ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 3, 2005

SUMMABILITY OF SPLICED SEQUENCES

JEFFREY A. OSIKIEWICZ

ABSTRACT. A spliced sequence is formed by combining all of the terms of two or more convergent sequences, in their original order, into a new "spliced" sequence. We investigate which nonnegative regular matrices will sum spliced sequences and to what value, and provide examples illustrating these results.

1. Preliminaries. Let c denote the set of convergent sequences and c_0 the set of null sequences. We write $e := \{1, 1, 1, \dots\}$ for the sequence all of whose terms are 1. If $x := \{x_n\}_{n=1}^{\infty}$ is a complex number sequence and $A := (a_{n,k})$ is a summability matrix, then Ax is the sequence whose nth term is given by $(Ax)_n := \sum_{k=1}^{\infty} a_{n,k}x_k$. The matrix A preserves zero limits if $x \in c_0$ implies $Ax \in c_0$, is regular if $\lim_n x_n = L$ implies $\lim_n (Ax)_n = L$, and is t-multiplicative, $t \in \mathbf{R}$, if $\lim_n x = L$ implies $\lim_n (Ax)_n = tL$. The well-known Silverman-Töeplitz theorem characterizes regular matrices, see [**1**].

Theorem 1.1 (Silverman, Töeplitz). *The matrix $A := (a_{n,k})$ is regular if and only if it satisfies the following three conditions*:

($\mathbf{Sp}_0$) $\lim_n a_{n,k} = 0$ *for each* $k = 1, 2, 3, \dots$;

($\mathbf{Zs}_1$) $\lim_n \sum_{k=1}^{\infty} a_{n,k} = 1$;

(Zn) $\sup_n \sum_{k=1}^{\infty} |a_{n,k}| < \infty$.

It can be shown, see [**1**], that A preserves zero limits if and only if it satisfies conditions (Sp_0) and (Zn), and is t-multiplicative if and only if it preserves zero limits and satisfies

(Zs) $\lim_n \sum_{k=1}^{\infty} a_{n,k} = t$.

Let A be a nonnegative regular matrix and E a subset of $\mathbf{N}$. Following Freedman and Sember [**3**], we define the A-density of E, denoted $\delta_A(E)$,

---

2000 AMS *Mathematics Subject Classification.* Primary 40C05, 40D05.
Received by the editors on September 27, 2002, and in revised form on November 18, 2002.

by

$$\delta_A(E) := \lim_{n\to\infty} \sum_{k\in E} a_{n,k} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\chi_E(k) = \lim_{n\to\infty} (A\cdot\chi_E)_n,$$

provided this limit exists. (Here, χ_E denotes the characteristic sequence of the set E.) If A is the Cesàro matrix C_1, then the C_1-density of E is called the natural density of E and is denoted $\delta(E)$. If E is considered as the range of a strictly increasing sequence of natural numbers, say $E := \{\nu(j)\}_{j=1}^{\infty}$, then an elementary result concerning natural density is that

$$\delta(E) = \lim_{j\to\infty} \frac{j}{\nu(j)}, \tag{1.1}$$

provided this limit exists, see [**6**].

Let $A := (a_{n,k})$ be an infinite matrix and $E := \{\nu(j)\}$ an infinite subset of $\mathbf{N}$. Define the column submatrix $A^{[E]} := (d_{n,k})$, where $d_{n,k} = a_{n,\nu(k)}$. Then if x is any complex number sequence,

$$\left(A^{[E]}x\right)_n = \sum_{k=1}^{\infty} d_{n,k}x_k = \sum_{k=1}^{\infty} a_{n,\nu(k)}x_k.$$

Theorem 1.2. *Let A be a nonnegative regular matrix and $E := \{\nu(j)\}$ an infinite subset of* $\mathbf{N}$. *If $\delta_A(E)$ exists, then $A^{[E]}$ is $\delta_A(E)$-multiplicative. Conversely, if $A^{[E]}$ is t-multiplicative, then $\delta_A(E)$ exists and equals t.*

Proof. Since $A^{[E]}$ is a column submatrix of A, $A^{[E]}$ satisfies conditions (Sp_0) and (Zn) of Theorem 1.1. Also note that, for any n,

$$\left(A^{[E]}e\right)_n = \sum_{k=1}^{\infty} a_{n,\nu(k)} = \sum_{k\in E} a_{n,k}.$$

Thus, if $\delta_A(E)$ exists, then $A^{[E]}$ is $\delta_A(E)$-multiplicative. Conversely, if $A^{[E]}$ is t-multiplicative, then

$$t = \lim_{n\to\infty} \left(A^{[E]}e\right)_n = \lim_{n\to\infty} \sum_{k\in E} a_{n,k} = \delta_A(E). \qquad \square$$

2. Finite splices.

Definition 2.1. Let N be a fixed natural number. An N-*partition* of $\mathbf{N}$ consists of N infinite sets $E_1 := \{\nu_1(j)\}_{j=1}^{\infty}$, $E_2 := \{\nu_2(j)\}_{j=1}^{\infty}, \ldots, E_N := \{\nu_N(j)\}_{j=1}^{\infty}$ such that $\mathbf{N} = \bigcup_{i=1}^{N} E_i$ and $E_i \cap E_k = \varnothing$ for $i \neq k$.

Definition 2.2. Let $\{E_1, \ldots, E_N\}$ be a fixed N-partition of $\mathbf{N}$. For $i = 1, \ldots, N$, let $\gamma^{(i)} := (\gamma_j^{(i)})_{j=1}^{\infty}$ be a convergent complex sequence with $\lim_j \gamma_j^{(i)} = \Gamma^{(i)}$. Then the N-*splice* of the sequences $\gamma^{(1)}, \ldots, \gamma^{(N)}$, over the N-partition $\{E_1, \ldots, E_N\}$ is the sequence x defined as follows: if $n \in E_i$, then $n = \nu_i(j)$ for some j. So let $x_n = x_{\nu_i(j)} := \gamma_j^{(i)}$.

