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ANALYTIC FUNCTIONS WITH H^p-DERIVATIVE

DANIEL GIRELA AND MARÍA AUXILIADORA MÁRQUEZ

ABSTRACT. For f an analytic function in the unit disc Δ, the following results are well known:

(1) If $f' \in H^p$ with $0 < p < 1$ then $f \in H^{p/(1-p)}$.

(2) If $f' \in H^1$, then f belongs to the disc algebra.

(3) If $f' \in H^p$ with $1 < p < \infty$, then f belongs to the Lipschitz space Λ_α with $\alpha = (p-1)/p$.

Both (1) and (2) have been shown to be sharp in a strong sense. We prove constructively that (3) is also very strongly sharp.

In spite of what we have just said, Aleman and Cima have recently obtained an improvement of (1) showing that a certain condition which is weaker than the condition $f' \in H^p$, $0 < p < 1$, is enough to conclude that $f \in H^{p/(1-p)}$. In this paper we also obtain the analogues of this for $p \geq 1$.

1. Introduction and main results. Let Δ denote the unit disc $\{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{T}$ the unit circle $\{\xi \in \mathbf{C} : |\xi| = 1\}$. For $0 \leq r < 1$ and f analytic in Δ, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad \text{if } 0 < p < \infty,$$
$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions f, analytic in Δ, for which

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0 \leq r < 1} M_p(r,f) < \infty.$$

There are a good number of classical and well-known results showing that the condition $f' \in H^p$ for a certain value of p implies that the function f belongs to a certain space of analytic functions.

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First, we recall the following result due to Hardy and Littlewood, see [**6**, Theorem 5.12].

Theorem A. *Let f be a function which is analytic in Δ. If $0 < p < 1$ and $f' \in H^p$ then $f \in H^q$, where $q = p/(1-p)$.*

Taking $f'(z) = (1-z)^{\varepsilon - 1/p}$ for small $\varepsilon > 0$ we see that for each value of $p \in (0,1)$ the index q is best possible. In [**14**] we proved that Theorem A is sharp in a much stronger sense.

Theorem A can be extended to $p = 1$. Indeed, a result of Privalov [**6**, Theorem 3.11] implies that

$$\boxed{f' \in H^1 \Longrightarrow f \in \mathcal{A} \subset H^\infty} \tag{1.1}$$

where, as usual, $\mathcal{A}$ denotes the disc algebra formed by all functions which are analytic in Δ and have a continuous extension to $\overline{\Delta}$. This result has also been shown to be sharp, see [**4, 9, 10, 17**] and [**13**].

Now we consider functions f which are analytic in Δ whose derivative f' belongs to H^p for some $p > 1$. In this case the function f belongs to a certain Lipschitz space.

If f is a function which is analytic in Δ and has a nontangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbf{T}$, we define

$$\omega_p(t; f) = \sup_{0<|h|\le t} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i(\theta+h)}) - f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad t > 0, \quad \text{if } 1 \le p < \infty,$$

$$\omega_\infty(t; f) = \sup_{0<|h|\le t} \left(\operatorname*{ess.sup}_{\theta \in [-\pi,\pi]} |f(e^{i(\theta+h)}) - f(e^{i\theta})| \right), \quad t > 0.$$

Then $\omega_p(\,.\,, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f. Given $1 \le p \le \infty$ and $0 < \alpha \le 1$, the mean Lipschitz space Λ^p_α is defined as

$$\Lambda^p_\alpha = \{ f \in H^p : \omega_p(t; f) = \mathrm{O}(t^\alpha), \text{ as } t \to 0 \}.$$

A theorem of Hardy and Littlewood [**15**], see also Chapter 5 of [**6**], asserts that for $1 \le p \le \infty$ and $0 < \alpha \le 1$, we have

(1.2)
$$\Lambda^p_\alpha = \left\{ f \text{ analytic in } \Delta : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}}\right), \text{ as } r \to 1 \right\}.$$

If $p = \infty$ we write Λ_α instead of Λ^∞_α. Thus, a function f which is analytic in Δ belongs to Λ_α if and only if it has a continuous extension to the closed unit disc $\overline{\Delta}$ and its boundary values satisfy a Lipschitz condition or order α.

Hardy and Littlewood also proved the following result.

