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KRULL RINGS, PRÜFER v-MULTIPLICATION RINGS AND THE RING OF FINITE FRACTIONS

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ABSTRACT. This paper deals with extending the notions of Krull domains and PvMDs to rings with zero divisors. Two of the problems to be addressed involve characterizing when the Nagata ring $R(x)$ will be a Krull ring and when it will be a PvMR. For both problems, the characterizations require consideration of how the ring in question sits in its associated ring of finite fractions.

1. Introduction. Throughout this paper, R will denote a commutative ring with identity, $T(R)$ will denote the total quotient ring of R and $Z(R)$ will denote the set of zero divisors of R. We use $Q_0(R)$ to denote the ring of finite fractions over R. One way to view the ring $Q_0(R)$ is to consider it as the subring of $T(R[X])$ which consists of those fractions $f = b(X)/a(X)$ where $a(X), b(X) \in R[X]$ with $\deg(b(X)) \leq \deg(a(X))$ such that $fa_i = b_i$ for each coefficient a_i of $a(X)$. Another is to view it as a set of equivalence classes of R-module homomorphisms on semiregular ideals, i.e., on those ideals of R which contain a finitely generated ideal that has no nonzero annihilators. Each class consists of those homomorphisms which agree on some semiregular ideal. In the next section we provide a few details of both these constructions. Note that if R is a McCoy ring, i.e., each finitely generated ideal containing only zero divisors has a nonzero annihilator, then $Q_0(R) = T(R)$. As each polynomial ring is a McCoy ring, [**42**, Proposition 6] and [**19**, Theorem 1], $Q_0(R[X]) = T(R[X])$.

While the results in this paper will hold for integral domains, the emphasis is on rings which have nonzero divisors of zero. Recall that an element of a ring R is said to be regular if it is not a zero divisor and an ideal is regular if it contains a regular element. Although an element is either regular or a zero divisor, an ideal need not be regular to have no

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nonzero annihilators. An ideal (or set) that has no nonzero annihilators is said to be dense. Thus the semiregular ideals are those which contain a finite dense set. We let $\mathcal{J}(R)$ denote the set of semiregular ideals of R and $\mathcal{F}(R)$ denote the set of finitely generated semiregular ideals of R. Even though a semiregular ideal may contain no regular elements of R, when the ideal is extended to the polynomial ring $R[X]$, it will contain regular elements of $R[X]$. For example, if I is a semiregular ideal of R and $A = (a_0, a_1, \dots, a_n)$ is a finitely generated dense ideal contained in I, then the polynomial $a(X) = \sum a_i X^i$ is a regular element of $R[X]$. The ideal A is referred to as the *content* of $a(X)$. In general, we denote the content of a polynomial $f(X)$ by $C(f)$, and say that f has unit content if $C(f) = R$. Please note that there are quite a few more terms that we need to define. To provide sufficient background for the problems at hand and keep the introduction somewhat brief, we postpone stating most of the definitions until later.

Recall that one of the many ways to characterize, or define, Krull domains is that an integral domain D is (said to be) a Krull domain if either D is a field or there is a family of discrete rank one valuation domains $\{V_\alpha\}$ for which

(i) $\cap V_\alpha = D$, and

(ii) each nonzero nonunit of D is a unit in all but finitely many of the V_αs.

The following statements are known to be equivalent for an integral domain D [**4, 9, 10**] and [**21**].

(1) D is a Krull domain.

(2) D_P is a discrete rank one valuation domain for each height one prime P, each nonzero nonunit is contained in only finitely many height one primes and $D = \cap\{D_P \mid P \text{ a height one prime of } D\}$.

(3) D is completely integrally closed and its set of divisorial ideals satisfies the ascending chain property.

(4) Each nonzero ideal is t-invertible.

(5) Each nonzero prime ideal is t-invertible.

(6) Each nonzero prime ideal contains a t-invertible prime ideal.

(7) $D[X]$ is a Krull domain.

(8) $D(X)$ is a Krull domain.

The ring $D(X)$ is the localization of the polynomial ring $D[X]$ at the set $\mathcal{U}(D) = \{f(X) \in D[X] \mid C(f) = D\}$.

In [**25**], Kennedy defined a ring R to be a Krull ring if it is not equal to its total quotient ring and there is a family of discrete rank one valuation rings $\{V_\alpha\}$ for which

(i) $R = \bigcap V_\alpha$, and

(ii) each nonunit $r \in R \backslash Z(R)$ is a unit in all but finitely many V_αs.

He proved that if R is a Krull ring, then it is completely integrally closed in $T(R)$ and its set of divisorial regular ideals satisfies the ascending chain condition. The converse was established by Matsuda [**36**, Theorem 1]. We will follow Huckaba in [**18**] and also allow a ring to be a Krull ring if it is equal to its total quotient ring. It is rather rare to have a Krull ring R for which the corresponding polynomial ring $R[X]$ is also a Krull ring. In fact the polynomial ring $R[X]$ is a Krull ring if and only if R is a reduced Krull ring with only finitely many minimal primes (and therefore, R is a finite direct sum of Krull domains) [**4**, Theorem 5.7]. An open problem has been to characterize those rings R for which the corresponding Nagata ring $R(X)$ is a Krull ring [**4**, p. 114]. We shall provide a solution to this problem in Theorem 6.7 below. It turns out the answer involves the ring of finite fractions and requires one to extend Kennedy's notion of a Krull ring to the pair $R \subseteq Q_0(R)$. We say that a ring R is a Q_0-*Krull ring* if either $R = Q_0(R)$ or there is a family of discrete rank one Q_0-valuation rings $\{V_\alpha\}$ for which the following hold:

(i) $R = \cap V_\alpha$.

(ii) For each finitely generated semiregular ideal, there are at most finitely many V_αs where no element of the ideal has value 0.

(iii) For each V_α, the prime at infinity is not semiregular.

Note that condition (iii) is necessary for $R(X)$ to be a Krull ring, without it $R(X)$ may not be completely integrally closed in $T(R[X])$. In Theorem 4.2 we prove that the following statements are equivalent.

(1) R is a Q_0-Krull ring.

(2) Each semiregular ideal I is contained in at most finitely many maximal t-ideals and for each maximal t-ideal M, $(R_{\{M\}}, \{M\}R_{\{M\}})$ is a discrete rank one Q_0-valuation pair for which the prime at infinity is not semiregular.

(3) The set of semiregular divisorial ideals of R satisfies the ascending chain condition and R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$.

(4) Each semiregular ideal of R is t-invertible.

(5) Each semiregular prime ideal of R is t-invertible.

(6) Each semiregular prime ideal of R contains a t-invertible semiregular prime ideal.

In statement (2), $R_{\{M\}}$ denotes the subring of $Q_0(R)$ consisting of those $t \in Q_0(R)$ for which there is an $r \in R\backslash M$ such that tr $\in R$ and $\{M\}R_{\{M\}}$ denotes those elements of $R_{\{M\}}$ which can be multiplied into M by some element of $R\backslash M$. Later we extend this notion to arbitrary primes of R. An integral domain D is a Prüfer v-multiplication domain if each finitely generated nonzero ideal is t-invertible. Huckaba and Papick [**20**] and Matsuda [**37**] extended this notion to rings with zero divisors by declaring a ring R to be a Prüfer v-multiplication ring if each finitely generated regular ideal is t-invertible. Huckaba and Papick studied this condition for additively regular McCoy rings and Matsuda worked under the assumption that the ring was both a McCoy ring and a Marot ring. A ring R is *additively regular* if, for each pair of elements $a, b \in R$ with a regular, there is an element $r \in R$ such that $ra + b$ is regular and R is a *Marot ring* if each regular ideal is generated by the regular elements it contains. Huckaba's book [**18**] is a good source for what was known about Krull rings and Prüfer v-multiplication rings before 1988 or so. Other sources for information on Krull rings include [**3, 22, 41**] and [**23**].

From our list of statements above concerning Q_0-Krull rings, we see that t-invertibility of all semiregular ideals is equivalent to R being a Q_0-Krull ring. It seems natural then to say that R is a Q_0-*PvMR* (short for Q_0-*Prüfer v-multiplication ring*) if each finitely generated semiregular ideal is t-invertible.

The following conditions are equivalent for an integral domain D, [**12, 16**] and Corollary 7.11, below.

(1) $D[X]$ is a PvMD.

(2) $D(X)$ is a PvMD.

(3) D is a PvMD.

(4) Either D is a field or D_P is a valuation domain for each maximal t-ideal P.

We wish to improve upon the work of Matsuda and Huckaba and Papick with an eye toward adapting these four statements about PvMDs to rings with zero divisors. Our ultimate goal is to find for each pair $1 \leq i < j \leq 4$, "minimal" conditions to add to statement (j) to make it equivalent to (i) (nothing "extra" will be needed to show (i) implies (j)).

Prüfer v-multiplication domains are integrally closed, and the conditions used in defining Prüfer v-multiplication rings and Q_0-Prüfer v-multiplication rings are sufficient to show that a PvMR is integrally closed in its total quotient ring and a Q_0-PvMR is integrally closed in its ring of finite fractions. Also, if R is a McCoy ring, then $Q_0(R) = T(R)$ and there is no difference between R being a Q_0-PvMR and (only) a PvMR. Thus the two notions coincide for polynomial rings. On the other hand, the ring R in Example 8.10 below is a PvMR but not a Q_0-PvMR even though $Q_0(R) = T(R)$. With regard to statement (1), a polynomial ring is never integrally closed if the base ring has nonzero nilpotent elements. Thus we have no choice but to add (some form of) the condition that R is a reduced ring to each of statements (2)–(4) to have any hope of reaching our goal when $i = 1$. Consider the following six statements.

(A) $R[X]$ is a PvMR.

(B) $R(X)$ is a PvMR.

(C) R is a Q_0-PvMR.

(C$'$) R is a PvMR.

(D) Either $R = Q_0(R)$ or $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair for each maximal t-ideal P.

(D$'$) Either $R = T(R)$ or $(R_{(P)}, (P)R_{(P)})$ is a valuation pair for each regular maximal t-ideal of P.

We will show that, if no other conditions are added, then (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D), (C) $\Rightarrow$ (C$'$) $\Leftrightarrow$ (D$'$), (D) $\Rightarrow$ (D$'$) and (D$'$) $\not\Rightarrow$ (D) $\not\Rightarrow$ (C) $\not\Rightarrow$ (B) $\not\Rightarrow$ (A). We shall not attempt to give any additional statements to add to either (C$'$) or (D$'$) to have a statement that is equivalent to any one of (B), (C) or (D). It turns out that simply adding that $T(R)$ is von Neumann regular to each of (B), (C), (C$'$), (D) and (D$'$) is exactly what is required to have a statement which is equivalent to (A). In Theorem 4.3, we show that a condition we may add to (D) to have a statement equivalent to (C) is that the prime at infinity of the valuation pair $(R_{\{P\}}, \{P\}R_{\{P\}})$ is not semiregular. Note that this is consistent with what is needed to have R be a Q_0-Krull domain. Since a PvMR is integrally closed in its total quotient ring, if $R(X)$ is a PvMR, it must contain the nilradical of $T(R[X])$. In Theorem 7.12, we show that adding the assumptions that $R(X)$ contains the nilradical of $T(R[X])$ and has no maximal t-ideals of type III to statement (C) gives necessary and sufficient conditions for $R(X)$ to be a PvMR. Adding the same two assumptions to (D) along with adding the statement that either $R = Q_0(R)$ or the prime at infinity of $(R_{\{P\}}, \{P\}R_{\{P\}})$ is not semiregular gives necessary and sufficient conditions for $R(X)$ to be a PvMR. A maximal t-ideal of *type* III is one which neither contracts to a minimal prime of R nor is extended from a maximal t-ideal of R. Necessary and sufficient conditions for $R(X)$ to contain the nilradical of $T(R[X])$ can be found in [**32**]. We shall recall these conditions at the appropriate time.