Example 2.3. Consider the 3-partition

$$\begin{aligned} E_1 &:= \{1, 3, 5, 7, 9, 11, 13, \ldots\}, \\ E_2 &:= \{2, 6, 10, 14, 18, 22, 26, \ldots\}, \\ E_3 &:= \{4, 8, 12, 16, 20, 24, 28, \ldots\}, \end{aligned}$$

and the convergent sequences $a := \{a_j\}, b := \{b_j\}, c := \{c_j\}$. Then the 3-splice of the sequences a, b, c, over the 3-partition $\{E_1, E_2, E_3\}$ is the sequence

$$x := \{a_1, b_1, a_2, c_1, a_3, b_2, a_4, c_2, a_5, b_3, a_6, c_3, a_7, b_4, a_8, c_4, a_9, \ldots\}.$$

Note that an N-splice is necessarily a bounded sequence with at most N limit points (the values of $\Gamma^{(i)}$ need not be unique). Conversely, if x is a bounded sequence with N distinct limit points $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$, then it is clear that there exists an N-partition $\{E_1, \ldots, E_N\}$ such that x is an N-splice over $\{E_1, \ldots, E_N\}$. Of course, this N-partition is not unique.

The formal definition of a spliced sequence is new to the literature, but the idea has appeared in earlier work. In [**7**] and [**8**], Rhoades examined matrix summability of splices over N-partitions of the form $E_i := \{Nj - (N - i)\}_{j=1}^{\infty}$, $i = 1, \ldots, N$. Also in [**2**] and [**4**], Cooke, Barnett and Henstock investigated to what value a regular matrix will sum a bounded divergent sequence. Their results have a natural tie-in to matrix summability of spliced sequences.

Definition 2.4. Let A be a regular matrix and consider a fixed N-partition $\{E_1, \dots, E_N\}$. Then A is said to have the *splicing property* over $\{E_1, \dots, E_N\}$ provided that A sums every N-splice over the N-partition $\{E_1, \dots, E_N\}$.

Theorem 2.5. *Let A be a nonnegative regular matrix and $\{E_1, \dots, E_N\}$ a fixed N-partition of $\mathbf{N}$. If $\delta_A(E_i)$ exists for each $i = 1, \dots, N$, then A has the splicing property over $\{E_1, \dots, E_N\}$ with*

$$\lim_{n\to\infty} (Ax)_n = \sum_{i=1}^{N} \delta_A(E_i)\Gamma^{(i)},$$

for every N-splice x over $\{E_1, \dots, E_N\}$.

Proof. Assume that $\delta_A(E_i)$ exists for each $i = 1, \dots, N$, and let x be an N-splice over the N-partition $\{E_1, \dots, E_N\}$. Then, for a given n,

$$\begin{aligned}
(Ax)_n = \sum_{k=1}^{\infty} a_{n,k}x_k &= \sum_{i=1}^{N}\left(\sum_{k\in E_i} a_{n,k}x_k\right)\\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{\infty} a_{n\nu_i(j)}x_{\nu_i(j)}\right)\\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{\infty} a_{n\nu_i(j)}\gamma_j^{(i)}\right)\\
&= \sum_{i=1}^{N}\left(A^{[E_i]}\gamma^{(i)}\right)_n.
\end{aligned}$$

By Theorem 1.2, $A^{[E_i]}$ is $\delta_A(E_i)$-multiplicative. Hence,

$$\lim_{n\to\infty}(Ax)_n = \lim_{n\to\infty}\sum_{i=1}^{N}\left(A^{[E_i]}\gamma^{(i)}\right)_n = \sum_{i=1}^{N}\lim_{n\to\infty}\left(A^{[E_i]}\gamma^{(i)}\right)_n$$
$$= \sum_{i=1}^{N}\delta_A(E_i)\Gamma^{(i)}.$$

Thus, x is A-summable to $\sum_{i=1}^{N}\delta_A(E_i)\Gamma^{(i)}$ and therefore A has the splicing property over $\{E_1,\dots,E_N\}$. □

We note that the hypotheses in Theorem 2.5 can be weakened to require only that A be regular and $A^{[E_i]}$ multiplicative for each $i = 1,\dots,N$. However, the requirement that A be nonnegative enables us to utilize the concept of A-density, which can be advantageous. The examples in Section 5 will illustrate this advantage. To see that the nonnegativity in Theorem 2.5 can be removed, we note that in [**2**], Cooke and Barnett proved that if x is a bounded sequence with N limit points $l_1, l_2, \dots, l_N$, then a sufficient condition that x should be summable by a regular matrix A is that A should sum each of N particular sequences of 0's and 1's suitably constructed from the given sequence x. While not using the idea of a spliced sequence per se, their proof of this result amounts to finding an N-partition $\{E_1,\dots,E_N\}$ such that x can be written as an N-splice over this partition. The requirement that A sums N particular sequences of 0's and 1's will then imply that $A^{[E_i]}$ is multiplicative for $i = 1,\dots,N$, and their result follows.

Recall that a matrix A is strongly regular if it sums every almost convergent sequence. That is, A is strongly regular if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=m+1}^{m+n}x_i = L, \quad \text{uniformly in } m,$$

implies $\lim_n(Ax)_n = L$, see [**1**]. In [**8**, Theorem 1], Rhoades proved the following theorem concerning summability of N-splices by matrices that are strongly regular.

Theorem 2.6 (Rhoades). *For a given N, consider the N-partition $E_i := \{Nj-(N-i)\}_{j=1}^{\infty}$, $i = 1,\dots,N$. Then any strongly regular* *matrix A has the splicing property over $\{E_1, \dots, E_N\}$ with*

$$\lim_{n\to\infty} (Ax)_n = \frac{1}{N}\sum_{i=1}^{N} \Gamma^{(i)},$$

for every N-splice x over $\{E_1, \dots, E_N\}$.