Theorem B. *Let f be a function which is analytic in Δ. If $1 < p \le \infty$ and $f' \in H^p$ then $f \in \Lambda_\alpha$, where $\alpha = 1 - 1/p$.*

Our first goal in this paper is to show that Theorem B is also sharp in a very strong sense. In order to do this we shall use the following generalization of the mean Lipschitz spaces Λ^p_α which occurs frequently in the literature. If $\omega\colon [0, \pi] \to [0, \infty)$ is a continuous and increasing function with $\omega(0) = 0$ and $w(t) > 0$ if $0 < t \le \pi$, then, for $1 \le p \le \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions $f \in H^p$ which satisfy
$$\omega_p(t; f) = \mathrm{O}\big((\omega(t)\big), \quad \text{as } t \to 0.$$
Let us notice that taking $0 < \alpha \le 1$ and $\omega(t) = t^\alpha$, we have

$$\Lambda^p_\alpha = \Lambda(p, t^\alpha), \quad 1 \le p \le \infty, \quad 0 < \alpha \le 1. \tag{1.3}$$

The question of finding conditions on ω so that it is possible to obtain results on the spaces $\Lambda(p, \omega)$ analogous to those proved by Hardy and Littlewood for the spaces Λ^p_α has been studied by several authors, see e.g., [**5**] and [**16**]. If $\omega\colon [0, \pi] \to [0, \infty)$ is a continuous and increasing function with $\omega(0) = 0$ and $w(t) > 0$ if $0 < t \le \pi$, we say that ω satisfies the Dini condition or that ω is a Dini-weight if there exists a positive constant C such that

$$\int_0^t \frac{\omega(s)}{s}\, ds \le C\omega(t), \quad 0 < t < \pi, \tag{1.4}$$

and we say that ω satisfies the condition b_1 or that $\omega \in b_1$ if there exists a positive constant C such that

$$\int_t^\pi \frac{\omega(s)}{s^2}\, ds \le C\, \frac{\omega(t)}{t}, \quad 0 < t < \pi. \tag{1.5}$$

Note that if $0 < \alpha < 1$ then $\omega(t) = t^\alpha$ is a Dini and b_1 weight.

Blasco and Soares de Souza obtained in [**5**] the following extension of (1.2).

Theorem C. *Let $1 \le p \le \infty$ and let $\omega\colon [0,\pi] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $w(t) > 0$ if $0 < t \le \pi$. If ω is a Dini-weight and satisfies the condition b_1, then*

(1.6)

$$\Lambda(p,\omega) = \left\{ f \text{ analytic in } \Delta : M_p(r,f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

Now let $1 < p < \infty$ and $\alpha = 1 - 1/p$, and observe that, using (1.2) and (1.3), one can write Theorem B in the form

$$f \in \Lambda_1^p = \Lambda(p,t) \Longrightarrow f \in \Lambda_\alpha,$$

or, equivalently,

$$\Lambda(p,t) \subset \Lambda_\alpha. \tag{1.7}$$

Our first result in this paper asserts that this is sharp.

Theorem 1. *Let $1 < p < \infty$ and $\alpha = 1 - 1/p$. If $\omega\colon [0,\pi] \to [0,\infty)$ is a continuous and increasing function with $\omega(0) = 0$ and $w(t) > 0$ if $0 < t \le \pi$, such that ω is a Dini-weight, satisfies the condition b_1 and*

$$\limsup_{\delta\to 0} \frac{\omega(\delta)}{\delta} = \infty, \tag{1.8}$$

then there exists a function $f \in \Lambda(p,\omega)$ such that $f \notin \Lambda_\alpha$.

The proof of this result will be presented in Section 2.

Even though, as we noticed above, Theorem A is sharp in a very strong sense, Aleman and Cima have recently proved in [**1**] an extension of this result as follows.

If $0 < p < \infty$ and p_1, p_2 are positive numbers such that $1/p = 1/p_1 + 1/p_2$, then given $g \in H^p$, g can be written in the form

$$g = Bh = Bh^{p/p_1} h^{p/p_2},$$

where B is the Blaschke product with the same zeros as g and $h \in H^p$. Then, setting $g_1 = Bh^{p/p_1}$ and $g_2 = h^{p/p_2}$ we see that g can be factored as

$$g = g_1 g_2, \quad \text{with} \quad g_1 \in H^{p_1}, g_2 \in H^{p_2}. \tag{1.9}$$

Using Hölder's inequality one can easily see that any function which can be written in this way belongs to H^p. Thus, we have

Lemma 1.1. *If $0 < p < \infty$ and p_1, p_2 are positive numbers such that $1/p = 1/p_1 + 1/p_2$, then $H^p = H^{p_1} \cdot H^{p_2}$.*

Now, as is well known, see, e.g., the lemma on page 36 of [**6**], if $g_2 \in H^{p_2}$ then

$$M_\infty(r, g_2) = \mathrm{O}\left(\frac{1}{(1-r)^{1/p_2}}\right), \quad \text{as } r \to 1, \tag{1.10}$$

but the class of functions which are subject to this growth restriction is much larger than H^{p_2}. Using this fact and Lemma 1.1, we see that the following result due to Aleman and Cima [**1**] is an improvement of Theorem A.