2. The ring of finite fractions, etc. A ring that plays a very significant role in this paper is the ring of finite fractions. We begin by giving a definition for the ring of finite fractions which is similar to the definition one finds for the complete ring of quotients in [**26**]. (For more on the ring of finite fractions see, for example, [**28**] or [**30**].)

First note that both $\mathcal{J}(R)$ (=the set of semiregular ideals of R) and $\mathcal{F}(R)$ (=the set of finitely generated semiregular ideals of R) are closed under finite intersections and finite products. Thus if I_1 and I_2 are semiregular ideals and $f_1 \in \mathrm{Hom}\,(I_1, R)$ and $f_2 \in \mathrm{Hom}\,(I_2.R)$, then the sum $f_1 + f_2$ and product $f_1 f_2$ make sense as R-module homomorphisms from $I_1 I_2$ into R. Also, it is easy to show that, if f_1 and f_2 agree on some dense ideal of R, then they agree on each semiregular ideal on which both are defined [**26**, Lemma 1, p. 38]. Define an equivalence relation on the set $\{f \mid f \in \operatorname{Hom}(I,R)$ for some $I \in \mathcal{J}(R)\}$ by setting $f \equiv g$ if they agree on some dense ideal of R. If $f_1 \equiv g_1$ and $f_2 \equiv g_2$, then $f_1+f_2 \equiv g_1+g_2$ and $f_1 f_2 \equiv g_1 g_2$. The *ring of finite fractions over* R consists of these equivalence classes and is denoted by $Q_0(R)$. As with the complete ring of quotients, both R and $T(R)$ embed naturally into $Q_0(R)$; for each pair of elements $r, s \in R$ with r regular and each $I \in \mathcal{J}(R)$, multiplication by s/r defines an R-module homomorphism from rI into R. But even more can be said. Let I be a semiregular ideal of R, and let $A = (a_0, a_1, \dots, a_n)$ be a finitely generated dense ideal contained in I. Let $f \in \operatorname{Hom}(I,R)$ and set $a(X) = \sum a_i X^i$ and $b(X) = \sum b_i X^i$ where $b_i = f(a_i)$ for each i. For each $r \in I$ we have $rb_i = rf(a_i) = a_i f(r)$, and it follows that $(b(X)/a(X))r = f(r)$. On the other hand, if $g = c(X)/a(X) \in T(R[X])$ (with $a(X) = \sum a_i X^i$) is such that $gr \in R$ for each $r \in I$, then multiplication by g defines an R-module homomorphism from I into R. Since $a(X)$ is a regular element of $T(R[X])$, we also have $ga_i = c_i$ for each i. Thus for each semiregular ideal I there is a natural one-to-one correspondence between the set $\operatorname{Hom}(I,R)$ and the set $\langle R : I\rangle = \{t \in T(R[X]) \mid tI \subseteq R\}$. Moreover, for semiregular ideals I_1 and I_2, $f_1 \in \operatorname{Hom}(I_1,R)$ and $f_2 \in \operatorname{Hom}(I_2,R)$ are in the same equivalence class in $Q_0(R)$ if and only if the corresponding elements of $T(R[X])$ are equal as fractions. It follows that there is a natural ring isomorphism between the ring of finite fractions over R defined as above and the subring $\cup\{\langle R : I\rangle \mid I \in \mathcal{J}(R)\}$ of $T(R[X])$. While we shall employ both views in the paper, in the proofs we shall most often represent individual finite fractions as elements of $T(R[X])$. Also, for each ideal I of R, we shall use I^{-1} to denote the set $\{t \in Q_0(R) \mid tI \subseteq R\}$ and refer to it as the *inverse of* I. With this notation, $I \operatorname{Hom}(I,R)$ and II^{-1} coincide when $I \in \mathcal{J}(R)$. Note that, with R and $T(R)$ embedded in $Q_0(R)$, we have that the total quotient ring of $Q_0(R)[X]$ and $R[X]$ are the same. Thus, for each ideal $I \in \mathcal{J}(R)$, I^{-1} is also equal to the set $\{t \in T(R[X]) \mid tI \subseteq R\}$.

For an ideal I of R, we call the set $(R : I) = \{t \in T(R) \mid tI \subseteq R\}$ the regular inverse of I. If I is a regular ideal of R, the regular inverse of I coincides with the set I^{-1}, but if I is only semiregular, I^{-1} may be larger than the regular inverse of I. For each semiregular ideal I, we let $I_v = \{r \in R \mid rI^{-1} \subseteq R\}$ and $I_t = \cup J_v$ where the union is taken over the finitely generated semiregular ideals $J \subseteq I$. If I is regular, then we have $I_v = \{s \in R \mid st \in R$ for each $t \in (R : I)\}$ and $I_t = \cup J_v$

where the union need be taken only over the set of finitely generated regular ideals J that are contained in I. Obviously I_v is an ideal for each semiregular ideal I, but so is I_t. Any union of ideals is closed under multiplication, that I_t is closed under addition follows from the fact that if A and J are finitely generated semiregular ideals, then so is $A+J$ and $(A+J)_v$ contains both A_v and J_v. If $I = I_v$, it is said to be divisorial, and if $I = I_t$, it is said to be a *t-ideal* of R. A *t-prime* is a prime ideal which is also a t-ideal and a *maximal t-ideal* is a proper t-ideal which is maximal in the set of proper t-ideals. We use $t \operatorname{Max}(R)$ to denote the set of all maximal t-ideals of the ring R. As we will see in Theorem 3.1, $t \operatorname{Max}(R)$ is nonempty if and only if $R \neq Q_0(R)$. For $I \in \mathcal{J}(R)$, we say that I is Q_0-invertible if $II^{-1} = R$ and *t-invertible* if $(II^{-1})_t = R$.

For a prime ideal P of R, we may form generalized rings of quotients $R_{\{P\}}$ and $R_{(P)}$ in $Q_0(R)$ and $T(R)$, respectively, by setting $R_{\{P\}} = \{t \in Q_0(R) \mid \mathrm{tr} \in R$ for some $r \in R\backslash P\}$ and $R_{(P)} = \{t \in T(R) \mid \mathrm{tr} \in R$ for some $r \in R\backslash P\}$. Obviously, the two rings coincide when $Q_0(R) = T(R)$ and $R_{\{P\}} \cap T(R) = R_{(P)}$. In particular, if M is a prime ideal of $R(X)$, then $R(X)_{\{M\}} = R(X)_{(M)}$. For an ideal I of R, we set $\{I\}R_{\{P\}} = \{t \in Q_0(R) \mid \mathrm{tr} \in I$ for some $r \in R\backslash P\}$ and $(I)R_{(P)} = \{t \in Q_0(R) \mid \mathrm{tr} \in I$ for some $r \in R\backslash P\}$. As with the rings themselves, $\{I\}R_{\{P\}} \cap T(R) = (I)R_{(P)}$. However, in general, the ideals $\{I\}R_{\{P\}}$ and $(I)R_{(P)}$ need not be simply extensions of the ideal I, even $\{P\}R_{\{P\}}$ and $(P)R_{(P)}$ need not be the same as $PR_{\{P\}}$ and $PR_{(P)}$, respectively, nor must either of $\{P\}R_{\{P\}}$ and $(P)R_{(P)}$ be maximal. But if R is a Marot ring and $Q_0(R) = T(R)$, then for each $t \in R_{\{P\}}$, the set $I = \{r \in R \mid \mathrm{rt} \in R\}$ is a regular ideal of R and thus generated by regular elements of R. It follows that $R_{\{P\}} = \{s/r \in T(R) \mid s \in R$ and $r \in R\backslash P$ is a regular element of $R\}$, also $PR_{\{P\}} = \{P\}R_{\{P\}}$. (For more about $R_{\{P\}}$ see [**31**] and for more about $R_{(P)}$ see, for example, [**13**] or [**18**].)

Recall that, for a pair of rings $R \subseteq T$, the ring R is said to be a valuation ring of T [**35**] if there is a totally ordered Abelian group $\langle G, + \rangle$, a symbol ∞ for which $g < \infty = g + \infty = \infty + \infty$ for each $g \in G$ and a function $\nu : T \to G \cup \{\infty\}$ satisfying the following properties.

(i) ν is surjective.

(ii) $\nu(ab) = \nu(a) + \nu(b)$.

(iii) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

(iv) $R = \{t \in T \mid \nu(t) \geq 0\}$.

Since ν is surjective and $\nu(b) = \nu(1 \cdot b) = \nu(1) + \nu(b)$ for each $b \in T$, $\nu(1) = 0$ and $\nu(0) = \infty$. The inverse image of ∞ is a common prime ideal of R and T, and the set $P = \{t \in T \mid \nu(t) > 0\}$ is a prime ideal of R. The pair (R, P) is said to be a *valuation pair* of T. Another consequence of having ν surjective is that the pair (R, P) has the property that for each $t \in T \backslash R$, there is an element $p \in P$ for which pt is in R and not P. For our purposes we need only consider valuation pairs of total quotient rings and of rings of finite fractions. When $T = Q_0(R)$ we refer to (R, P) as a Q_0-*valuation pair*, and when $T = T(R)$ we simply say that (R, P) is a *valuation pair*. If the group G is isomorphic to the integers, then R is a *discrete rank one valuation ring* when $T = T(R)$ and it is a *discrete rank one* Q_0-*valuation ring* when $T = Q_0(R)$.

Following the terminology in [**33**], we say that an element $t \in Q_0(R)$ is *almost integral over* R *as an element of* $T(R[X])$ if there is a finitely generated R-submodule J of $T(R[X])$ such that each positive power of t is contained in J. Also from [**33**], we say that R is *completely integrally closed in* $Q_0(R)$ *as a subring of* $T(R[X])$ if the only elements of $Q_0(R)$ which are almost integral over R as elements of $T(R[X])$ are the elements of R. Theorem 5 of [**33**] states that $t \in Q_0(R)$ is almost integral over R as an element of $T(R[X])$ if and only if there is a semiregular ideal I of R for which $tI \subseteq I$. Example 10 of [**30**] shows that in general we cannot restrict J to being a submodule of $Q_0(R)$.

For a valuation pair and a Q_0-valuation pair, the prime at infinity is never a regular ideal, but might be semiregular, see Examples 8.9 and 8.12 below. For Krull rings it does not matter. But for Q_0-Krull rings, a different theory will emerge if we allow the prime at infinity to be semiregular in the defining valuation pairs. In particular, if (R, P) is a discrete rank one Q_0-valuation pair, then R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$ if and only if the prime at infinity is not semiregular. It is known that if R is not completely integrally in $Q_0(R)$ as a subring of $T(R[X])$, then the associated Nagata ring $R(X)$ is not completely integrally closed [**33**, Corollaries 10 and 15]. An example of a discrete rank one Q_0-valuation pair where the prime at infinity is semiregular is given in Example 8.9. As both

Krull domains and Krull rings are completely integrally closed, we have chosen to define Q_0-Krull rings in such a way that they are completely integrally closed (in the sense we stated above with respect to $T(R[X])$). This is the reason we require the prime at infinity for each defining Q_0-valuation pair be a prime ideal which is not semiregular. A bonus is that this allows us to extend the associated valuation maps to the ring $T(R[X])$ [**15**, Theorem 3.3].