The next theorem shows that a regular matrix A cannot have the splicing property over every N-partition, where $N \geq 2$.

Theorem 2.7. *Let A be a regular matrix. Then for any $N \geq 2$, there exists an N-partition $\{E_1, \dots, E_N\}$ such that A does not have the splicing property over $\{E_1, \dots, E_N\}$.*

Proof. By a theorem of Steinhaus [**10**], there exists a sequence x consisting of 0's and 1's that is not A-summable. So, for a fixed $N \geq 2$, consider the N-partition $\{E_1, \dots, E_N\}$ where $E_1 := \{\, n : x_n = 1\,\}$ and $E_2, \dots, E_N$, are constructed so that they are disjoint, infinite, and $\cup_{i=2}^{N} E_i = \mathbf{N} \setminus E_1$. Then x can be treated as an N-splice of the constant sequences $\gamma^{(1)} := 1$ and $\gamma^{(i)} := 0$ for $i = 2, \dots, N$, over the N-partition $\{E_1, \dots, E_N\}$. Hence, A does not have the splicing property over $\{E_1, \dots, E_N\}$. $\square$

It is now natural to ask the following question: if A is a regular matrix that sums an N-splice over an N-partition $\{E_1, \dots, E_N\}$, with $\Gamma^{(i)}$ all distinct, will A sum all N-splices over $\{E_1, \dots, E_N\}$? That is, will A necessarily have the splicing property over $\{E_1, \dots, E_N\}$? The following results address this question. We begin, however, with a theorem regarding matrices that preserve zero limits.

Theorem 2.8. *Let A be a matrix that preserves zero limits. If there exists a sequence $\gamma \in c \setminus c_0$ that is A-summable to L, then A is L/Γ-multiplicative, where $\lim_n \gamma_n = \Gamma \neq 0$.*

Proof. Since A preserves zero limits, it satisfies conditions (Sp_0) and (Zn) of Theorem 1.1. Hence we need only show that $\lim_n \sum_k a_{n,k} = L/\Gamma$. Since $\lim_n \gamma_n = \Gamma \neq 0$, we may write $\gamma_n := \Gamma e_n + \varepsilon_n$, where

$e := \{1, 1, 1, \dots\}$ and $\lim_n \varepsilon_n = 0$. Then, for a given n,

$$(A\gamma)_n = \sum_{k=1}^{\infty} a_{n,k}\gamma_k = \sum_{k=1}^{\infty} a_{n,k}\left(\Gamma e_k + \varepsilon_k\right) = \Gamma \sum_{k=1}^{\infty} a_{n,k} e_k + \sum_{k=1}^{\infty} a_{n,k}\varepsilon_k.$$

Since γ is A-summable to L and A preserves zero limits, we have

$$\begin{aligned} L = \lim_{n\to\infty} (A\gamma)_n &= \Gamma \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} + \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\varepsilon_k \\ &= \Gamma \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} + 0. \end{aligned}$$

That is, $\lim_n \sum_k a_{n,k} = L/\Gamma$ and hence A is L/Γ-multiplicative. $\square$

Theorem 2.9. *Let x be a 2-splice of the sequences $\gamma^{(1)}$ and $\gamma^{(2)}$ over the 2-partition $\{E_1, E_2\}$, with $\Gamma^{(1)} \neq \Gamma^{(2)}$. If A is a nonnegative regular matrix that sums x to the value L, then both $\delta_A(E_1)$ and $\delta_A(E_2)$ exist with*

$$\delta_A(E_1) = \frac{\Gamma^{(2)} - L}{\Gamma^{(2)} - \Gamma^{(1)}} \qquad \textit{and} \qquad \delta_A(E_2) = \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}}.$$

Proof. Let x be a 2-splice of the sequences $\gamma^{(1)}$ and $\gamma^{(2)}$ over the 2-partition $\{E_1, E_2\}$, with $\Gamma^{(1)} \neq \Gamma^{(2)}$, and assume x is A-summable to L. Since $A^{[E_1]}$ and $A^{[E_2]}$ are column submatrices of A, they satisfy conditions (Sp_0) and (Zn) of Theorem 1.1. Hence, $A^{[E_1]}$ and $A^{[E_2]}$ preserve zero limits. We write $E_1 := \{\nu_1(j)\}$ and $E_2 := \{\nu_2(j)\}$. Then, for a given n,

$$\begin{aligned} \left(A\left(x - \Gamma^{(1)}\right)\right)_n &= \sum_{k=1}^{\infty} a_{n,k}\left(x_k - \Gamma^{(1)}\right) \\ &= \sum_{j=1}^{\infty} a_{n,\nu_1(j)}\left(\gamma_j^{(1)} - \Gamma^{(1)}\right) + \sum_{j=1}^{\infty} a_{n,\nu_2(j)}\left(\gamma_j^{(2)} - \Gamma^{(1)}\right) \\ &= \left(A^{[E_1]}\left(\gamma^{(1)} - \Gamma^{(1)}\right)\right)_n + \left(A^{[E_2]}\left(\gamma^{(2)} - \Gamma^{(1)}\right)\right)_n. \end{aligned}$$

Since $\gamma^{(1)} - \Gamma^{(1)} \in c_0$, we have

$$\begin{aligned}\left(A^{[E_2]}\left(\gamma^{(2)} - \Gamma^{(1)}\right)\right)_n &= \left(A\left(x - \Gamma^{(1)}\right)\right)_n - \left(A^{[E_1]}\left(\gamma^{(1)} - \Gamma^{(1)}\right)\right)_n \\ &= L - \Gamma^{(1)} + o(1).\end{aligned}$$

Hence, $\gamma^{(2)} - \Gamma^{(1)}$ is $A^{[E_2]}$-summable to $L - \Gamma^{(1)}$. Since $\gamma^{(2)} - \Gamma^{(1)} \in c \setminus c_0$, Theorem 2.8 implies that $A^{[E_2]}$ is t-multiplicative, where $t = (L - \Gamma^{(1)})/(\Gamma^{(2)} - \Gamma^{(1)})$. By Theorem 1.2, $\delta_A(E_2)$ exists with

$$\delta_A(E_2) = \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}}.$$

Lastly, it is clear that

$$\delta_A(E_1) = 1 - \delta_A(E_2) = 1 - \frac{L - \Gamma^{(1)}}{\Gamma^{(2)} - \Gamma^{(1)}} = \frac{\Gamma^{(2)} - L}{\Gamma^{(2)} - \Gamma^{(1)}}. \qquad \square$$

We note that, as in Theorem 2.5, the requirement that A be nonnegative in Theorem 2.9 enables us to make use of A-density. Accordingly, with the appropriate modifications, this requirement may be dropped yielding the following corollary.

