

$$\frac{l}{\log l} < F(B, n_1) \cdot \log^3 g,$$

where

$$\begin{aligned} F(B, n_1) &= 2.967 \times 10^{17} \cdot \log\left(\sqrt{2}n_1\right) \cdot \log^2(2Bn_1) \cdot \log\left(\sqrt{2}n_1 B^{5/7}\right) \\ &< 2.967 \times 10^{17} \cdot \log^4(2Bn_1) = 2.967 \times 10^{17} \cdot \log^4(2m_1). \end{aligned}$$

By Proposition 2, **1)**, we have $g/m_1 < l$, which implies

$$\begin{aligned} \frac{l}{\log l} &< 2.967 \times 10^{17} \cdot \log^4(2m_1) \cdot \log^3(lm_1) \\ &< 7.133 \times 10^{18} \cdot \log^7(2m_1) \cdot \log^3 l. \end{aligned} \tag{25}$$

If (25) implies $l < l_0$, then

$$g < l \cdot m_1 < l_0 \cdot m_1.$$

For every $m_1 \geq 1$, (25) has form

$$l < K \cdot \log^4 l, \tag{26}$$

where $K = K(m_1) = 7.133 \times 10^{18} \cdot \log^7(2m_1) > 5.483 \times 10^{17}$. We have

$$K < K \log^4 K,$$

which implies $K < l_0$. Therefore, we can assume that $l_0 = K^{1+t_0}$, where $t_0 > 0$. Let us define a function $t_{\min}(K)$, which is implicitly given by

$$K \log^4 K^{1+t_{\min}} = K^{1+t_{\min}},$$

with $t_{\min}(K) > 0$. We find that $t_{\min}(K)$ is a continuous and decreasing function, so we have $t_{\min}(K) < t_{\min}\left(5.483 \times 10^{17}\right) < 0.396$. Hence, we may take $t_0 = 3/7$, i.e.,

$$l_0 = K^{10/7} < 8.576 \times 10^{26} \log^{10}(2m_1),$$

which implies

$$g < K^{10/7} m_1 < 8.576 \times 10^{26} \cdot m_1 \cdot \log^{10}(2m_1). \tag{27}$$

Assume that $g = m^{\varepsilon}$, $0 < \varepsilon \le 1$. Then, from (27), we have

$$m^{\varepsilon} < G \cdot m^{(1-\varepsilon)} \cdot \log^{10}\left(2m^{(1-\varepsilon)}\right), \tag{28}$$

where $G = 8.576 \times 10^{26}$. Let us define a function $M_0(\varepsilon)$ which is implicitly given by

$$M_0^{\varepsilon} = G \cdot M_0^{1-\varepsilon} \cdot \log^{10}\left(2M_0^{(1-\varepsilon)}\right), \tag{29}$$

with $M_0(\varepsilon) \ge 2$. We find that $M_0(\varepsilon)$ is defined for every $0.5 < \varepsilon \le 1$ and $M_0(\varepsilon)$ is a strictly decreasing function of ε. Hence $M_0(\varepsilon) \ge M_0(1) > 2.195 \times 10^{25}$.

If $0 < \varepsilon \le 0.5$ then (28) is satisfied for every m, which again implies that (27) is satisfied for every $1 < g \le \sqrt{m}$. This case is not interesting.

If $\varepsilon = 1$, i.e., $m = n = g$, then (28) implies that for $g \ge 2.195 \times 10^{25}$ we have only the trivial solutions. This case is completely solved in [**13**], as we mentioned before.

Let us suppose $\varepsilon_0 \in (0.5, 1)$ and $M_0(\varepsilon_0) = m_0$. Then

$$m^{\varepsilon_0} < G \cdot m^{(1-\varepsilon_0)} \log^{10}\left(2m^{(1-\varepsilon_0)}\right)$$

implies $m < m_0$. So, for $g = m^{\varepsilon}$, $m \ge m_0$ and $g \ge m^{\varepsilon_0} \ge m_0^{\varepsilon_0}$ we have only the trivial solutions $(\pm 1, 0)$ and $(0, \pm 1)$ of equation (1). Let us give some special cases:

- if $\varepsilon_0 = 0.999$, which implies $m_0 < 10^{26}$, then for $m \ge 10^{26}$ and $\varepsilon \ge 0.999$, i.e.,

$$g \ge m^{0.999} = \max\left\{m^{0.999}, n^{0.999}\right\} \ge 10^{25.974} > (m_0)^{0.999},$$

we have only the trivial solutions.