In Section 6 we consider the question of when $R(X)$ is a Krull ring. Note that each ideal of $R(X)$ can be generated by the set of polynomials it contains. Moreover, if the ideal is finitely generated, then either it has a nonzero annihilator or its generators can be assumed to be regular, see, for example, [**18**, Theorem 14.2]. Hence $Q_0(R[X]) = T(R[X])$, and there is no difference between $R(X)$ being a Krull ring and it being a Q_0-Krull ring.

3. t-Ideals. For an integral domain D, each proper t-ideal is contained in a maximal t-ideal and $D = \bigcap\{D_P \mid P \text{ a maximal } t\text{-ideal}\}$, see, for example, [**24**, Exercise 20, p. 42]. We wish to establish similar results for the t-ideals and the maximal t-ideals of a ring which is not a domain. If $R = Q_0(R)$, then it can contain no proper t-ideals (because the inverse of each semiregular ideal is simply R). On the other hand, if $R \neq Q_0(R)$, then there is an element $s \in Q_0(R)\backslash R$. For the element s, there is a finitely generated semiregular ideal I such that $sI \subseteq R$. Since $I^{-1} \neq R$, $I_v = I_t \neq R$, i.e., I_t is a proper t-ideal of R. In contrast, if J is a semiregular ideal for which $J_t = R$, then there must a finitely generated semiregular ideal $A \subseteq J$ with $1 \in A_v$, or equivalently, with $A^{-1} = R$. Most of the results in this section extend known results about t-ideals and t-invertible ideals of integral domains, see, for example, [**14, 34**] and [**17**].

Theorem 3.1. *Let R be a commutative ring which contains at least one proper t-ideal. Then the following hold.*

(a) *Maximal t-ideals exist and each of these is prime.*

(b) *Each proper t-ideal is contained in a maximal t-ideal.*

(c) *If I and J are semiregular ideals of R, then $(IJ)_t = (IJ_t)_t = (I_tJ_t)_t$.*

(d) *If J is a proper t-ideal of R and P is a prime minimal over J, then P is a t-prime.*

(e) $R = \bigcap\{R_{\{P\}} \mid P$ *a maximal t-ideal of R*$\}$.

Proof. For the most part, the proofs of the these statements are not much different from the corresponding proofs dealing with the ideals of an integral domain. The main difference is that we must restrict to semiregular ideals.

To prove (a) and (b), we first show that Zorn's Lemma applies to the set $\mathcal{T}$ of proper t-ideals of R. To this end, let $\{J_\alpha\}$ be a chain in $\mathcal{T}$. We must show that $J = \cup J_\alpha$ is a proper t-ideal. Let I be finitely generated semiregular ideal contained in J. Then some J_α contains I and, since J_α is a t-ideal, it also contains I_v. Hence $I_v \subseteq J$ and J is a proper t-ideal. Now apply Zorn's lemma to get maximal t-ideals. This also shows that each proper t-ideal is contained in a maximal t-ideal. It remains to show that each maximal t-ideal is prime.

Let P be a maximal t-ideal and let $r \in R\backslash P$. We will show that there is a finitely generated semiregular ideal $I \subseteq P$ such that $(rR+I)^{-1} = R$. If no such ideal exists, then $Q = \cup\{(rR+I)_v \mid I \subseteq P$ and $I \in \mathcal{F}(R)\}$ is a proper t-ideal of R that properly contains P since $(rR+I)_v$ contains I_v. This contradicts P being a maximal t-ideal so there is a finitely generated semiregular ideal $I \subseteq P$ such that $(rR+I)^{-1} = R$.

Now let $ab \in P$. By way of contradiction, assume neither a nor b is in P. By the above, there must be a finitely generated semiregular ideal $I \subseteq P$ such that both $(aR+I)^{-1}$ and $(bR+I)^{-1}$ are equal to R. But for each finitely generated semiregular ideal $J \subseteq P$, $(abR+J)^{-1} \neq R$. Thus there is an element $s \in (abR+I)^{-1}\backslash R$. Since $(aR+I)^{-1} = (bR+I)^{-1} = R$, we have a contradiction: $sa \in (bR+I)^{-1} = R$ and $sb \in (aR+I)^{-1} = R$ implies $s \in (bR+I)^{-1} = (aR+I)^{-1} = R$.

To prove (c), follow the same outline as in the related statement for integral domain. The only difference is we must use semiregular ideals instead of simply nonzero ones. For those unfamiliar with the proof for integral domains we provide a brief sketch. First we show that for finitely generated semiregular ideals A and B, $(AB)_v = (A_vB_v)_v$. To establish this equality, let $t \in (AB)^{-1}$. Then $(tA)B \subseteq R$. Hence $tA \subseteq B^{-1}$ and we have $tAB_v \subseteq R$. Now apply the same reasoning to show $tB_v \subset A^{-1}$ and therefore $tA_vB_v \subseteq R$. That $(AB)_v = (A_vB_v)_v$

now follows from the fact that the v operation preserves order. To finish the proof let I and J be a pair of semiregular ideals of R and let C be a finitely generated semiregular ideal contained in I_tJ_t. Then there are finitely generated semiregular ideals $A \subseteq I_t$ and $B \subseteq J_t$ such that $C \subseteq AB$. Since the v operation preserves containment, there are finitely generated semiregular ideals $A' \subseteq I$ and $B' \subseteq J$ such that $A \subseteq= A'_v$ and $B \subseteq B'_v$. Hence $C_v \subseteq (AB)_v \subseteq (A'_vB'_v)_v = (A'B')_v \subseteq (IJ)_t$. The conclusion follows from the fact that the t operation preserves order.

The statement in (d) is a direct consequence of (c). Let J be a proper t-ideal of R and let P be a prime minimal over J. Let I be a finitely generated semiregular ideal which is contained in P. Since P is minimal over J, there is a positive integer n and an element $r \in R\backslash P$ such that $rI^n \subseteq J$. Thus $(rR + J)$ is a semiregular ideal that is not contained in P but does multiply I^n into J. By (c), taking the "t" of both sides of $(rR + J)I^n \subseteq J$ yields $(rR + J)_t(I^n)_t \subseteq J_t = J \subseteq P$. Since $rR + J$ is not contained in P, $(I^n)_t$ must be contained in P. The result now follows from (c) since we have $(I_v)^n = (I_t)^n \subseteq ((I_t)^n)_t = (I^n)_t$.

For (e), let $t \in Q_0(R)\backslash R$ and let $J = \{r \in R \mid tJ \subseteq R\}$. There is a finitely generated semiregular ideal I of R for which $tI \subseteq R$, so the ideal J is semiregular. Moreover, since $tJ \subseteq R$ and t is not in R, $J_t \neq R$. Thus by (b), there is a maximal t-ideal P that contains J. Since P contains every element of R that multiplies t into R, t is not in $R_{\{P\}}$. It follows that $R = \bigcap\{R_{\{P\}} \mid P \text{ a maximal } t\text{-ideal of } R\}$. □

Theorem 3.2. *The following are equivalent for a ring R.*

(1) *$R(X)$ has no proper t-ideals.*

(2) *$R(X) = T(R[X])$.*

(3) *R is a McCoy ring and $R = T(R) = Q_0(R)$.*

Proof. [(1)⇒(2)]. Assume $R(X)$ has no proper t-ideals. Then $R(X)$ can contain no regular elements which are not units. Hence $R(X) = T(R[X])$.

[(2)⇔(3)]. In general, $R(X) \cap T(R) = R$. With regard to McCoy rings, if R is a McCoy ring, then $T(R) = Q_0(R)$. Also R is a McCoy ring if and only if $T(R)(X) = T(R[X])$ [**18**, Theorem 16.4].

It follows that $R(X) = T(R[X])$ if and only if R is a McCoy ring and $R = T(R) = Q_0(R)$.

[(2)⇒(1)]. As each regular element of $T(R[X])$ is a unit and $T(R[X])$ is a McCoy ring [**42**, Proposition 6] and [**19**, Theorem 1], it has no (proper) t-ideals. Hence, if $R(X) = T(R[X])$, then $R(X)$ has no t-ideals. □

The following lemma is one of several which will be useful in establishing characterizations of Q_0-Krull rings and Q_0-Prüfer v-multiplication rings in terms of t-invertibility.

Lemma 3.3. *Let I be a semiregular ideal of R. Then*

(a) *I is t-invertible if and only if there is a finitely generated semiregular ideal $J \subseteq I$ and a finite dense subset A of I^{-1} such that $(AJ)^{-1} = R$.*

(b) *If I is finitely generated, then I is t-invertible if and only if there is a finite dense subset $A \subset I^{-1}$ for which $(AI)^{-1} = R$.*

Proof. Since $(AI)^{-1} \subseteq (AJ)^{-1}$ whenever $J \subseteq I$, statement (b) follows easily from (a).

To prove (a), first assume that I is t-invertible. Then II^{-1} contains a finitely generated semiregular ideal B for which $B^{-1} = R$. It follows that there is a finite set $\{b_1, b_2, \ldots, b_n\} \subset I$ and a finite set $A = \{a_1, a_2, \ldots, a_n\} \subset I^{-1}$ such that $B \subseteq AJ$ where J is the ideal generated by the b_is. Since B is semiregular and $B^{-1} = R$, J is semiregular, A is dense and $(AJ)^{-1} = R$.

Conversely, if there is a finitely generated semiregular ideal $J \subseteq I$ and a finite (dense) subset A of I^{-1} such that $(AJ)^{-1} = R$, then the ideal AJ is a finitely generated semiregular ideal contained in II^{-1} whose v is R. Hence $(II^{-1})_t = R$. □

Since every proper semiregular t-ideal is contained in a maximal t-ideal, a semiregular ideal I is t-invertible if and only if no maximal t-ideal contains II^{-1}. As each maximal t-ideal is prime and $(I^n)^{-1}$ contains $(I^{-1})^n$, each positive power of I is t-invertible whenever I is.

Lemma 3.4. *Let $I \in \mathcal{J}(R)$. Then the following hold.*

(a) If I is t-invertible, then I^n is t-invertible for each positive integer n, $\operatorname{Hom}(I,I) = R$, and there is a finitely generated semiregular ideal $J \subseteq I$ such that $J_v = I_v = I_t$.

(b) *If I is a t-invertible prime ideal, then each ideal B which properly contains I is such that $B_t = R$.*

(c) *If I is a t-invertible prime ideal and $I_t \neq R$, then I is a maximal t-ideal which is also a divisorial ideal and no semiregular prime which is properly contained in I is t-invertible.*

Proof. Assume I is t-invertible. Then $(II^{-1})^{-1} = R$. Now note that $f(II^{-1}) \subseteq II^{-1}$ for each $f \in \operatorname{Hom}(I,I)$ [**5**]. Hence $\operatorname{Hom}(I,I) = R$.

By Lemma 3.3, there is a finitely generated semiregular ideal $J \subseteq I$ and a finite dense subset A of I^{-1} such that $(JA)^{-1} = R$. We will show that $I_v = J_v$. Since $J \subseteq I$, it suffices to show $J^{-1} \subseteq I^{-1}$. Let $t \in J^{-1}$ and consider the product $tIJA$. Since $A \subseteq I^{-1}$, $tIJA \subseteq R$. As $(JA)^{-1} = R$, we must have $tI \subseteq R$. Thus $I_v = J_v$.