Corollary 2.10. *Let x be a 2-splice of the sequences $\gamma^{(1)}$ and $\gamma^{(2)}$ over the 2-partition $\{E_1, E_2\}$, with $\Gamma^{(1)} \neq \Gamma^{(2)}$. If A is a regular matrix that sums x, then A has the splicing property over $\{E_1, E_2\}$.*

One may now conjecture that, for $N \geq 3$, if a regular matrix A sums an N-splice over an N-partition $\{E_1, \dots, E_N\}$, with $\Gamma^{(i)}$ all distinct, then A will sum all N-splices over $\{E_1, \dots, E_N\}$ (and hence have the splicing property over $\{E_1, \dots, E_N\}$). Unfortunately this conjecture is shown to be false in the following example.

Example 2.11. We construct a 3-splice x over a 3-partition $\{E_1, E_2, E_3\}$, with $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$ all distinct, such that x is C_1-summable to 0, but $\delta(E_1), \delta(E_2)$, and $\delta(E_3)$ do not exist.

Let $E_1 := \{\nu_1(n)\}$ be an infinite subset of $\mathbf{N}$ such that $\nu_1(n) + 1 < \nu_1(n+1)$ and $\delta(E_1)$ does not exist. (We note that it is not difficult to construct a set E_1 having these properties.) Define $E_2 := \{\nu_2(n)\}$ by $\nu_2(n) := \nu_1(n) + 1$. Then E_2 is infinite, $\delta(E_2)$ does not exist, and $E_1 \cap E_2 = \varnothing$. Lastly, let $E_3 := \{\nu_3(n)\} = \mathbf{N} \setminus (E_1 \cup E_2)$ and note that E_3 is infinite and does not have natural density.

Define a 3-splice x over $\{E_1, E_2, E_3\}$ by $\gamma_j^{(1)} := 1$, $\gamma_j^{(2)} := -1$, and $\gamma_j^{(3)} := 0$. We claim that x is C_1-summable to 0. Observe that, for a given n,

$$\begin{aligned}
|(C_1x)_n| &= \left| \frac{1}{n} \sum_{\substack{k\in E_1\\ 1\le k\le n}} x_k + \frac{1}{n} \sum_{\substack{k\in E_2\\ 1\le k\le n}} x_k + \frac{1}{n} \sum_{\substack{k\in E_3\\ 1\le k\le n}} x_k \right| \\
&= \frac{1}{n} \left| \sum_{\{j:\nu_1(j)\le n\}} \gamma_j^{(1)} + \sum_{\{j:\nu_2(j)\le n\}} \gamma_j^{(2)} + \sum_{\{j:\nu_3(j)\le n\}} \gamma_j^{(3)} \right| \\
&= \frac{1}{n} \left| \sum_{\{j:\nu_1(j)\le n\}} 1 + \sum_{\{j:\nu_2(j)\le n\}} (-1) + \sum_{\{j:\nu_3(j)\le n\}} 0 \right| \\
&= \frac{1}{n} \left| \sum_{\{j:\nu_1(j)\le n\}} 1 - \sum_{\{j:\nu_1(j)+1\le n\}} 1 \right| \\
&= \frac{1}{n} \left| \sum_{\{j:\nu_1(j)\le n\}} 1 - \sum_{\{j:\nu_1(j)\le n-1\}} 1 \right| \\
&\le \frac{1}{n}.
\end{aligned}$$

Thus x is C_1-summable to 0, but $\delta(E_1)$, $\delta(E_2)$, and $\delta(E_3)$ do not exist.

3. Infinite splices.

Definition 3.1. An ∞-*partition* of $\mathbf{N}$ consists of an infinite number of infinite sets $E_i := \{\nu_i(j)\}_{j=1}^{\infty}$, $i \in \mathbf{N}$, such that $\mathbf{N} = \bigcup_{i=1}^{\infty} E_i$ and $E_i \cap E_k = \varnothing$ for $i \neq k$.

Definition 3.2. Let $\{E_i\}$ be a fixed ∞-partition of $\mathbf{N}$. For $i \in \mathbf{N}$, let $\gamma^{(i)} := (\gamma_j^{(i)})_{j=1}^{\infty}$ be a convergent complex sequence with $\lim_j \gamma_j^{(i)} = \Gamma^{(i)}$. Then the ∞-*splice* of the sequences $\gamma^{(i)}$, $i \in \mathbf{N}$, over the ∞-partition $\{E_i\}$ is the sequence x defined as follows: if $n \in E_i$, then $n = \nu_i(j)$ for some j. So let $x_n = x_{\nu_i(j)} := \gamma_j^{(i)}$.

Note that an ∞-splice is not necessarily bounded.

Definition 3.3. Let A be a regular matrix and consider a fixed ∞-partition $\{E_i\}$. Then A is said to have the *splicing property* over $\{E_i\}$ provided that A sums every bounded ∞-splice over the ∞-partition $\{E_i\}$.