- Similarly we find that, for $g \ge m^{0.99} \ge 10^{28.71}$, $g \ge m^{0.9} \ge 10^{42.3}$, $g \ge m^{0.8} \ge 10^{56.8}$, $g \ge m^{0.7} \ge 10^{80.5}$, $g \ge m^{0.6} \ge 10^{152.4}$, $g \ge m^{0.51} \ge 10^{1591.2}$, $g \ge m^{0.501} \ge 10^{18453.834}$, we have only the trivial solutions.

2) If we suppose that $1 \le m < n$ and $g = \gcd(m, n) > 1$, then in the similar manner, we can find that for every $\varepsilon_0 \in (0.5, 1)$ if

$$g \ge n^{\varepsilon_0} = \max\{m^{\varepsilon_0}, n^{\varepsilon_0}\},$$

and $n \geq n_0$, where $n_0 = N_0(\varepsilon_0)$ and $N_0(\varepsilon)$ is a strictly decreasing function implicitly given by

$$N_0^{\varepsilon} = 1.875 \times 10^{27} \cdot N_0^{(1-\varepsilon)} \cdot \log^{10}\left(2N_0^{(1-\varepsilon)}\right)$$

with $N_0(\varepsilon) \geq 2$, then equation (1) has only the trivial solutions. In particular, we find that, for $g \geq n^{0.999} \geq 10^{26.973}$, $g \geq n^{0.99} \geq 10^{29.7}$, $g \geq n^{0.9} \geq 10^{43.2}$, $g \geq n^{0.8} \geq 10^{56.8}$, $g \geq n^{0.7} \geq 10^{81.2}$, $g \geq n^{0.6} \geq 10^{153}$, $g \geq n^{0.51} \geq 10^{1600.38}$, $g \geq n^{0.501} \geq 10^{18454.836}$ we have only the trivial solutions. (For details, see [**13**, Section 5.4.3].)

Let us remark that $N_0(\varepsilon) > M_0(\varepsilon)$ for every $\varepsilon \in (0.5, 1)$, so the combination of **1)** and **2)** finishes the proof of Theorem 1.

8. Bound for the number of the solutions. In [**4**], Bennett proved that the system of simultaneous Pell equations

$$x^2 - az^2 = 1, \qquad y^2 - bz^2 = 1 \tag{30}$$

where a and b are distinct positive integers, possess at most three solutions in positive integers (x, y, z). Recently, Yuan [**31**] proved that if $\max(a, b) > 1.4 \cdot 10^{57}$, then the system (30) possesses at most two solutions.

In this section we follow Bennett [**4**] in proving that system (2) and (3) has at most 7 solutions in positive integers (V, Z, U).

Let $m \geq 0$, $n \geq 2$ and $V = V_{k_i}$, $Z = Z_{l_i}$, $U = U_{k_i} = T_{l_i}$ be positive solutions of systems (2) and (3) for $i = 1, 2, 3$, where $U_{k_1} < U_{k_2} < U_{k_3}$. We consider the determinant Δ defined as follows

$$\begin{aligned}\Delta &= \begin{vmatrix} V_{k_1} & Z_{l_1} & U_{k_1} \\ V_{k_2} & Z_{l_2} & U_{k_2} \\ V_{k_3} & Z_{l_3} & U_{k_3} \end{vmatrix} \\ &= \begin{vmatrix} V_{k_1} - \sqrt{m^2+2}\,U_{k_1} & Z_{l_1} - \sqrt{n^2-2}\,T_{l_1} & U_{k_1} \\ V_{k_2} - \sqrt{m^2+2}\,U_{k_2} & Z_{l_2} - \sqrt{n^2-2}\,T_{l_2} & U_{k_2} \\ V_{k_3} - \sqrt{m^2+2}\,U_{k_2} & Z_{l_3} - \sqrt{n^2-2}\,T_{l_3} & U_{k_3}. \end{vmatrix}\end{aligned}$$