In addition to being t-invertible, assume that I is prime. There is nothing to prove if $I_t = R$, so we may assume $I_t \neq R$. As $(II^{-1})_t = R$, we must have $I^{-1} \neq R$ and $I_v \neq R$. Let B be an ideal that properly contains I and let $r \in B \backslash I$. Consider the ideal $J' = J + rR \subseteq B$. For $s \in J'^{-1}$, $sr \in R$ so $srJ \subseteq I$ implies $sJ \subseteq I$ since I is prime. But then $sJA \subseteq IA \subseteq R$, i.e., $s \in (JA)^{-1} = R$. Hence $B_t = R$ and, therefore, $I_t \neq R$ implies I is a maximal t-ideal which is also divisorial and no semiregular prime which is properly contained in I is t-invertible. □

Theorem 3.5. *A ring R satisfies* a.c.c. *on semiregular divisorial ideals if and only if for each semiregular ideal I there is a finitely generated semiregular ideal $A \subseteq I$ such that $A_v = I_v$.*

Proof. For a semiregular ideal I, if no finitely generated semiregular ideal $A \subseteq I$ is such that $A_v = I_v$, then we may build an infinite ascending chain $\{A_i\}$ where each A_i is the "v" of a finitely generated semiregular sub-ideal of I and $(A_i)_v \neq (A_{i+1})_v$ for each i.

Conversely, let $J = \cup A_n$ where $\{A_n\}$ is an ascending chain of semiregular divisorial ideals of R. If $J_v = A_v$ for some finitely generated semiregular ideal $A \subseteq J$. Then some A_k must contain A and we have $A_v = J_v = J = A_k$. $\square$

Our next lemma deals with general valuation pairs. It is particularly useful with regard to t-invertibility of semiregular ideals.

Lemma 3.6. *Let (R, P) be a nontrivial valuation pair of a ring T. If J is a finitely generated ideal of R which is not contained in the prime at infinity, then there is an element $t \in T$ such that tJ is contained in R but not in P.*

Proof. Let $v : T \to G \cup \{\infty\}$ be a valuation map corresponding to the valuation pair (R, P) and let $J = (a_1, a_2, \ldots , a_n)$ be a finitely generated ideal of R which is not contained in $v^{-1}(\infty)$. Set $g = \min\{v(a_1), v(a_2), \ldots , v(a_n)\}$. Then $g \geq 0$ and, since v is surjective, there is an element $t \in T$ such that $v(t) = -g$. It follows that $0 = \min\{v(ta_1), v(ta_2), \ldots , v(ta_n)\}$. Hence $tJ \subseteq R$ but some ta_i is not in P. $\square$

Note that having tJ not contained in P does not necessarily imply that $tJ = R$ as P need not be a maximal ideal of R, even in the more restricted settings where T is $T(R)$ or $Q_0(R)$, see, for example, [**18**, Example 27.7].

Theorem 3.7. *Let M be a prime ideal of a ring R. Then the pair (R, M) is a discrete rank one Q_0-valuation pair with prime at infinity not semiregular if and only if M is the only maximal t-ideal of R and each semiregular ideal of R is t-invertible.*

Proof. Assume (R, M) is a discrete rank one Q_0-valuation pair with prime at infinity not semiregular. Denote the corresponding valuation map by ν. Since ν is discrete and rank one, for each ideal I of R, $I^{-1} = \{t \in Q_0(R) \mid \nu(t) \geq min\{\nu(b) \mid b \in I\}\}$. Thus $I^{-1} = R$ if and only if I is not contained in M. Since the value group of ν is not trivial, $R \neq Q_0(R)$. Thus, R has at least one maximal t-ideal. Moreover, for each $s \in Q_0(R)\backslash R$, the set $\{r \in R \mid rs \in R\}$ must be contained in M. It follows that M is a maximal t-ideal of R.

Let J be a semiregular ideal of R. If J is not contained in M, then there is an element $j \in J$ such that $\nu(j) = 0$. Thus there is a finitely generated semiregular ideal $A \subseteq J$ such that $A^{-1} = R$, simply add jR to any finitely generated semiregular ideal that is contained in J. Hence $J_t = R$ and J is t-invertible. If J is contained in M, then it is not contained in the prime at infinity. Thus $\min\{\nu(b) \mid b \in J\}$ is positive (and less than ∞). Let $r \in J$ be such that $\nu(r) = \min\{\nu(b) \mid b \in J\}$, and let $I \subseteq J$ be a finitely generated semiregular ideal. Then $J^{-1} = (I + rR)^{-1} = \{t \in Q_0(R) \mid \nu(t) \geq -\nu(r)\}$. As some $s \in J^{-1}$ is such that $\nu(s) = -\nu(r) < 0$, $J^{-1} \neq R$ and JJ^{-1} is not contained in M. It follows that each semiregular ideal of R is t-invertible. That M is the unique maximal t-ideal of R follows from Lemma 3.4.

Assume M is the unique maximal t-ideal of R and that every semiregular ideal of R is t-invertible. By Lemma 3.4, M is divisorial. Moreover, if P is a prime which is properly contained in M, then P is not semiregular.

For each $t \in Q_0(R)\backslash R$, the set $(R :_R t)$ must be contained in M since such a set is always a semiregular divisorial ideal. Set $I = (R :_R t)$. As I is semiregular, it must be t-invertible. We will show that tI is not contained in M. Consider the ideal $J = tI + I$. For each $g \in J^{-1}$, we have both gtI and gI contained in R. It follows that $gI \subseteq I$ and we have $g \in R$ by Lemma 3.4. Thus $J_v = J$. Since M is divisorial, we have that tI is not contained in M. Therefore (R, M) is a Q_0-valuation pair.

Let G be the totally ordered Abelian group and ν the valuation map associated with the valuation pair (R, M). Since M is t-invertible and divisorial, there is an element $t \in M^{-1}$ such that tM is not contained in M. Thus $\nu(tp) = 0$ for some $p \in M$ (and $\nu(tr) \geq 0$ for all $r \in M$). It follows that G has a smallest positive member. If G does not have rank one, then it has a nontrivial subgroup H such that for each $h \in H$ and each $g \in G$, if $0 < g < h$, then $g \in H$. For such a subgroup H, the set $\{r \in R \mid \nu(r) > h \text{ for each } h \in H\}$ is a prime ideal of R. By the argument in the previous paragraph, such a prime would have to be semiregular. As M properly contains no semiregular prime of R, no such subgroup can exist, i.e., G has rank one. It follows that G is isomorphic to $\mathbf{Z}$ and, therefore, (R, M) is a discrete rank one Q_0-valuation pair. $\square$

Theorem 3.8. *Let P be a semiregular prime ideal of R. Then the following hold.*

(a) $(R_{\{P\}}, \{P\}R_{\{P\}})$ *is a Q_0-valuation pair if and only if, for each $t \in Q_0(R)\backslash R_{\{P\}}$, there is an element $p \in P$ for which $tp \in R_{\{P\}}\backslash\{P\}R_{\{P\}}$.*

(b) *If $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a nontrivial Q_0-valuation pair and I is a t-invertible ideal of R, then P is a t-prime of R and I is not contained in the prime at infinity.*

(c) *If P is a maximal t-ideal and each semiregular ideal contained in P is t-invertible, then $(R_{\{P\}}, \{P\}R_{\{P\}})$ is discrete rank one Q_0-valuation pair whose prime at infinity is not semiregular.*

Proof. For each $p \in \{P\}R_{\{P\}}$, there is an element $r \in R\backslash P$ such that $rp \in P$. Thus p is in $\{P\}R_{\{P\}} \cap R$ if and only if it is in P.

[Proof of (a)]. By definition, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair if and only if, for each $t \in Q_0(R)\backslash R_{\{P\}}$, there is an element $s \in \{P\}R_{\{P\}}$ such that $st \in R_{\{P\}}\backslash\{P\}R_{\{P\}}$. It follows that there is an element $r \in R\backslash P$ such that $rs \in P$ and $rst \in R$. We cannot have $rst \in \{P\}R_{\{P\}}$ since st is not in $\{P\}R_{\{P\}}$.

[Proof of (b)]. Assume $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a nontrivial Q_0-valuation pair, and let I be a t-invertible ideal of R. To show that P is a t-prime it suffices to show that $J_v \subseteq P$ for each finitely generated semiregular ideal $J \subseteq P$. As J is finitely generated and $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a nontrivial Q_0-valuation pair, there is an element $t \in Q_0(R)\backslash R_{\{P\}}$ such that $tJ \subseteq R_{\{P\}}$. No element of $R\backslash P$ can multiply t into $R_{\{P\}}$, but some element $r \in R\backslash P$ must multiply tJ into R since J is finitely generated. Thus $rt \in J^{-1}\backslash R_{\{P\}}$. It follows that $J_v \subseteq \{P\}R_{\{P\}} \cap R = P$.

A necessary and sufficient condition for an ideal A to be contained in the prime at infinity of $(R_{\{P\}}, \{P\}R_{\{P\}})$ is that $AQ_0(R) \subseteq \{P\}R_{\{P\}}$. Since P is a t-prime and I is t-invertible, P does not contain II^{-1}. Thus II^{-1} is not contained in $\{P\}R_{\{P\}}$. It follows that I is not contained in the prime at infinity of $(R_{\{P\}}, \{P\}R_{\{P\}})$.

[Proof of (c)]. Let P be a maximal t-ideal of R such that each semiregular ideal contained in P is t-invertible. Thus P is t-invertible. By Lemma 3.4, P is divisorial and no semiregular prime which is properly contained in P is t-invertible. Thus no prime which is properly contained in P is semiregular.

Consider the pair $(R_{\{P\}}, \{P\}R_{\{P\}})$. Let $h \in Q_0(R)\backslash R_{\{P\}}$. The set $J = (R :_R h)$ is always a semiregular ideal of R. Since h is not in $R_{\{P\}}$, J is contained in P, and, therefore J is t-invertible. Consider the ideal $hJ + J$. For $g \in (hJ + J)^{-1}$, we must have $gJ \subseteq J$ since $J = (R :_R h)$ and both gJ and ghJ are contained in R. But as J is t-invertible, $\mathrm{Hom}\,(J, J) = R$. Hence $g \in R$. It follows that P cannot contain hJ. Thus $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair. The prime at infinity cannot be semiregular since no prime which is properly contained in P is semiregular. That $(R_{\{P\}}, \{P\}R_{\{P\}})$ is discrete rank one follows the same line of reasoning used in the proof of Theorem 3.7. $\square$

4. Q_0-Krull rings and Q_0-Prüfer v-multiplication rings. A key step in establishing that Krull rings (domains) can be characterized as those rings (integral domains) which are completely integrally closed and satisfy a.c.c. on regular divisorial ideals is being able to prove that having a.c.c. on regular divisorial ideals is enough to guarantee that each regular divisorial ideal is contained in at most finitely many maximal t-ideals. A standard proof of this runs as follows: First pick a regular (nonzero) element r from the regular divisorial ideal I. Next consider the chain of regular ideals $rM_1^{-1} \subseteq r(M_1 \cap M_2)^{-1} \subseteq r(M_1 \cap M_2 \cap M_3)^{-1} \subseteq \cdots$ where $M_1, M_2, \ldots$ are the maximal t-ideals that contain I. Each of the ideals in this chain is divisorial and $r^{-1}(M_1 \cap \cdots \cap M_n)$ is the inverse of $r(M_1 \cap \cdots \cap M_n)^{-1}$. So there is an integer n for which $r(M_1 \cap \cdots \cap M_n)^{-1} = r(M_1 \cap \cdots \cap M_{n+1})^{-1}$ and so $r^{-1}(M_1 \cap \cdots \cap M_n) = r^{-1}(M_1 \cap \cdots \cap M_{n+1})$. Now simply cancel r^{-1} and use the fact that the M_is are distinct maximal t-ideals. As we are dealing with semiregular ideals rather than regular ones, the same proof does not carry over directly to our situation since I need not contain a regular element. All is not lost however. A similar style of proof will work under the additional assumption that the ideal I contains a t-invertible ideal J, the ideal J can be substituted for the regular element r. Moreover, one of our characterizations of Q_0-Krull rings is that R is a Q_0-Krull ring if and only if each semiregular ideal is t-invertible. In Example 8.10 we construct a Krull ring which has a.c.c. on semiregular divisorial ideals but has a semiregular divisorial ideal which is contained in infinitely many maximal t-ideals (so this ring is not a Q_0-Krull ring).