Theorem 3.4. *Let A be a nonnegative regular matrix and $\{E_i\}$ an ∞-partition of $\mathbf{N}$. If $\delta_A(E_i)$ exists for every i and $\sum_i \delta_A(E_i) = 1$, then A has the splicing property over $\{E_i\}$ with*

$$\lim_{n\to\infty} (Ax)_n = \sum_{i=1}^{\infty} \delta_A(E_i)\Gamma^{(i)},$$

for every bounded ∞-splice x over $\{E_i\}$.

Proof. Assume that $\delta_A(E_i)$ exists for every i, $\sum_i \delta_A(E_i) = 1$, and let x be a bounded ∞-splice over $\{E_i\}$. Then, for a given n,

$$\begin{aligned}
(Ax)_n = \sum_{k=1}^{\infty} a_{n,k}x_k &= \sum_{i=1}^{\infty} \left(\sum_{k\in E_i} a_{n,k}x_k \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{n\nu_{i(j)}} x_{\nu_{i(j)}} \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{n\nu_{i(j)}} \gamma_j^{(i)} \right) \\
&= \sum_{i=1}^{\infty} \left(A^{[E_i]}\gamma^{(i)} \right)_n .
\end{aligned}$$

For a fixed n, define $f_n : \mathbf{N} \to \mathbf{C}$ and $g_n : \mathbf{N} \to \mathbf{C}$ by

$$f_n(i) := \left(A^{[E_i]}\gamma^{(i)}\right)_n \qquad \text{and} \qquad g_n(i) := M\left(A^{[E_i]}e\right)_n,$$

where $M := \|x\|_\infty$. Since $\delta_A(E_i)$ exists for every i, by Theorem 1.2, $A^{[E_i]}$ is $\delta_A(E_i)$-multiplicative. Thus,

$$f(i) := \lim_{n\to\infty} f_n(i) = \lim_{n\to\infty}\left(A^{[E_i]}\gamma^{(i)}\right)_n = \delta_A(E_i)\Gamma^{(i)},$$

and

$$g(i) := \lim_{n\to\infty} g_n(i) = \lim_{n\to\infty} M\left(A^{[E_i]}e\right)_n = M\delta_A(E_i).$$

If μ represents counting measure, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbf{N}} g_n(i)\,d\mu &= \lim_{n\to\infty}\sum_{i=1}^{\infty} M\left(A^{[E_i]}e\right)_n \\ &= M\lim_{n\to\infty}\sum_{i=1}^{\infty}\left(\sum_{k\in E_i} a_{n,k}\right) \\ &= M\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}.\end{aligned}$$

Since A is regular and $\sum_i \delta_A(E_i) = 1$,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbf{N}} g_n(i)\,d\mu &= M\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} = M\cdot 1 = M\sum_{i=1}^{\infty}\delta_A(E_i) \\ &= \int_{\mathbf{N}} g(i)\,d\mu.\end{aligned}$$

That is,

$$\lim_{n\to\infty}\int_{\mathbf{N}} g_n(i)\,d\mu = \int_{\mathbf{N}}\lim_{n\to\infty} g_n(i)\,d\mu. \tag{3.1}$$

Also, for every n,

$$\begin{aligned}|f_n(i)| &= \left|\left(A^{[E_i]}\gamma^{(i)}\right)_n\right| = \left|\sum_{j=1}^{\infty} a_{n\nu_{i(j)}}\gamma_j^{(i)}\right| \\ &\le M\sum_{j=1}^{\infty} a_{n\nu_{i(j)}} = M\left(A^{[E_i]}e\right)_n = g_n(i).\end{aligned} \tag{3.2}$$

Thus, (3.1) and (3.2) enable us to invoke the Lebesgue Dominated Convergence theorem to yield

$$\lim_{n\to\infty}(Ax)_n = \lim_{n\to\infty}\sum_{i=1}^{\infty}\left(A^{[E_i]}\gamma^{(i)}\right)_n = \sum_{i=1}^{\infty}\lim_{n\to\infty}\left(A^{[E_i]}\gamma^{(i)}\right)_n$$
$$= \sum_{i=1}^{\infty}\delta_A(E_i)\Gamma^{(i)}.$$

Hence x is A-summable to $\sum_i \delta_A(E_i)\Gamma^{(i)}$ and consequently A has the splicing property over $\{E_i\}$. □

We note that Henstock [**2**, Theorem I] has proved the following theorem concerning matrix summability of bounded sequences.

Theorem 3.5 (Henstock). *Let $A := (a_{n,k})$ be a real regular matrix, and let $\{z_k\}$ be a real sequence such that $|z_k| < B$ for every k. Suppose that $\{x(k)\}$, $k = 1, 2, \dots,$ is the subsequence of the positive integers such that $z_{x(k)} \leq x$, and let $g_n(x) := \sum_{k=1}^{\infty} a_{n,x(k)}$. If $g_n(x)$ tends to a limit $g(x)$ as $n \to \infty$ for all x in $(-B, B)$, then*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} z_k = \int_{-B}^{B} x\, dg(x).$$

Cooke and Barnett's result concerning matrix summability of sequences with a finite number of limit points is a corollary of this theorem.

While Henstock's result does yield the limiting value of the A-transform of a bounded sequence, it requires the evaluation of a possibly difficult Riemann-Stieltjes integral. In contrast, Theorem 3.4 gives the limiting value of the A-transform of a bounded ∞-splice in terms of the sum of the A-density of the sets $\{E_i\}$, which can be beneficial.

Thus, a nonnegative regular matrix A will have the splicing property over $\{E_i\}$ provided $\delta_A(E_i)$ exists for every i and $\sum_i \delta_A(E_i) = 1$. We next investigate the existence of those matrices that have these properties. Let $s := \{s_n\}$ be a given sequence. Then for each point y in the interval $(0,1]$, associate the sequence $\{s_n\alpha_n(y)\}_{n=1}^{\infty}$, where $\alpha_n(y)$ is the nth digit in the nonterminating binary expansion $0.\alpha_1\alpha_2\alpha_3\cdots\alpha_n\cdots$ of y. In [**5**, Theorem 5.7], Hill proved the following result.