Theorem D. *Let $0 < p < 1$ and $p_1 > 0$, $p_2 > 1$ be such that $1/p = 1/p_1 + 1/p_2$. Let f be an analytic function in Δ such that f' can be factored in the form $f' = g_1 g_2$ where $g_1 \in H^{p_1}$ and g_2 is an analytic function in Δ which satisfies* (1.10)*. Then $f \in H^q$, where $q = p/(1-p)$.*

Since (1.1) is a natural extension of Theorem A to $p = 1$, it is natural to ask whether the condition

(1.11)

$$\boxed{f' = g_1 g_2 \text{ with } g_1 \in H^{p_1}, g_2 \text{ satisfying (1.10) and } 1 = \frac{1}{p_1} + \frac{1}{p_2}}$$

implies that $f \in H^\infty$ or not. We shall see that the answer to this question is negative.

Theorem 2. *Let p_1 and p_2 be positive numbers with $1 = 1/p_1 + 1/p_2$. Then there exists a function f which is analytic and unbounded in Δ such that its derivative f' can be factored in the form $f' = g_1 g_2$ where $g_1 \in H^{p_1}$ and g_2 is an analytic function in Δ which satisfies* (1.10).

However, we can prove that (1.11) implies that f belongs to the space $VMOA$. Let us recall that $BMOA$ is the space of all functions f in H^1 whose boundary values have bounded mean oscillation, see [**3, 8**] or [**11**], and $VMOA$ is the subspace of $BMOA$ which consists of all functions $f \in H^1$ whose boundary values have vanishing mean oscillation. It is well known that $H^\infty \subset BMOA$, and $\mathcal{A} \subset VMOA$. We have

Theorem 3. *Let p_1 and p_2 be positive numbers with $1 = 1/p_1 + 1/p_2$. Let f be an analytic function in Δ such that $f' = g_1 g_2$ where $g_1 \in H^{p_1}$ and g_2 is an analytic function in Δ which satisfies* (1.10). *Then $f \in VMOA$.*

The proofs of Theorems 2 and 3 will be presented in Section 3 where we shall also see that an analogue of Theorem D for $p > 1$ can be trivially obtained.

2. A proof of Theorem 1 and some complementary results. Before embarking into the proof of Theorem 1, let us remark that analytic functions given by power series with Hadamard gaps are often useful to construct examples of functions which show that a certain result is sharp. So, it is natural to ask whether it is possible to prove

Theorem 1 using a power series with Hadamard gaps or not. The answer is negative. Indeed, let p, α and ω be as in Theorem 1, and suppose that w is smaller than t^α, in the sense that

$$\omega(t) = \mathrm{O}(t^\alpha), \quad \text{as } t \to 0. \tag{2.1}$$

Suppose that there exists a function $f \in \Lambda(p, \omega)$ such that $f \notin \Lambda_\alpha$, and f is given by a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

having Hadamard gaps, that is,

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1, \quad \text{for } k = 1, 2, \dots .$$

Then, see Theorem 1 of [**12**], the condition $f \in \Lambda(p, \omega)$ is equivalent to

$$a_k = \mathrm{O}\left(\omega\left(\frac{1}{n_k}\right)\right), \quad \text{as } k \to \infty,$$

which, with (2.1), implies that

$$a_k = \mathrm{O}\left(\frac{1}{n_k^\alpha}\right), \quad \text{as } k \to \infty.$$

This implies that $f \in \Lambda_\alpha$, see [**7**] or [**12**], which leads to a contradiction.