Lemma 4.1. *Let R be a ring with* a.c.c. *on semiregular divisorial ideals. If I is a divisorial semiregular ideal that contains a t-invertible ideal J, then the set of maximal t-ideals that contain I is finite.*

Proof. Assume I is a divisorial semiregular ideal that contains a t-invertible ideal J. Construct a chain $(JM_1^{-1})_v \subseteq (J(M_1 \cap M_2)^{-1})_v \subseteq (J(M_1 \cap M_2 \cap M_3)^{-1})_v \subseteq \cdots$ where M_1, M_2, M_3, etc., are the maximal t-ideals that contain I. A consequence of Theorem 3.5 is that each M_n must be divisorial. Since R has a.c.c. on semiregular divisorial ideals, the chain stabilizes at some n. Since each intersection $M_1 \cap M_2 \cap \cdots \cap M_n$ is a semiregular divisorial ideal of R and an intersection of distinct maximal t-ideals, it suffices to show that if A and B are comparable divisorial ideals that contain I, then $(JA^{-1})_v = (JB^{-1})_v$ implies $A = B$. By way of contradiction assume A is properly contained in B and $(JA^{-1})_v = (JB^{-1})_v$. Then $A^{-1}B$ is not contained in R, but we do have $J^{-1}B \subseteq (JB^{-1})^{-1} = (JA^{-1})^{-1}$ and therefore, $JJ^{-1}BA^{-1} \subseteq R$. Since J is t-invertible, $(JJ^{-1})_t = R$. Thus, by Theorem 3.5, $(JJ^{-1})_v = R$ as well. But the latter equality is equivalent to saying that $(JJ^{-1})^{-1} = R$ which then implies that $BA^{-1} \subseteq R$. Therefore $(JA^{-1})_v = (JB^{-1})_v$ implies $A = B$. Hence I is contained in only finitely many maximal t-ideals. □

Lemma 4.1 together with Lemma 3.4 and Theorems 3.5 and 3.8 provide enough tools to give several ways of characterizing Q_0-Krull rings. If one deletes the reference to the prime at infinity being semiregular, relaxes the condition about R being completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$ to simply R being completely integrally closed in $T(R)$ and replaces "semiregular" by "regular" and "$Q_0(R)$" by "$T(R)$", then statements (2) through (6) in Theorem 4.2 are all equivalent to R being a Krull ring, see [**36**, Theorem 1], [**3**, Proposition 2.11] and [**23**, Theorem 3.6].

Theorem 4.2. *The following are equivalent for a ring R.*

(1) *R is a Q_0-Krull ring.*

(2) *Each semiregular ideal I is contained in at most finitely many maximal t-ideals and for each maximal t-ideal M, $(R_{\{M\}}, \{M\}R_{\{M\}})$*

is a discrete rank one Q_0-valuation pair whose prime at infinity is not semiregular.

(3) *The set of semiregular divisorial ideals of R satisfies the ascending chain condition and R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$.*

(4) *Each semiregular ideal of R is t-invertible.*

(5) *Each semiregular prime ideal of R is t-invertible.*

(6) *Each semiregular prime ideal of R contains a t-invertible semiregular prime ideal.*

Proof. All of the statements hold if $R = Q_0(R)$, so throughout the proof we will assume that $R \neq Q_0(R)$. Obviously (2) implies (1), (4) implies (5) and (5) implies (6). A simple consequence of Lemma 3.4 is that (6) implies (5). To complete the proof we prove that (3) and (4) are equivalent, (4) implies (2), (5) implies (4) and, finally, (1) implies (3).

[(3)⇒(4)]. Assume that the set of semiregular divisorial ideals of R satisfies the ascending chain condition and R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$. Since R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$, $Hom(I, I) = R$ for each semiregular ideal I. Combining this statement with the fact that $\mathrm{Hom}\,(II^{-1}, II^{-1}) = (II^{-1})^{-1}$ [**5**], we also have $(II^{-1})^{-1} = R$ and, therefore, $(II^{-1})_v = R$.

Let I be a semiregular ideal of R. There is nothing to prove if $I_t = R$, so we may assume $I_t \neq R$. Let $A_1 \subset A_2 \subset A_3 \cdots$ be a chain of finitely generated semiregular ideals contained in I. Then $(A_1)_v \subseteq (A_2)_v \subseteq (A_3)_v \subseteq \cdots$ is a chain of semiregular divisorial ideals which are all contained in I_t. By a.c.c. this chain must stabilize at some integer n. It follows that $I_t = A_v$ for some finitely generated semiregular ideal $A \subseteq I$. In particular we have that each proper t-ideal is a semiregular divisorial ideal. It follows that $(II^{-1})_t = (II^{-1})_v = R$.

[(4)⇒(3)]. Assume that each semiregular ideal of R is t-invertible. By Lemma 3.4, if B is t-invertible, $\mathrm{Hom}\,(B, B) = R$ and there is a finitely generated semiregular ideal $A \subseteq B$ for which $A_v = B_t = B_v$. That R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$ follows from [**33**, Theorem 5].

Let $\{J_\alpha\}$ be a chain of semiregular divisorial ideals. Then $I = \cup J_\alpha$ is a t-ideal of R. Hence I is t-invertible. Thus, by Lemma 3.4, there is a finitely generated semiregular ideal $J \subseteq I$ such that $J_v = I = I_v$ and it follows that the chain stabilizes.

[(4)$\Rightarrow$(2)]. Assume that each semiregular ideal of R is t-invertible. Then R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$ and satisfies a.c.c. on semiregular divisorial ideals. By Theorem 4.1, each semiregular divisorial ideal is contained in at most finitely many maximal t-ideals. By Theorem 3.8, for each maximal t-ideal P, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a discrete rank one Q_0-valuation pair with prime at infinity not semiregular.

[(5)$\Rightarrow$(4)]. Assume that each semiregular prime ideal of R is t-invertible. Let J be a semiregular ideal of R. Since $\{P\}R_{\{P\}} \cap R = P$ for each maximal t-ideal P, it suffices to show that $\{P\}R_{\{P\}}$ never contains $\{JJ^{-1}\}R_{\{P\}}$. There is nothing to prove for those maximal t-ideals which do not contain J. Let $\mathcal{P}(J)$ be the set of maximal t-ideals which contain J. Note that, by Lemma 3.4, no prime which is properly contained in a maximal t-ideal is semiregular. Thus each $P \in \mathcal{P}(J)$ is minimal over J. Let P be a fixed element in the set $\mathcal{P}(J)$. As P is a t-invertible maximal t-ideal, PP^{-1} is not contained in P and, therefore, PR_P must be a principal (regular) ideal of R_P. It follows that there is a positive integer n such that $JR_P = P^n R_P$. Hence for each $r \in J$ and each $p \in P^n$, there are elements $s, t \in R \backslash P$ such that $rs \in P^n$ and $pt \in J$. It follows that $\{J\}R_{\{P\}} = \{P^n\}R_{\{P\}}$.

Note that, for each $r \in \{P^n\}R_{\{P\}}$, there is an element $t \in R \backslash P$ such that tr $\in P^n$, Hence for each $s \in (P^n)^{-1}$, $tsr \in R$. Therefore $s \in (\{P^n\}R_{\{P\}})^{-1} = (\{J\}R_{\{P\}})^{-1}$. As P does not contain $P^n(P^n)^{-1}$, $(P^n)^{-1}\{J\}R_{\{P\}}$ is contained in $R_{\{P\}}$ but not in $\{P\}R_{\{P\}}$. In particular, there are elements $p \in P^n$, $t \in (P^n)^{-1}$ and $s \in R \backslash P$ such that $sp \in J$ and $tp \in R \backslash P$. Thus $spt \in R \backslash P$. To complete this portion of the proof, all we need do is show that J^{-1} contains $(P^n)^{-1}$. As $R = \cap\{R_{\{Q\}} \,|\, Q \in t\mathrm{Max}\,(R)\}$, it suffices to show $(P^n)^{-1}\{J\}R_{\{Q\}} \subseteq R_{\{Q\}}$ for each maximal t-ideal Q. We have taken care of the case $Q = P$. Hence it only remains to consider the case where $Q \neq P$. But, for Q different from P, $(P^n)^{-1} \subseteq R_{\{Q\}}$ since P^n is not contained in Q. Thus $(P^n)^{-1}\{J\}R_{\{Q\}} \subseteq R_{\{Q\}}$. It follows that J^{-1} contains $(P^n)^{-1}$ and, therefore, J is t-invertible.

[(1)⇒(3)]. Assume R is a Q_0-Krull ring, and let $\{(V_\alpha, M_\alpha)\}$ be a family of discrete rank one Q_0-valuation pairs such that $R = \cap V_\alpha$, each finitely generated semiregular ideal is contained in at most finitely many of the M_αs, and for each pair, the prime at infinity is not semiregular. It follows that each semiregular ideal of V_α is t-invertible. By combining Lemma 3.4 and Theorem 3.5 we see that each V_α is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$ and each satisfies a.c.c. on semiregular divisorial ideals. Since $R = \cap V_\alpha$, R is completely integrally closed in $Q_0(R)$ as a subring of $T(R[X])$. To complete the proof all we need prove is that R satisfies a.c.c. on semiregular divisorial ideals.

By Theorem 3.5, it suffices to show that each semiregular divisorial ideal is the "v" of some finitely generated semiregular ideal. Let $I = I_v$ be a semiregular divisorial ideal of R. Then, for each V_α, IV_α is t-invertible. Moreover, for all but finitely many V_α, $(IV_\alpha)_t = V_\alpha$. For those V_α where $(IV_\alpha)_v \neq V_\alpha$, we can choose a finitely generated semiregular $J_\alpha \subseteq I$ such that $(J_\alpha V_\alpha)_v = (IV_\alpha)_v$. As we need do this only finitely many times, the sum J of these J_αs is again finitely generated and $(JV_\alpha)_v = (IV_\alpha)_v$ whenever $(IV_\alpha)_v \neq V_\alpha$. It is possible that $(JV_\beta)_v \neq V_\beta$ while $(IV_\beta)_v = V_\beta$. But this can happen at most finitely many times and, for each such V_β, there is a finitely generated semiregular ideal $I_\beta \subseteq I$ such that $(I_\beta V_\beta)_v = (IV_\beta)_v$. Let J' be the sum of J and the I_βs. We will show that $I_v = J'_v$. To this end let $t \in J'^{-1}$. Then $t \in (J'V_\alpha)^{-1}$ for each α. As $(IV_\alpha)_v = (J'V_\alpha)_v$ for each α, $tIV_\alpha \subseteq V_\alpha$. That t is in I^{-1} now follows from the fact that $R = \bigcap V_\alpha$. □

A consequence of our next theorem is that, if R is a Q_0-PvMR, then either $R = Q_0(R)$ or $(R_{\{M\}}, \{M\}R_{\{M\}})$ is a Q_0-valuation pair for each maximal t-ideal M, thus establishing the implication (C) ⇒ (D) from our list in the introduction. Statement (3) shows what must be added to (D) for it to gain equivalence with (C).

Theorem 4.3. *The following are equivalent for a ring R.*

(1) *R is a Q_0-*PvMR.