Theorem 3.6 (Hill). *Let $A := (a_{n,k})$ be any matrix method, and let $\{s_n\}$ be A-summable to $s \neq 0$. If*

$$\sum_{k=1}^{\infty} a_{n,k}^2 s_k^2 = o\left(\frac{1}{\log n}\right),$$

then almost all of the sequences $\{s_n\alpha_n(y)\}$ are A-summable to $s/2$.

The next theorem gives a sufficient condition guaranteeing the existence of an ∞-partition $\{E_i\}$ such that $\delta_A(E_i)$ exists for every i and $\sum_i \delta_A(E_i) = 1$.

Theorem 3.7. *Let A be a nonnegative regular matrix such that*

$$\sum_{k=1}^{\infty} a_{n,k}^2 = o\left(\frac{1}{\log n}\right). \tag{3.3}$$

Then there exists an ∞-partition $\{E_i\}$ such that $\delta_A(E_i)$ exists for every i and $\sum_i \delta_A(E_i) = 1$.

Proof. Let A be a nonnegative regular matrix satisfying (3.3). Since $e := \{1,1,1,\ldots\}$ is A-summable to 1 and $\sum_k a_{n,k}^2 e_k^2 = o(1/\log n)$, by Theorem 3.6, almost all sequences of 0's and 1's are A-summable to $1/2$; choose one, say χ_1. Then the sequence χ_1 determines a set $E_1 \subset \mathbf{N}$ where $E_1 := \{\, m : \chi_1(m) = 1 \,\}$. Since A is regular and χ_1 is A-summable to $1/2$, E_1 and its complement $\overline{E}_1$ must be infinite. We write $E_1 := \{\nu_1(j)\}$ and $\overline{E}_1 := \{\overline{\nu_1}(j)\}$. Then

$$\delta_A(E_1) = \lim_{n\to\infty} \sum_{k\in E_1} a_{n,k} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\chi_1(k) = \frac{1}{2}.$$

Now consider the matrix $A^{[\overline{E}_1]}$. It is clear that e is $A^{[\overline{E}_1]}$-summable to $1/2$ and $\sum_k a_{n,\overline{\nu_1}(k)}^2 e_k^2 = o(1/\log n)$. Hence Theorem 3.6 implies the existence of a sequence χ_2 of 0's and 1's such that χ_2 is $A^{[\overline{E}_1]}$-summable to $(1/2)/2 = 1/4$. Then χ_2 determines a set $E_2 \subset \overline{E}_1$ where $E_2 := \{\, \overline{\nu_1}(j) : \chi_2(j) = 1 \,\}$. Since $A^{[\overline{E}_1]}$ is 1/2-multiplicative and χ_2 is $A^{[\overline{E}_1]}$-summable to 1/4, E_2 and its complement $\overline{E}_2$ must be infinite. Also note that $E_1 \cap E_2 = \varnothing$ and

$$\delta_A(E_2) = \lim_{n\to\infty} \sum_{k\in E_2} a_{n,k} = \lim_{n\to\infty} \sum_{j=1}^{\infty} a_{n,\overline{\nu_1}(j)}\chi_2(j) = \frac{1}{4}.$$

By induction, for a given i, we have the existence of a set $E_i \subset \mathbf{N}$ such that E_i is infinite, $E_i \cap E_k = \varnothing$ for all $k < i$, and $\delta_A(E_i)$ exists with $\delta_A(E_i) = 1/2^i$. Thus,

$$\sum_{i=1}^{\infty} \delta_A(E_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Now consider the set $E_0 := \mathbf{N} \setminus \bigcup_{i=1}^{\infty} E_i$. We claim that $\delta_A(E_0) = 0$. Note that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \sum_{k\in E_0} a_{n,k} = \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} a_{n,k} - \sum_{k\notin E_0} a_{n,k} \right) \\ &= 1 - \liminf_{n\to\infty} \sum_{k\notin E_0} a_{n,k} \\ &= 1 - \liminf_{n\to\infty} \sum_{i=1}^{\infty} \left(\sum_{k\in E_i} a_{n,k} \right) \\ &= 1 - \liminf_{n\to\infty} \sum_{i=1}^{\infty} \left(A^{(E_i)} e \right)_n . \end{aligned}$$

By Fatou's lemma,

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \sum_{k\in E_0} a_{n,k} \le 1 - \sum_{i=1}^{\infty} \liminf_{n\to\infty} \left(A^{[E_i]} e \right)_n \\ &= 1 - \sum_{i=1}^{\infty} \delta_A(E_i) = 1 - 1 = 0. \end{aligned}$$

Thus,

$$\delta_A(E_0) = \lim_{n\to\infty} \sum_{k\in E_0} a_{n,k} = 0.$$

If E_0 is infinite, consider the ∞-partition $\{E_i\}_{i=0}^{\infty}$. If E_0 is finite (or empty), take $E_1^* := E_0 \cup E_1$, and $E_1^* \cup \{E_i\}_{i=2}^{\infty}$ is the partition we seek. $\square$

Hence, if a nonnegative regular matrix satisfies (3.3), then it will determine an ∞-partition $\{E_i\}$ such that it has the splicing property over $\{E_i\}$. Conversely, the following corollary shows that for a given ∞-partition $\{E_i\}$, there exists a nonnegative regular matrix that has the splicing property over $\{E_i\}$. However, we first prove the following theorem.