If we expand Δ along the third column and define $\alpha_i = V_{k_i} + \sqrt{m^2+2}\,U_{k_i}$ and $\beta_i = Z_{l_i} + \sqrt{n^2-2}\,T_{l_i}$, as in [**4**, Lemmas 6.1 and 6.2], we find

$$\Delta = \frac{1}{\sqrt{(m^2+2)(n^2-2)}\,U_{k_1}U_{k_2}U_{k_3}} \sum_{1 \leq i \leq 3} U_{k_i}^2 \delta_i$$

where

$$\begin{aligned}
\delta_1 &= 2\left(\alpha_3^{-2}-\alpha_2^{-2}\right)-2\left(\beta_2^{-2}-\beta_3^{-2}\right)+4\left(\alpha_2^{-2}\beta_3^{-2}-\alpha_3^{-2}\beta_2^{-2}\right)\\
\delta_2 &= 2\left(\alpha_1^{-2}-\alpha_3^{-2}\right)-2\left(\beta_3^{-2}-\beta_1^{-2}\right)+4\left(\alpha_3^{-2}\beta_1^{-2}-\alpha_1^{-2}\beta_3^{-2}\right)\\
\delta_3 &= 2\left(\alpha_2^{-2}-\alpha_1^{-2}\right)-2\left(\beta_1^{-2}-\beta_2^{-2}\right)+4\left(\alpha_1^{-2}\beta_2^{-2}-\alpha_2^{-2}\beta_1^{-2}\right).
\end{aligned}$$

We can prove $|\delta_i| < 1/U_{k_1}^2 + 1/9U_{k_1}^4$ for $i = 1, 2, 3$, which implies that

$$\begin{aligned}
|\Delta| &\leq \frac{1}{\sqrt{(m^2+2)(n^2-2)}\,U_{k_1}U_{k_2}U_{k_3}}\sum_{1\leq i\leq 3} U_{k_i}^2\,|\delta_i|\\
&< \frac{10}{3\sqrt{(m^2+2)(n^2-2)}\cdot U_{k_1}^3U_{k_2}}\,U_{k_3}.
\end{aligned}$$

Since Δ is an integer and $\Delta \neq 0$, it follows that

$$U_{k_3} > \frac{3\sqrt{(m^2+2)(n^2-2)}}{10}\,U_{k_1}^3U_{k_2}.$$

If $m \geq n$, except for $m = n = 2$, we have

$$U_{k_3} > 1.4\cdot U_{k_1}^3U_{k_2}. \tag{31}$$

From recurrences (14) we obtain

$$U_5 < 8\left(m^2+1\right)^3U_2 < 1.4\cdot\left(2m^2+1\right)^3U_2 = 1.4\cdot U_1^3U_2,$$

which implies $k_3 \geq 6$. If $m = n = 2$ we have only a trivial positive solution $(V, Z, U) = (2, 2, 1)$. For details, see [**13**, Proposition 8.8].

If $m < n$, except for $n = 2$, using $U_{k_i} = T_{l_i}$, we find $T_{l_3} > 1.9\cdot T_{l_1}^3T_{l_2}$. Similarly, from recurrences (16), we obtain

$$T_5 < 8\left(n^2-1\right)^3T_2 < 1.9\cdot\left(2n^2-1\right)^3T_2 = 1.9\cdot T_1^3T_2,$$

which implies $l_3 \geq 6$. If $m = 0$ and $n = 2$, it's easy to show that we have only the trivial nonnegative solution $(V, Z, U) = (0, 2, 1)$. If $m = 1$ and $n = 2$ we have two positive solutions as we mentioned at the end of Section 5.

Let $M = \max\left\{m^2+2, n^2-2\right\}$ and $(V_{k_i}, Z_{l_i}, U_{k_i})$, $(V_{k_j}, Z_{l_j}, U_{k_j})$ be two positive solutions of (2) and (3) which satisfy $U_{k_i} < U_{k_j}$. We will use the following theorem of Bennett [**4**, Theorem 3.2].