(2) *Either $R = Q_0(R)$ or $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a nontrivial Q_0-valuation pair for each prime t-ideal P.*

(3) *Either* $R = Q_0(R)$ *or for each maximal* t*-ideal* M *of* R, $(R_{\{M\}}, \{M\}R_{\{M\}})$ *is a* Q_0*-valuation pair whose corresponding prime at infinity is not semiregular.*

Proof. [(1)⇒(2)]. Assume $R \neq Q_0(R)$ is a Q_0-PvMR, and let P be a prime t-ideal. First note that $R \cap \{P\}R_{\{P\}} = P$ for if $t \in R \cap \{P\}R_{\{P\}}$, then there is an element $r \in R\backslash P$ such that $rt \in P$. If $R_{\{P\}} = Q_0(R)$, then $\{P\}R_{\{P\}}$ is an (prime) ideal of $Q_0(R)$. It follows that if J is a finitely generated semiregular contained in P, then $JJ^{-1} \subseteq JQ_0(R) \subseteq \{P\}R_{\{P\}}$. But, as R is a Q_0-PvMR and P is a t-ideal, P does not contain JJ^{-1}. Hence, $\{P\}R_{\{P\}}$ cannot contain $JQ_0(R)$, and therefore $R_{\{P\}} \neq Q_0(R)$.

Let $s \in Q_0(R)\backslash R_{\{P\}}$. Then for each finitely generated semiregular ideal B of R, if $sB \subseteq R$, then $B \subseteq P$. Also for some finitely generated semiregular ideal J, $sJ \subseteq R$. Let $I = sJ + J$. If I is not contained in P, then there is an element $b \in J \subseteq \{P\}R_{\{P\}}$ such that $bs \in R_{\{P\}}\backslash\{P\}R_{\{P\}}$. Thus we may assume that P contains sJ. Since R is a Q_0-PvMR and J is finitely generated, there is a finite dense set A contained in I^{-1} such that $(AI)^{-1} = R$. Thus $AJ \subseteq R$ and $sAJ \subseteq R$. Since P is a t-ideal, AI is not contained in P. We must have $AJ \subseteq P$, for otherwise s is in $R_{\{P\}}$. Thus sAJ is not contained in P and there is an element $r \in AJ \subseteq P$ such that $sr \in R_{\{P\}}\backslash\{P\}R_{\{P\}}$. Therefore, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair.

[(2)⇒(3)]. Assume $R \neq Q_0(R)$ and that $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a nontrivial Q_0-valuation ring for each t-prime P. Let M be a maximal t-ideal of R. We must show that the prime at infinity of $(R_{\{M\}}, \{M\}R_{\{M\}})$ is not semiregular. Let P_∞ be the prime at infinity of $(R_{\{M\}}, \{M\}R_{\{M\}})$, and let $P = P_\infty \cap R$. By way of contradiction, assume P is semiregular. We first show that if such is the case, then P is a t-ideal. To this end, let J be a finitely generated semiregular contained in P and let $t \in Q_0(R)$. Since $J \subseteq P_\infty$, $tJ \subseteq R_{\{M\}}$. Thus there is an element $s \in R\backslash M$ such that $stJ \subseteq R$ since J is finitely generated. Hence, $st \in J^{-1}$. It follows that $stJ_v \subseteq R$. As s is not in M, $tJ_v \subseteq R_{\{M\}}$. Thus $J_vQ_0(R) \subset R_{\{M\}}$ is a common ideal of $Q_0(R)$ and $R_{\{M\}}$. This is only possible if J_v is contained in P_∞. Hence, $J_v \subseteq P$ and P is a t-ideal.

Since $\{P\}R_{\{M\}} \subseteq \{P\}R_{\{P\}}$, $\{P\}R_{\{P\}}$ must be the prime at infinity for the Q_0-valuation pair $(R_{\{P\}}, \{P\}R_{\{P\}})$, i.e., $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a trivial Q_0-valuation pair contradicting our assumption that each such valuation pair is nontrivial. Therefore, P_∞ is not semiregular and R is a Q_0-PvMR.

[(3)⇒(1)]. Assume $R \neq Q_0(R)$, and let M be a maximal t-ideal of R. Further assume that the prime at infinity of $(R_{\{M\}}, \{M\}R_{\{M\}})$ is not semiregular. Then, for each finitely generated semiregular $J \subseteq M$, JJ^{-1} is not contained in M. Let $J \subseteq M$ be a finitely generated semiregular. As the prime at infinity of $(R_{\{M\}}, \{M\}R_{\{M\}})$ is not semiregular, it does not contain J. Hence there is an element $t \in Q_0(R)$ such that tJ is contained in $R_{\{M\}}$ but is not contained in $\{M\}R_{\{M\}}$. Since J is finitely generated, there is an element $s \in R \backslash M$ such that $stJ \subseteq R$ but is not contained in M. Thus $st \in J^{-1}$ and M does not contain JJ^{-1}. $\square$

As with Q_0-Krull rings, a Q_0-PvMR is always a PvMR, but the converse is false. In Example 8.9, we construct a ring R for which $(R_{\{M\}}, \{M\}R_{\{M\}})$ is a nontrivial Q_0-valuation pair for each maximal t-ideal M, yet R is not Q_0-PvMR. What fails, of course, is that the prime at infinity of $(R_{\{M\}}, \{M\}R_{\{M\}})$ is a semiregular ideal.

5. Maximal t-ideals of $R[X]$ and $R(X)$. For an integral domain D, the maximal t-ideals of $D[X]$ are of two types. There are maximal t-ideals of the form $PD[X]$ where P is a maximal t-ideal of D and there are maximal t-ideals of the form $f(X)K[X] \cap D[X]$ where $f(X)$ is an irreducible polynomial of $K[X]$, with coefficients in D if one desires, where K is the quotient field of D [**17**, Proposition 1.1]. These second types are referred to as "uppers to zero" since such an ideal contracts to (0) when intersected with D. Note that for each irreducible polynomial $f(X)$, the ideal $f(X)K[X] \cap D[X]$ yields an upper to zero but it need not yield a maximal t-ideal. It turns out that an upper to zero is a maximal t-ideal if and only if it contains a polynomial $g(X)$ for which $C(g)^{-1} = D$, by the aforementioned [**17**, Proposition 1.1]. In the ring $D(X)$ we also have two types of maximal t-ideals, those of the form $PD(X)$ where P is a maximal t-ideal of D, and those which are extensions of uppers to zero of $D[X]$ that do not contain polynomials with unit content. Similar things happen for rings which are not domains. However, the second type of maximal t-ideal can come in two different forms.

When $R \neq Q_0(R)$, there are three types of maximal t-ideals in the rings $R[X]$ and $R(X)$; there are two types when $R = Q_0(R)$. For a maximal t-ideal M of $R(X)$ ($R[X]$), we say that M is of *type* I, if $M = PR(X)$, $PR[X]$, for some maximal t-ideal P of R. We say that M is of *type* II, if $P = M \cap R$ is a minimal prime of R, and M is of *type* III, if $P = M \cap R$ is neither a minimal prime nor a semiregular ideal of R. If $R = Q_0(R)$, then R has no proper t-ideals. Thus in this case there are no maximal t-ideals of type I in $R[X]$ and $R(X)$. The next several results will provide proofs for these assertions.

For an integral domain D, it is well known that $(ID[X])^{-1} = I^{-1}D[X]$ for each nonzero ideal I of D [**39**, Proposition 7.1]. Our next lemma establishes a similar result for semiregular ideals.

Lemma 5.1. *Let A be a semiregular ideal of R. Then the following hold.*

(a) $(AR[X])^{-1} = A^{-1}R[X]$ *and* $(AR(X))^{-1} = A^{-1}R(X)$.

(b) $A_vR[X] = (AR[X])_v$ *and* $A_vR(X) = (AR(X))_v$.

Proof. Let $f(X)/g(X)$ be an element of $(AR[X])^{-1}$. It suffices to show that $f(X)/g(X)$ is in $I^{-1}R[X]$ for each finitely generated semiregular ideal $I \subseteq A$. Let $I = (a_0, a_1, \dots, a_n) \subseteq A$ be a finitely generated semiregular ideal. Let $a(X) = \sum a_iX^i$. Since $a(X) \in AR[X]$ is a regular element, we may assume that $g(X) = a(X)$. Thus we have $[f(X)/a(X)]A \subseteq R[X]$. We first show that $f(X)/a(X)$ is in $I^{-1}R[X]$. Let $d_i(X) = a_if(X)/a(X)$. Then, for each i and j, $d_i(X)a_j = d_j(X)a_i$. Hence the coefficients d_{ik} of $d_i(X)$ and the coefficients d_{jk} of $d_j(X)$ are such that $d_{ik}a_j = d_{jk}a_i$. For each k, set $r_k(X) = \sum d_{ik}X^k$. Since $d_{ik}a_j = d_{jk}a_i$, $h_k = r_k(X)/a(X) \in I^{-1}$. Moreover, $h_ka_i = d_{ik}$ for each i and k. Hence $h(X) = \sum h_kX^k \in I^{-1}R[X]$. That $h(X)$ is equal to $f(X)/a(X)$ follows from the fact that for each i, $a_ih(X) = d_i(X) = a_if(X)/a(X)$.

To see that $(AR(X))^{-1} = A^{-1}R(X)$, let $f(X)/g(X) \in (AR(X))^{-1}$ and again let $a(X) = \sum a_iX^i$ where $I = (a_0, a_1, \dots, a_n) \subseteq A$ is an arbitrary finitely generated semiregular ideal. In this case, we may assume that $g(X) = a(X)u(X)$ for some $u(X) \in \mathcal{U}$. Since I is finitely generated, there is a polynomial $v(X) \in \mathcal{U}$ for which $v(X)u(X)[f(X)/g(X)]I \subseteq R[X]$. From part (a), we have $v(X)f(X)/a(X) \in I^{-1}R[X]$. As both $u(X)$ and $v(X)$ are units of $R(X)$, $f(X)/g(X)$ is in $I^{-1}R(X)$. Therefore, $(AR(X))^{-1} = A^{-1}R(X)$.

Now suppose $j(X) \in R[X]$ is in $(A^{-1}R[X])^{-1}$. Then $tj(X) \in R[X]$ for each $t \in A^{-1}$. It follows that each coefficient of $j(X)$ is in A. Hence $A_vR[X] = (AR[X])_v$. Since each ideal of $R(X)$ is generated by polynomials, the same proof is enough to verify that $A_vR(X) = (AR(X))_v$. $\square$

For polynomials $f(X) \in R[X]$ and $h(X) \in Q_0(R)[X]$, the Dedekind-Mertens formula [**10**, Theorem 28.1] guarantees the existence of a positive integer n for which $C(fh)C(f)^n = C(f)^{n+1}C(h)$. In the formula, $C(h)$ denotes the R-submodule of $Q_0(R)$ generated by the coefficients of $h(X)$. We will make use of this formula in the proof of our next lemma, and in several others.

Lemma 5.2. *Let $A = (a_0, a_1, \dots, a_n)$ be a finitely generated semiregular ideal of a ring R, and let $f(X) \in R[X]\backslash\{0\}$. Then the following hold.*

(a) $([A, f(X)]R[X])^{-1} \neq R[X]$ *if and only if* $[A, C(f)]^{-1} \neq R$.

(b) $[(A, f(X))R(X)]^{-1} \neq R(X)$ *if and only if* $[A, C(f)]^{-1} \neq R$.

Proof. If $[A, C(f)]^{-1} \neq R$, then $([A, f(X)]R[X])^{-1} \neq R[X]$ and $[(A, f(X))R(X)]^{-1} \neq R(X)$ since $R = Q_0(R) \cap R[X] = Q_0(R) \cap R(X)$.