Theorem 3.8. *Let $\{E_i\}$ be an ∞-partition of* $\mathbf{N}$ *and $\{t_i\}$ a sequence of nonnegative numbers such that $\sum_i t_i = 1$. Then there exists a nonnegative regular matrix A such that, for every i, $\delta_A(E_i)$ exists and equals t_i.*

Proof. Let $\{E_i\}$ be an ∞-partition of $\mathbf{N}$, $E_i := \{\nu_i(j)\}$, and $\{t_i\}$ a sequence of nonnegative numbers such that $\sum_i t_i = 1$. Define a matrix $A := (a_{n,k})$ by

$$a_{n,k} := \begin{cases} t_i & \text{if } k = \nu_i(n), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, for every k, $\lim_n a_{n,k} = 0$, and, for every n,

$$\sum_{k=1}^{\infty} a_{n,k} = \sum_{i=1}^{\infty} a_{n,\nu_i(n)} = \sum_{i=1}^{\infty} t_i = 1.$$

Hence A is regular and, for a given i and n,

$$(A\cdot\chi_{E_i})_n = \sum_{k\in E_i} a_{n,k} = \sum_{j=1}^{\infty} a_{n,\nu_i(j)} = a_{n,\nu_i(n)} = t_i.$$

Thus,

$$\delta_A(E_i) = \lim_{n\to\infty} (A\cdot\chi_{E_i})_n = \lim_{n\to\infty} t_i = t_i. \qquad \square$$

Corollary 3.9. *Let $\{E_i\}$ be an ∞-partition of* $\mathbf{N}$ *and $\{t_i\}$ a sequence of nonnegative numbers such that $\sum_i t_i = 1$. Then there exists a nonnegative regular matrix A such that A has the splicing property over $\{E_i\}$ with*

$$\lim_{n\to\infty}(Ax)_n = \sum_{i=1}^{\infty} t_i \Gamma^{(i)},$$

for every bounded ∞-splice x over $\{E_i\}$.

The proof of this corollary is a direct application of Theorems 3.4 and 3.8.

4. Examples. In this final section, we present examples illustrating the ideas presented in the previous sections. We limit our examples to the Cesàro matrix C_1 and its associated natural density. We begin with three examples of ∞-partitions.

Example 4.1. For every $i \in \mathbf{N}$, let $E_i := \{2^{i-1}(2j-1)\}_{j=1}^{\infty}$. Then clearly each E_i is infinite, $\mathbf{N} = \bigcup_{i=1}^{\infty} E_i$, and $E_i \cap E_k = \varnothing$ for $i \neq k$. By (1.1), for every i,

$$\delta(E_i) = \lim_{j\to\infty} \frac{j}{2^{i-1}(2j-1)} = \frac{1}{2^i}.$$

Hence, $\{E_i\}$ is an ∞-partition of $\mathbf{N}$ such that for every i, $\delta(E_i) = 1/2^i$, and $\sum_i \delta(E_i) = 1$.

Example 4.2. Construct an ∞-partition of $\mathbf{N}$ as follows: let E_1 be the set of squares, E_2 the set of squares plus one, and E_3 the set of squares plus two. The set E_4 is the set of squares plus three, less any previously used terms. In general, the set E_i will be the set of squares plus $i-1$, less any previously used terms. That is, for $i > 1$, $E_i := \{j^2 + (i-1)\}_{j=[i/2]}^{\infty}$, where $[i/2]$ denotes the greatest integer less than or equal to $i/2$. Then for every i, E_i is infinite, $\mathbf{N} = \bigcup_{i=1}^{\infty} E_i$, and $E_i \cap E_k = \varnothing$ for $i \neq k$. Also, by (1.1),

$$\delta(E_1) = \lim_{j\to\infty} \frac{j}{j^2} = 0,$$

and for $i > 1$,

$$\delta(E_i) = \lim_{j\to\infty} \frac{j}{j^2 + (i-1)} = 0.$$

Hence, $\{E_i\}$ is an ∞-partition of $\mathbf{N}$ such that for every i, $\delta(E_i) = 0$.

Example 4.3. Construct an ∞-partition of $\mathbf{N}$ as follows: let $E_1 := \{2j-1\}_{j=1}^{\infty}$ and $E_2 := \{\nu_2(j)\}_{j=1}^{\infty}$, where $\nu_2(1) := 2$ and $\nu_2(j) := [2(j-1)]^2$ for $j = 2, 3, \ldots$. The sets E_3, E_4, E_5, E_6, and E_7 are the set of even squares plus $2, 4, 6, 8$, and 10, respectively. The set E_8 is the set of even squares plus 12, less any previously used terms. In general, the set E_i will be the set of even squares plus $2(i-2)$, less any previously used terms. That is, for $i > 3$, $E_i := \{(2j)^2 + 2(i-2)\}_{j=[i/4]}^{\infty}$, where $[i/4]$ denotes the greatest integer less than or equal to $i/4$. Then for every i, E_i is infinite, $\mathbf{N} = \bigcup_{i=1}^{\infty} E_i$, and $E_i \cap E_k = \varnothing$ for $i \neq k$. Also, by (1.1),

$$\delta(E_1) = \lim_{j\to\infty} \frac{j}{2j-1} = \frac{1}{2},$$
$$\delta(E_2) = \lim_{j\to\infty} \frac{j}{[(2(j-1))]^2} = 0,$$

and for $i \geq 3$,

$$\delta(E_i) = \lim_{j\to\infty} \frac{j}{(2j)^2 + 2(i-2)} = 0.$$

Hence, $\{E_i\}$ is an ∞-partition of $\mathbf{N}$ such that $\delta(E_1) = 1/2$ and $\delta(E_i) = 0$ for $i \geq 2$.

We next present examples illustrating C_1-summability of spliced sequences.