Proof of Theorem 1. Let p, α and ω be as in Theorem 1, and set

$$f(z) = \int_0^\pi \frac{\omega(t)}{t(1+t-z)^{1/p}}\, dt, \quad z \in \Delta.$$

Then f is an analytic function in Δ, and

$$f'(z) = \frac{1}{p} \int_0^\pi \frac{\omega(t)}{t(1+t-z)^{1+1/p}}\, dt, \quad z \in \Delta.$$

Take r, with $0 < r < 1$, and denote by C a positive constant which does not depend on r, and can be different at each appearance. Using

Minkowski's inequality and the fact that for each $\gamma > 1$ there exists a constant $c > 0$, which only depends on γ, such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|1-re^{i\theta}|^{\gamma}} \le \frac{c}{(1-r)^{\gamma-1}}, \quad 0<r<1,$$

we have

$$\begin{aligned}
M_p(r,f') &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p\,d\theta\right)^{1/p}\\
&\le C\biggl(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^{\pi}\frac{\omega(t)}{t|1+t-re^{i\theta}|^{1+1/p}}\,dt\right)^p d\theta\biggr)^{1/p}\\
&\le C\int_0^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\omega(t)}{t|1+t-re^{i\theta}|^{1+1/p}}\right)^p d\theta\right)^{1/p} dt\\
&= C\int_0^{\pi}\frac{\omega(t)}{t}\biggl(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|1+t-re^{i\theta}|^{p+1}}\biggr)^{1/p} dt\\
&= C\int_0^{\pi}\frac{\omega(t)}{t}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|1-(r/(1+t))e^{i\theta}|^{p+1}}\right)^{1/p}\\
&\qquad\qquad\qquad\times\frac{1}{(1+t)^{1+1/p}}\,dt\\
&\le C\int_0^{\pi}\frac{\omega(t)}{t}\left(\frac{C}{\bigl(1-(r/(1+t))\bigr)^p}\right)^{1/p}\frac{1}{1+t}\,dt\\
&= C\int_0^{\pi}\frac{\omega(t)}{t(1+t-r)}\,dt\\
&= C\int_0^{1-r}\frac{\omega(t)}{t(1+t-r)}\,dt + C\int_{1-r}^{\pi}\frac{\omega(t)}{t(1+t-r)}\,dt.
\end{aligned}$$

Now, ω is a Dini-weight, so using (1.4), we obtain

$$\int_0^{1-r}\frac{\omega(t)}{t(1+t-r)}\,dt \le \frac{1}{1-r}\int_0^{1-r}\frac{\omega(t)}{t}\,dt \le C\frac{\omega(1-r)}{1-r}.$$

Also, ω satisfies the condition b_1 and then, using (1.5), we see that

$$\begin{aligned}
\int_{1-r}^{\pi}\frac{\omega(t)}{t(1+t-r)}\,dt &= \int_{1-r}^{\pi}\frac{\omega(t)}{t^2}\frac{t}{1+t-r}\,dt \le \int_{1-r}^{\pi}\frac{\omega(t)}{t^2}\,dt\\
&\le C\frac{\omega(1-r)}{1-r}.
\end{aligned}$$

Thus, we have proved that

$$M_p(r, f') \leq C\, \frac{\omega(1-r)}{1-r}, \quad 0 < r < 1,$$

which, using Theorem C, gives $f \in \Lambda(p, \omega)$.

Now, given r with $0 < r < 1$, bearing in mind that ω is increasing and using the trivial inequality

$$1 + t - r \leq 3(1-r) \quad \text{if} \quad 1 - r \leq t \leq 2(1-r),$$

we deduce that

$$\begin{aligned} f'(r) &= C \int_0^{\pi} \frac{\omega(t)}{t(1+t-r)^{1+1/p}}\, dt \\ &\geq C \int_{1-r}^{2(1-r)} \frac{\omega(t)}{t(1+t-r)^{1+1/p}}\, dt \\ &\geq C\omega(1-r) \int_{1-r}^{2(1-r)} \frac{dt}{t(1+t-r)^{1+1/p}} \\ &\geq C\omega(1-r) \frac{1}{(1-r)^{1+1/p}} \int_{1-r}^{2(1-r)} \frac{dt}{t} \\ &= C\, \frac{\omega(1-r)}{(1-r)^{1+1/p}} \log 2, \end{aligned}$$

that is,

$$f'(r) \geq C\, \frac{\omega(1-r)}{(1-r)^{1+1/p}},$$

which implies that

$$(1-r)^{1/p} f'(r) \geq C\, \frac{\omega(1-r)}{1-r}.$$

Then, using (1.8), we obtain

$$\sup_{0<r<1} (1-r)^{1/p} f'(r) = \infty.$$

Hence,

$$M_\infty(r, f') \neq \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1,$$

which, using (1.2) for $p = \infty$, yields $f \notin \Lambda_\alpha$. This finishes the proof. $\square$