Let $g \in [(A, f(X))R(X)]^{-1}\backslash R(X)$. Since A is finitely generated, there is a polynomial $u(X) \in \mathcal{U}$ such that $gu(X)A \subseteq R[X]$ and $gu(X)f(X) \in R[X]$. It follows that $gu(X) \in ([A, f(X)]R[X])^{-1}\backslash R[X]$. As $A^{-1}R[X] = (AR[X])^{-1}$, we may assume g is a polynomial with coefficients in $Q_0(R)$. Applying the Dedekind-Mertens formula, we have that $C(f)^nC(fg) = C(f)^{n+1}C(g) \subseteq R$ for some minimal integer $n \geq 0$. Thus each coefficient of g is in $(C(f)^{n+1})^{-1}$ and some coefficient is not in $(C(f)^n)^{-1}$. As $C(g)C(f)^n \subseteq [A, C(f)]^{-1}$, we have $[A, C(f)]^{-1} \neq R$. $\square$

Lemma 5.3. *Let I be an ideal of R and let $\overline{R} = R/I$. If $f(X) \in R[X]\backslash Z(R[X])$ is such that $C(f)\overline{R} = \overline{R}$, then there is a polynomial $u(X) \in \mathcal{U}$ such that for each $s(X) \in R[X]$, $\bar{s}(X)/\bar{f}(X) = \bar{s}(X)/\overline{u}(X)$.*

Proof. Let $f(X) \in R[X]$ be such that $C(f)\overline{R} = \overline{R}$. Write $f(X) = f_n X^n + \cdots + f_1 X + f_0$ and let $r_0, r_1, \ldots, r_n \in R$ be such that $\sum \bar{f}_i \, \bar{r}_i = \bar{1}$. Thus there is an element $b \in B$ such that $b + \sum f_i r_i = 1$. Set $u(X) = bX^{n+1} + f(X)$. The result now follows easily from the fact that $\bar{b} = 0$. $\square$

Lemma 5.4. *Let $f(X) = \sum f_j X^j \in R[X]$ be a regular element of $R[X]$, and let B be an ideal of R which is contained in $f(X)R(X) \cap R$. For the ring $\overline{R} = R/B$, if $\overline{C(f)}^{\,-1} \neq \overline{R}$, then $C(f)^{-1} \neq R$.*

Proof. Let $\bar{h} \in \overline{C(f)}^{-1}\backslash\overline{R}$ and for each coefficient f_i of $f(X)$, let $r_i \in R$ be such that $\bar{h}\,\bar{f}_i = \bar{r}_i$. It follows that, for each i and j, there is an element $b_{i,j} \in B$ such that $f_i r_j = f_j r_i + b_{i.j}$. Set $r(X) = \sum r_j X^j$ and $b_i(X) = \sum b_{i.j} X^j$ for each i. As $BR(X) \subseteq f(X)R(X)$, $(r(X)/f(X))f_i = r_i + b_i(X)/f(X) \in R(X)$. Since $\bar{r}(X)/\bar{f}(X) = \bar{h} \notin \overline{R}$, $r(X)/f(X) \in (R(X) : C(f))\backslash R(X)$. Therefore by Lemma 5.1, $C(f)^{-1} \neq R$. $\square$

For an ideal I of the polynomial ring $R[X]$, the content of I is simply the ideal of R generated by the content of each member of I. As each ideal of $R(X)$ can be generated, as an ideal of $R(X)$) by polynomials, we can define a similar ideal for each ideal J of $R(X)$. In this case, we set $C(J) = \cup\{C(g) \mid g(X) \in J \cap R[X]\}$.

Theorem 5.5. *Let P be a maximal t-ideal of R. Then $PR[X]$ is a maximal t-ideal of $R[X]$ and $PR(X)$ is a maximal t-ideal of $R(X)$.*

Proof. Let I be a finitely generated regular ideal contained in $PR[X]$. Then there is finitely generated semiregular ideal $A \subseteq P$ such that $I \subseteq AR[X]$. Thus $I_v \subseteq (AR[X])_v = A_v R[X] \subseteq PR[X]$ by Lemma 5.1. Thus $PR[X]$ is a t-ideal of $R[X]$. A similar argument shows that $PR(X)$ is a t-ideal of $R(X)$.

To see that $PR[X]$ is a maximal t-ideal let M be a maximal t-ideal that contains $PR[X]$. Then, for each polynomial $f(X) \in M$ and each finitely generated semiregular ideal $A \subseteq P$, $([A, f(X)]R[X])^{-1} \neq R[X]$ since M is a t-ideal. But then by Lemma 5.2 we also must have $(A, C(f))^{-1} \neq R$. As P is a maximal t-ideal of R, it follows that $f \in PR[X]$ and hence $PR[X]$ is a maximal t-ideal of $R[X]$. As above, similar reasoning shows that $PR(X)$ is a maximal t-ideal of $R(X)$. $\square$

Theorem 5.6. *Let M be a maximal t-ideal of $R[X]$, and let $P = M \cap R$. Then the following hold.*

(a) *P is semiregular if and only if $M = PR[X]$ and P is a maximal t-ideal of R. Hence, if $R = Q_0(R)$, then each finitely generated ideal contained in P has a nonzero annihilator.*

(b) *If P is not a maximal t-ideal, then each finitely generated ideal contained in P has a nonzero annihilator, M contains a regular polynomial $f(X)$ for which $C(f)^{-1} = R$ and M is the contraction to $R[X]$ of a maximal t-ideal N of $Q_0(R)[X]$ such that $R[X]_{(M)} = Q_0(R)[X]_{(N)}$.*

(c) *If P is not a maximal t-ideal of $R[X]$ and $(R[X]_{(M)}, (M)R[X]_{(M)})$ is a valuation pair, then P is a minimal prime of R.*

(d) *If P is a minimal prime of R and R is reduced, then $(R[X]_{(M)}, (M)R[X]_{(M)})$ is a discrete rank one valuation pair.*

Proof. [Proof of (a)]. By Theorem 5.6, if P is a maximal t-ideal or R, then $PR[X]$ is a maximal t-ideal of $R[X]$. Hence $M = PR[X]$ and P is semiregular.

For the converse, assume P is a semiregular ideal of R. Let $A = (a_0, a_1, \dots, a_n)$ be a finitely generated semiregular ideal contained in P, and let $a(X) = \sum a_i X^i$. Then, for each polynomial $h(X) \in M$, there is an integer $m \geq 0$ such that the polynomial $r(X) = h(X) + X^m a(X)$ is regular. Since M is a t-ideal, $([A, r(X)]R[X])^{-1} \neq R[X]$. Thus, by Lemma 5.2, $[A, C(r)]^{-1} \neq R$. It follows that $C(M)_t \neq R$ and, therefore, $M \subseteq C(M)_t R[X] \subseteq (C(M)R[X])_t \neq R[X]$. As M is a maximal t-ideal of $R[X]$, we must have $M = C(M)_t R[X] = (C(M)R[X])_t$ with $C(M)_t = P$ a maximal t-ideal of R.

As $Q_0(R)$ has no maximal t-ideals, each maximal t-ideal of $Q_0(R)$ must contract to a prime ideal which is not semiregular, i.e., an ideal all of whose finitely generated subideals have nonzero annihilators.

[Proof of (b)]. If P is not a maximal t-ideal of R, then by (a) it cannot be semiregular. Hence, each finitely generated ideal contained in P must have a nonzero annihilator. If each regular $f(X) \in M \cap R[X]$ is such that $C(f)^{-1} \neq R$, then $(C(M))_t \neq R$. Hence M would be properly contained in $(C(M))_t R[X] \subseteq (C(M)R[X])_t \neq R[X]$. Thus some regular $f(X)$ in $M \cap R[X]$ must be such that $C(f)^{-1} = R$. Since P is not semiregular, for each $s \in Q_0(R)$, the ideal $(R :_R s)$ is not contained in M. Hence each element of $Q_0(R)$ is contained in $R[X]_{(M)}$. So $R[X]_{(M)}$ contains $Q_0(R)[X]$.

Now suppose $g(X) \in Q_0(R)[X]$ is such that $g(X)f(X) \in R[X]$. Then by the Dedekind-Mertens formula we have $C(g)C(f)^{n+1} = C(gf)C(f)^n$ for some integer $n \geq 0$. As both $C(gf)$ and $C(f)$ are ideals of R, $C(g)$ must be contained in $[C(f)^{n+1}]^{-1}$. But $[C(f)^{n+1}]^{-1} = R$ since $C(f)^{-1} = R$, and it follows that each finitely generated regular ideal $J \subseteq M$ that contains $f(X)$ is such that $J^{-1} \cap Q_0(R)[X] = R[X]$.

Let $N = (M)R[X]_{(M)} \cap Q_0(R)[X]$. Then $N \cap R[X] = (M)R[X]_{(M)} \cap R[X] = M$. Obviously, $R[X]_{(M)} \subseteq Q_0(R)[X]_{(N)}$. To establish the reverse containment, let $r \in Q_0(R)[X]_{(N)}$. Then there is an element $s \in Q_0(R)[X] \backslash N$ such that $rs \in Q_0(R)[X]$. Since $R[X]_{(M)}$ contains $Q_0(R)[X]$, there is an element $t \in R[X] \backslash M$ such that both tsr and ts are in $R[X]$. As $N \cap R[X] = M$, $ts \in R[X] \backslash M$. Thus $r \in R[X]_{(M)}$.

To complete the proof, we need to show that N is a maximal t-ideal of $Q_0(R)[X]$.

We first show that each finitely generated regular ideal contained in N has a nontrivial inverse. To this end, let $I = (a_1(X), a_2(X), \ldots, a_n(X)) \subseteq N$ be a finitely generated regular ideal of $Q_0(R)[X]$. Since $R[X]$ is a Marot ring and I is finitely generated, there is a regular element $s(X) \in R[X] \backslash M$ such that $s(X)I \subseteq M$. Consider the ideal $B = s(X)I + f(X)R[X]$. As M is a maximal t-ideal of $R[X]$, there is an element $t(X)/r(X) \in B^{-1} \backslash R[X]$. Since $C(f)^{-1} = R$, $t(X)/r(X)$ cannot be a polynomial in $Q_0(R)[X]$. Neither can $s(X)t(X)/r(X)$ unless it is also in $R[X]$. As $B_v \subseteq M$ and $s(X)$ is not in M, $s(X)B^{-1}$ is not contained in $R[X]$, nor is it contained in $Q_0(R)[X]$.

It follows that the inverse of I as an ideal of $Q_0(R)[X]$ properly contains $Q_0(R)[X]$. Hence $N_t \neq Q_0(R)[X]$.

Let $M' = N_t \cap R[X]$ and assume $M' \neq M$. Since M is a maximal t-ideal of $R[X]$, we must have $M'_t = R[X]$. Thus there is a finitely generated regular ideal $J \subseteq M'$ such that $J^{-1} = R[X]$. Set $J' = JQ_0(R)[X]$. Since $N_t \neq Q_0(R)[X]$, $J'^{\,-1} \neq Q_0(R)[X]$. Let $t \in J'^{\,-1} \backslash Q_0(R)[X]$. As $tJ \subseteq Q_0(R)[X]$ and J is finitely generated, there is a finitely generated semiregular ideal A of R such that $tJA \subseteq R[X]$. Since $J^{-1} = R[X]$, we must have $tA \subseteq R[X]$. But then we have $t \in (AR[X])^{-1} = A^{-1}R[X] \subseteq Q_0(R)[X]$, a contradiction.