Example 4.4. Find the C_1-limit of the sequence x defined by

$$x_n := \begin{cases} \sqrt{(2m+1)/m} & \text{if } n = 3m-2 \text{ for some } m = 1, 2, \ldots, \\ \arctan m & \text{if } n = 3m-1 \text{ for some } m = 1, 2, \ldots, \\ 1/m, & \text{if } n = 3m \text{ for some } m = 1, 2, \ldots. \end{cases}$$

That is, find the C_1-limit of

$$x := \left\{ \sqrt{\frac{3}{1}}, \arctan(1), 1, \sqrt{\frac{5}{2}}, \arctan(2), \frac{1}{2}, \sqrt{\frac{7}{3}}, \arctan(3), \frac{1}{3}, \dots \right\}.$$

Observe that x is a 3-splice of the sequences $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$, over the 3-partition $\{E_1, E_2, E_3\}$, where

$$\gamma_j^{(1)} := \sqrt{\frac{2j+1}{j}}, \quad \gamma_j^{(2)} := \arctan j, \quad \gamma_j^{(3)} := \frac{1}{j},$$

and

$$E_1 := \{3j-2\}_{j=1}^{\infty}, \quad E_2 := \{3j-1\}_{j=1}^{\infty}, \quad \text{and} \quad E_3 := \{3j\}_{j=1}^{\infty}.$$

By (1.1), $\delta(E_1) = \delta(E_2) = \delta(E_3) = 1/3$. Since $\lim_j \gamma_j^{(1)} = \sqrt{2}$, $\lim_j \gamma_j^{(2)} = \pi/2$, and $\lim_j \gamma_j^{(3)} = 0$, Theorem 2.5 yields

$$\lim_{n\to\infty} (C_1 x)_n = \sum_{i=1}^{3} \delta(E_i)\Gamma^{(i)} = \frac{1}{3}\cdot\sqrt{2} + \frac{1}{3}\cdot\frac{\pi}{2} + \frac{1}{3}\cdot 0 = \frac{1}{3}\left(\sqrt{2} + \frac{\pi}{2}\right).$$

Alternatively, since C_1 is strongly regular, see [**1**], this limit also follows from Theorem 2.6.

Example 4.5. Let x be the sequence defined as follows: for every $n \in \mathbf{N}$, there exists a unique pair of integers i, j such that $n = 2^{i-1}(2j-1)$. So let $x_n := 1/i$. Show that x is C_1-summable to $\log 2$.

Observe that x is a bounded ∞-splice of the sequences $\gamma^{(i)}$ over the ∞-partition $\{E_i\}$ where, for each i, $\gamma^{(i)}$ is the constant sequence $1/i$ and $E_i := \{2^{i-1}(2j-1)\}_{j=1}^{\infty}$. From Example 4.1, for every i, $\delta(E_i) = 1/2^i$ and $\sum_i \delta(E_i) = 1$. Hence, Theorem 3.4 yields

$$\lim_{n\to\infty} (C_1 x)_n = \sum_{i=1}^{\infty} \delta(E_i)\Gamma^{(i)} = \sum_{i=1}^{\infty} \frac{1}{2^i}\cdot\frac{1}{i} = \log 2.$$

That is, x is C_1-summable to $\log 2$.

The last example illustrates the use of statistical convergence to determine the C_1-limit of a bounded ∞-splice. Recall that a sequence x is statistically convergent to L provided that for every $\varepsilon > 0$, $\delta(K_\varepsilon) = 0$, where $K_\varepsilon := \{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}$. We first note the following lemma proved by Schoenberg [**9**].

Lemma 4.6 (Schoenberg). *If x is a bounded sequence that is statistically convergent to L, then x is C_1-summable to L.*

Example 4.7. Consider the ∞-partition given in Example 4.2. Construct a bounded ∞-splice x over $\{E_i\}$ by setting, for a given i, $\gamma_j^{(i)} := 1/i$. That is, if $n \in E_i$, then $x_n := 1/i$. Show that x is C_1-summable to 0.

From Example 4.2, $\delta(E_i) = 0$ for every i. Thus Theorem 3.4 is not applicable in this case. However, we shall show that x is statistically convergent to 0, and hence, by Lemma 4.6, x is C_1-summable to 0.

Let $\varepsilon > 0$. Then there exists an $M \in \mathbf{N}$ such that $1/M < \varepsilon$. If $K_\varepsilon := \{\, k \in \mathbf{N} : |x_k| \geq \varepsilon \,\}$, then

$$\delta\left(K_\varepsilon\right) \leq \delta\left(\left\{\, k \in \mathbf{N} : |x_k| \geq \frac{1}{M} \,\right\}\right) = \delta\left(\bigcup_{i=1}^{M} E_i\right) = \sum_{i=1}^{M} \delta\left(E_i\right) = 0.$$

Hence, x is statistically convergent to 0 and therefore by Lemma 4.6, x is C_1-summable to 0.

Acknowledgments. The author is grateful to the referee for calling to his attention the results in [**7**] and [**8**].

REFERENCES

1. J. Boos, *Classical and modern methods in summability*, Oxford Univ. Press, Oxford, 2000.

2. R.G. Cooke and A.M. Barnett, *The 'right' value for the generalized limit of a bounded divergent sequence*, J. London Math. Soc. **23** (1948), 211–221.

3. A.R. Freedman and J.J. Sember, *Densities and summability*, Pacific J. Math. **95** (1981), 293–305.

4. R. Henstock, *The efficiency of matrices for bounded sequences*, J. London Math. Soc. **25** (1950), 27–33.

5. J.D. Hill, *Remarks on the Borel property.* Pacific J. Math. **4** (1954), 227–242.

6. I. Niven and H.S. Zuckerman, *An introduction to the theory of numbers*, John Wiley & Sons, Inc., New York, 1960.

7. B.E. Rhoades, *The convergence of matrix transforms for certain Markov chains*, Stochastic Process. Appl. **9** (1979), 85–93.

8. ———, *Some applications of strong regularity to Markov chains and fixed point theorems*, Approximation Theory, III (Proc. Conf. Univ. of Texas, Austin, TX, 1980), Academic Press, New York-London, 1980, pp. 735–740.

9. I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

10. H. Steinhaus, *Some remarks on the generalizations of the notion of limit*, Prace Mat. Fiz. **22** (1911), 121–134 (in Polish).

Dept. of Mathematical Sciences, Kent State University, Tuscarawas Campus, 330 University Dr NE, New Philadelphia OH 44663-9403
E-mail address: `josikiewicz@tusc.kent.edu`