3. Results related to Theorem D. In this section we shall prove Theorems 2 and 3 but, before doing so, it is worth noticing that the condition $p_2 > 1$ is really needed in Theorem D. Indeed, set

$$g_1(z) = 1, \quad g_2(z) = \sum_{k=1}^{\infty} 2^k z^{2^k - 1}, \quad f(z) = \sum_{k=1}^{\infty} z^{2^k}, \quad (z \in \Delta).$$

Then $f' = g_1 g_2$. Clearly, $g_1 \in H^{p_1}$ for every $p_1 > 0$. On the other hand, since f is given by a power series with Hadamard gaps and bounded coefficients, f is a Bloch function [**2**], that is,

$$M_\infty(r, f') = \mathrm{O}\left(\frac{1}{1-r}\right), \quad \text{as } r \to 1.$$

Since $g_2 = f'$, it follows that $M_\infty(r, g_2) = \mathrm{O}(1/1-r)$, as $r \to 1$, and hence g_2 satisfies (1.10) for every $p_2 \in (0, 1]$. However, it is well known that f does not belong to any of the Hardy spaces.

Proof of Theorem 2. Let p_1 and p_2 be positive numbers with $1 = 1/p_1 + 1/p_2$. Take α with $0 < \alpha < 1$ and such that $p_1 \alpha > 1$, and let

$$g_1(z) = \frac{1}{(1-z)^{1/p_1}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\alpha}}, \qquad g_2(z) = \frac{1}{(1-z)^{1/p_2}}.$$

Let f be the primitive of $g_1 g_2$ with $f(0) = 0$. Using Exercise 3 in Chapter 1 of [**6**], we easily see that $g_1 \in H^{p_1}$, and it is clear that g_2 satisfies (1.10). Now, for $0 < r < 1$, we have

$$f'(r) = g_1(r) g_2(r) = \frac{1}{(1-r)\left(\frac{1}{r}\log\frac{1}{1-r}\right)^{\alpha}},$$

and then

$$f(r) = \int_0^r \frac{1}{(1-\rho)\left(\frac{1}{\rho}\log\frac{1}{1-\rho}\right)^{\alpha}}\, d\rho,$$

which tends to infinity when r tends to 1. Hence, $f \notin H^\infty$. This finishes the proof. $\square$

In order to prove Theorem 3 we shall use the characterization of $VMOA$ in terms of Carleson measures, see, e.g., Theorem 6.6 of [**11**]:

If $\theta \in \mathbf{R}$ and $0 < h \leq 1$, the Carleson cube $S(\theta, h)$ is defined by

$$S(\theta,h) = \{z = re^{it} \in \Delta : 1-h \leq r < 1,\ \theta < t < \theta + h\}. \tag{3.1}$$

If f is an analytic function in Δ, then $f \in VMOA$ if and only if the Borel measure in Δ defined by

$$d\mu_f(z) = (1-|z|^2)|f'(z)|^2\, dx\, dy \tag{3.2}$$

satisfies that

$$\frac{\mu_f\,(S(\theta,h))}{h} \longrightarrow 0, \quad \text{as } h \to 0, \text{ uniformly in } \theta. \tag{3.3}$$

Proof of Theorem 3. Let p_1, p_2, f, g_1 and g_2 be as in Theorem 3. Let μ_f be the Borel measure in Δ defined in (3.2), Take $\theta \in \mathbf{R}$ and $0 < h \leq 1$. Let C be a positive constant which may be different at each occurrence. Then, using (3.2), we have

(3.4)

$$\begin{aligned}
\mu_f(S(\theta,h)) &= \iint_{S(\theta,h)} (1-|z|^2)|f'(z)|^2\, dx\, dy \\
&= \int_{1-h}^{1} \int_{\theta}^{\theta+h} r(1-r^2)|f'(re^{it})|^2\, dt\, dr \\
&\leq C \int_{1-h}^{1} \int_{\theta}^{\theta+h} (1-r)|f'(re^{it})|^2\, dt\, dr \\
&= C \int_{1-h}^{1} \int_{\theta}^{\theta+h} (1-r)|g_1(re^{it})|^2 |g_2(re^{it})|^2\, dt\, dr \\
&\leq C \int_{1-h}^{1} (1-r) \left(\frac{1}{(1-r)^{1/p_2}} \right)^2 \int_{\theta}^{\theta+h} |g_1(re^{it})|^2\, dt\, dr \\
&= C \int_{1-h}^{1} \frac{1}{(1-r)^{2/p_2 - 1}} \int_{\theta}^{\theta+h} |g_1(re^{it})|^2\, dt\, dr.
\end{aligned}$$