Theorem 4. *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $N > M_1^9$, where*

$$M_1 = \max_{0 \le i \le 2}\{|a_i|\},$$

then we have

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} (a_2-a_0)^2(a_2-a_1)^2/\left(2a_2-a_0-a_1\right) & \text{if } a_2-a_1 \ge a_1-a_0, \\ (a_2-a_0)^2(a_1-a_0)^2/\left(a_1+a_2-2a_0\right) & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

We will apply Theorem 4 with $a_0 = 0$, $a_1, a_2 \in \left\{2\left(m^2+2\right), 2\left(n^2-2\right)\right\}$, $M_1 = 2M$, $N = \left(m^2+2\right)\left(n^2-2\right)U_{k_i}^2$, $q = \left(m^2+2\right) \times \left(n^2-2\right)U_{k_i}U_{k_j}$, $p_1 = \left(m^2+2\right)V_{k_i}V_{k_j}$, $p_2 = \left(n^2-2\right)Z_{l_i}Z_{l_j}$. Since $N \ge M_1U_{k_i}^2$, we may apply Theorem 4 if $U_{k_i} > 16M^4$. We have, see [**4**, Lemma 6.2]

$$\max\left\{\left\{\left|\sqrt{1+\frac{a_1}{N}} - \frac{p_1}{q}\right|, \left|\sqrt{1+\frac{a_2}{N}} - \frac{p_2}{q}\right|\right\}\right\} < \frac{2}{U_{k_j}^2}.$$

Also $N\gamma \le 8 \cdot M^5 U_{k_i}^2$ and

$$N^2 \prod_{0\le i<j\le 2}(a_i-a_j)^{-2} \ge \frac{U_{k_i}^4}{2^8 \cdot M^2}.$$

So, for $m \geq n$ and $m \geq 3$, $U_{k_i} > 33.88 \cdot M^6 > 6 \times 10^7$ implies $\lambda < 1.8635$. It follows from Theorem 4 that

$$(130 \cdot 8M^5 U_{k_i}^2)^{-1} (M^2 U_{k_i} U_{k_j})^{-1.8635} < \frac{2}{U_{k_j}^2},$$

which implies

$$U_{k_j} < 2080^{7.327} M^{63.935} U_{k_i}^{28.305} < 1.0129 \times 10^8 \cdot U_{k_i}^{38.97} < U_{k_i}^{40}. \tag{32}$$

For $m < n$, except for $n = 2$, we can prove, using $U_{k_i} = T_{l_i}$ and $U_{k_j} = T_{l_j}$ that $T_{l_i} > 148.6 \cdot M^6 > 1.7 \times 10^7$ implies $\lambda < 1.842$. So, it follows from Theorem 4 that $T_{l_j} < T_{l_i}^{34}$.

Let $(V_{k_i}, Z_{l_i}, U_{k_i})$, where $U_{k_i} = T_{l_i}$ is a sequence of all positive solutions of (2) and (3) with $U_{k_i} < U_{k_j}$ for $i < j$.

Let $m \geq n$. Since $U_{k_3} \geq U_6 > 33.88 \cdot M^6$, except for $m = n = 2$, we have $U_{k_i} < U_{k_3}^{40}$ for $i \geq 4$. Using (31) successively, we obtain: $U_{k_5} > U_{k_4} U_{k_3}^3 > U_{k_3}^4$, $U_{k_6} > U_{k_5} U_{k_4}^3 > U_{k_3}^7$, $U_{k_7} > U_{k_6} U_{k_5}^3 > U_{k_3}^{19}$, $U_{k_8} > U_{k_7} U_{k_6}^3 > U_{k_3}^{40}$. So we have, by (32), a contradiction; hence, k_8 doesn't exist.

For $m < n$, except for $n = 2$, since $T_{l_3} \geq T_6 > 148.6 \cdot M^6$, we have $T_{l_i} < T_{l_3}^{34}$ for $i \geq 4$. Now we obtain, as before, that l_8 doesn't exist.

Therefore, we proved Theorem 2.

Remark 3. The result from Theorem 2 on the system of Pellian equations (2) and (3) is weaker than the corresponding Bennett's result on systems of Pell equations. This is mainly because, at present, there is no analogon of Bennett's doubly exponential gap principle [**4**, Lemma 2.2] for systems of Pellian equations. The difference between Pell and Pellian equations is also the reason why we had to use linear forms in three logarithms in Section 5, while in [**4**, Section 4] linear forms in two logarithms were used.

Acknowledgments. The author would like to thank Professor Andrej Dujella for helpful suggestions.

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