First, suppose $p_1 \le 2$. Then, using that

$$g_1 \in H^{p_1} \Longrightarrow M_\infty(r, g_1) = \mathrm{O}\left(\frac{1}{(1-r)^{1/p_1}}\right), \quad \text{as } r \to 1,$$

we deduce

$$\begin{aligned}
&\mu_f(S(\theta,h)) \\
&\quad \le C \int_{1-h}^{1} \frac{1}{(1-r)^{2/p_2-1}} \int_\theta^{\theta+h} |g_1(re^{it})|^{p_1} |g_1(re^{it})|^{2-p_1}\, dt\, dr \\
&\quad \le C \int_{1-h}^{1} \frac{1}{(1-r)^{2/p_2-1}} \int_\theta^{\theta+h} |g_1(re^{it})|^{p_1} \left(\frac{1}{(1-r)^{1/p_1}}\right)^{2-p_1} dt\, dr \\
&\quad = C \int_{1-h}^{1} \int_\theta^{\theta+h} |g_1(re^{it})|^{p_1}\, dt\, dr.
\end{aligned}$$

Take $\varepsilon > 0$. Using the fact that $g_1 \in H^{p_1}$, it is easy to see that there exists $\delta \in (0,1)$ such that

$$\int_\theta^{\theta+h} |g_1(re^{it})|^{p_1}\, dt < \varepsilon, \quad \theta \in \mathbf{R}, \quad 1-h < r < 1, \quad 0 < h < \delta.$$

Then, we have

$$\mu_f(S(\theta,h)) < C\varepsilon\, h, \quad \text{for every } \theta, \text{ if } 0 < h < \delta.$$

Consequently, we have shown that f satisfies (3.3) and, hence, $f \in VMOA$.

Suppose now that $p_1 > 2$. Then, using (3.4) and Hölder's inequality, we obtain

$$\mu_f(S(\theta,h)) \le C \int_{1-h}^{1} (1-r)^{1-2/p_2} \left(\int_\theta^{\theta+h} |g_1(re^{it})|^{p_1}\, dt\right)^{2/p_1} h^{1-2/p_1}\, dr.$$

Take $\varepsilon > 0$. Since $g_1 \in H^{p_1}$, it is easy to see that there exists $\delta \in (0,1)$ such that

$$\left(\int_\theta^{\theta+h} |g_1(re^{it})|^{p_1}\, dt\right)^{2/p_1} < \varepsilon, \quad \theta \in \mathbf{R}, \quad 1-h < r < 1, \quad 0 < h < \delta.$$

Then we have, for every $\theta \in \mathbf{R}$ and $0 < h < \delta$,

$$\begin{aligned}\mu_f(S(\theta,h)) &\le C \int_{1-h}^{1} (1-r)^{1-2/p_2}\, \varepsilon\, h^{1-2/p_1}\, dr \\ &= C\varepsilon\, h^{1-2/p_1}\, h^{2-2/p_2} = C\varepsilon\, h.\end{aligned}$$

Hence, also in this case we have that f satisfies (3.3) or, equivalently, that $f \in VMOA$.

Finally let us notice that it is trivial to obtain an analogue of Theorem D for $p > 1$. In fact, we have

If $1 < p < \infty$ and p_1, p_2 are positive numbers with $1/p = 1/p_1 + 1/p_2$, and f is an analytic function in Δ such that $f' = g_1 g_2$, where g_1 and g_2 are analytic functions in Δ satisfying that

$$M_\infty(r, g_j) = \mathrm{O}\left(\frac{1}{(1-r)^{1/p_j}}\right), \quad \text{as } r \to 1, \quad j = 1, 2,$$

then $f \in \Lambda_\alpha$, where $\alpha = 1 - 1/p$.

Indeed, if this is the case then $M_\infty(r, f') = \mathrm{O}(1/(1-r)^{1/p})$, as $r \to 1$ which, by (1.2), is equivalent to $f \in \Lambda_{1-1/p}$.

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: girela@uma.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: marquez@anamat.cie.uma.es