Finally, we show that $N = N_t$. By way of contradiction, let $a(X) \in N_t \backslash N$. Since $Q_0(R)[X]$ is a Marot ring and N is a regular ideal of $Q_0(R)[X]$, we may assume $a(X)$ is regular. Since $N = (M)R[X]_{(M)} \cap Q_0(R)[X]$ and $Q_0(R)[X]$ is contained in $R[X]_{(M)}$, $a(X)$ must be a unit of $R[X]_{(M)}$. Thus there are regular elements $r(X), s(X) \in R[X]\backslash M$ such that $a(X) = r(X)/s(X)$. Thus $r(X) = a(X)s(X) \in (N_t \cap R[X])\backslash M$, a contradiction.

[Proof of (c)]. Assume $(R[X]_{(M)}, (M)R[X]_{(M)})$ is a valuation pair. By way of contradiction, assume P is neither minimal nor semiregular. Then there is a polynomial $f(X) = \sum f_i X^i \in M \cap R[X]$ which is regular and such that $C(f)^{-1} = R$. Let Q be a minimal prime ideal which is contained in P and let $a \in P\backslash Q$. Since $R[X]$ is a Marot ring, the regular elements of $(M)R[X]_{(M)}$ map onto the positive elements of the corresponding value group. From this it follows that $PR[X]_{(M)} = (P)R[X]_{(M)}$ must be contained in the prime at infinity of $(R[X]_{(M)}, (M)R[X]_{(M)})$, otherwise there is an element $b \in P$ with positive value under the valuation ν and a regular element a with the same value. In such a case b/a would have value 0 and therefore not be in $(M)R[X]_{(M)}$ so not in $PR[X]_{(M)}$. But we would have $b = (b/a)a$ in the prime ideal $PR[X]_{(M)}$ forcing a to be in $PR[X]_{(M)}$, which is impossible. Hence we have $PR[X]_{(M)} = PT(R[X]) \supset QT(R[X]) = QR[X]_{(M)}$. Thus there is a element $r(X) \in R[X]\backslash M$ such that $r(X)(a/f(X)) = s(X) \in R[X]$. Thus $r(X)/f(X) \in (R[X] : (f(X), a))\backslash R[X]$.

Set $\overline{R} = R/Q$. Then $\bar{r}(X)/\bar{f}(X) \in (\overline{R}[X] : (\bar{f}(X), \bar{a}))\backslash \overline{R}[X]$. Since Q is prime and a is not in Q, the Dedekind-Mertens formula implies that $\bar{s}(X)/\bar{a} = \bar{r}(X)/\bar{f}(X) \in (\overline{R} : \overline{C(f)}^{m} + a\overline{R})\overline{R}[X]$ for some m.

Let $\{g_0, g_1, \ldots, g_n\}$ be a set of generators for $C(f)^m$. For each i, let $t_i \in R$ be such that $(\bar{r}(X)/\bar{f}(X))\bar{g}_i = \bar{t}_i$. Then $\bar{r}(X)\bar{g}_i = \bar{t}_i\bar{f}(X)$. So, for each pair i and j, there is an element $q_{i,j} \in Q$ such that $g_i r_j = f_j t_i + q_{i,j}$. Set $g(X) = \sum g_j X^j$, $t(X) = \sum t_j X^j$ and $q_i(X) = \sum q_{i,j} X^j$. Then for each i, $(r(X)/f(X))g_i = t_i + q_i(X)/f(X)$. Since $QT(R[X]) \subseteq R(X)_{(M)}$, there is an element $h(X) \in R[X]\backslash M$ such that $h(X)q_i(X)/f(X) \in R[X]$ for each i. Thus $h(X)r(X)/f(X) \in (R[X] : C(g)) = C(g)^{-1}R[X]$. Since neither $h(X)$ nor $r(X)$ are in M but $f(X)$ is, $h(X)r(X)/f(X)$ is not in $R[X]$, i.e., we would have $h(X)r(X)/f(X) \in C(g)^{-1}R[X]\backslash R[X]$. But, as $C(g) = C(f)^m$ and $C(f)^{-1} = R$, we also have $C(g)^{-1}R[X] = R[X]$. Thus P must be a minimal prime of R, i.e., M is of type II.

[Proof of (d)]. Assume P is a minimal prime of R and that R is reduced. We first show that $PR[X]_{(M)} = PT(R[X])$. This is rather easy since R being reduced implies that $r \in P$ if and only if P does not contain $\mathrm{Ann}\,(r)$. Hence, for each $r \in P$, M does not contain $\mathrm{Ann}\,(r)$, and therefore $r/t \in R(X)_{(M)}$ for each regular element $t \in R$. It follows that $PR(X)_{(M)} = PT(R[X])$.

Set $\overline{R} = R/P$, and let $\overline{K}$ denote the quotient field of $\overline{R}$. Then both $R(X)_{(M)}/PT(R[X])$ and $T(R[X])/PT(R[X])$ embed naturally into $\overline{K}(X)$. In $\overline{R}[X]$, $\overline{M}$ is an upper to zero and therefore, $\overline{R}[X]_{\overline{M}}$ is a discrete rank one valuation domain (with quotient field $\overline{K}(X)$). Let $\overline{\nu}$ denote the valuation associated with $\overline{R}[X]_{\overline{M}}$. Define a map ν from $T(R[X])$ to $\mathbf{Z} \cup \{\infty\}$ as follows: Let $g(X)/f(X) \in T(R[X])$ with $f(X), g(X) \in R[X]$, and $f(X)$ regular. If $g(X)$ is not in $PR[X]$, set $\nu(g(X)/f(X)) = \overline{\nu}(\bar{g}(X)/\bar{f}(X))$. If $g(X) \in PR[X]$, set $\nu(g(X)/f(X)) = \infty$.

To complete the proof all we need show is that $(R[X]_{(M)}, (M)R[X]_{(M)})$ is the valuation pair associated to ν. Since $PT(R[X])$ is a common prime of $T(R[X])$ and $R[X]_{(M)}$, we simply need to show that $\overline{R}[X]_{\overline{M}} \cap (T(R[X])/PT(R[X])) = R[X]_{(M)}/PT(R[X])$. Let $g(X)/f(X) \in T(R[X])$ be such that $\bar{g}(X)/\bar{f}(X) \in \overline{R}[X]_{\overline{M}}$. Thus there are polynomials $h(X), j(X) \in R[X]$ with $j(X) \in R[X]\backslash M$ such that $\bar{g}(X)\bar{f}(X) = \bar{h}(X)/\bar{j}(X)$. Thus $\bar{g}(X)\bar{j}(X) = \bar{f}(X)\bar{h}(X)$. It follows that there is a polynomial $p(X) \in PR[X]$ such that $g(X)j(X) = f(X)h(X) + p(X)$. Thus $(g(X)/f(X))j(X) = h(X) + p(X)/f(X)$. As $PT(R[X]) \subset R[X]_{(M)}$ and $j(X)$ is not in M, $g(X)/f(X)$ is in $R[X]_{(M)}$. $\square$

A similar result holds for the maximal t-ideals of $R(X)$. For part (d) it seems critical that we assume $R(X)$ contains the nilradical of $T(R[X])$. Otherwise, it may be that $R(X)_{(M)}$ does not contain the nilradical of $T(R[X])$. This would prevent it from being a valuation ring.

Theorem 5.7. *Let M be a maximal t-ideal of $R(X)$, and let $P = M \cap R$. Then the following.*

(a)*P is semiregular if and only if $M = PR(X)$ and P is a maximal t-ideal of R. Hence, if $R = Q_0(R)$, then each finitely generated ideal contained in P has a nonzero annihilator.*

(b) *If P is not a maximal t-ideal, then each finitely generated ideal contained in P has a nonzero annihilator and M contains a regular polynomial $f(X)$ for which $C(f)^{-1} = R$.*

(c) *If P is not a maximal t-ideal of $R(X)$ and $(R(X)_{(M)}, (M)R(X)_{(M)})$ is a valuation pair, then P is a minimal prime of R.*

(d) *If P is a minimal prime of R and $R(X)$ contains the nilradical of $T(R[X])$, then $(R(X)_{(M)}, (M)R(X)_{(M)})$ is a discrete rank one valuation pair.*

Proof. [Proof of (a)]. By Theorem 5.6, if P is a maximal t-ideal of R, then $PR(X)$ is a maximal t-ideal of $R(X)$, in which case $M = PR(X)$. For the converse we use the fact that each ideal of $R(X)$ can be generated by polynomials. Thus all we need do is apply Lemma 5.1(b) and Lemma 5.2(b) in place of Lemma 5.1(a) and Lemma 5.2(a).

[Proof of (b)]. The proof here is no different from that given for Theorem 5.6(b).

[Proof of (c)]. A slight modification is needed to adapt the proof of Theorem 5.6(c) to the ring $R(X)$. We begin as above by assuming $(R(X)_{(M)}, (M)R(X)_{(M)})$ is a valuation pair and, by way of contradiction, that P is neither minimal nor semiregular. Then there is a polynomial $f(X) = \sum f_i X^i \in M \cap R[X]$ which is regular and such that $C(f)^{-1} = R$. Let Q be a minimal prime ideal which is contained in P, and let $a \in P\backslash Q$. Since $R(X)$ is a Marot ring and P is not semiregular, $PR(X)_{(M)} = (P)R(X)_{(M)}$ must be contained in the prime at infinity of $(R(X)_{(M)}, (M)R(X)_{(M)})$. Hence we have

$PR(X)_{(M)} = PT(R[X]) \supset QT(R[X]) = QR(X)_{(M)}$. Thus, there is an element $r(X) \in R(X)\backslash M$ such that $r(X)(a/f(X)) = s(X) \in R(X)$. (Here is where we need to make a modification.) It follows that there is a polynomial $u(X) \in \mathcal{U}$ such that both $u(X)r(X)$ and $u(X)s(X)$ are polynomials. Since $u(X)$ is a unit of $R(X)$, we may (now) assume both $r(X)$ and $s(X)$ are polynomials. Thus, similar to the above, $r(X)/f(X) \in (R(X) : (f(X), a))\backslash R(X)$.

To complete the proof simply make the usual modifications when dealing with the ring $R(X)$ rather than $R[X]$ and conclude that P must be a minimal prime ideal of R so that M is type II.

[Proof of (d)]. Assume $R(X)$ contains the nilradical of $T(R[X])$ and that P is a minimal prime of R. The first part of the proof deals with the nilradical of $T(R[X])$.

Let p be a nonzero element of P, and let N be the nilradical of R. Since P is a minimal prime, it does not contain the ideal $(N :_R p)$ and, therefore, neither does M. But since $R(X)$ contains the nilradical of $T(R[X])$, $n/f(X) \in R(X)$ for each nilpotent element n and each regular element $f(X) \in R(X)$. Thus, there is an element $s \in (N :_R p)\backslash M$ such that $sp/f(X) \in R(X)$ for each regular element $f(X)$. Hence $PT(R[X]) = PR(X)_{(M)}$ (an equality which is crucial to the proof).

Continue to the completion of the proof by following the same steps used in establishing Theorem 5.6(d). $\square$

We next show that each maximal t-ideal of $Q_0(R)[X]$ contracts to a maximal t-ideal of $R[X]$ which is either of type II or III. This provides a converse to the statement in Theorem 5.6(b).

Theorem 5.8. *Let N be a maximal t-ideal of $Q_0(R)[X]$, and let $M = N \cap R[X]$. Then M is a maximal t-ideal of $R[X]$ and $M \cap R$ is not semiregular.*

Proof. By Theorem 5.6, each finitely generated ideal of $Q_0(R)$ contained in N has a nonzero annihilator. Hence, $M \cap R$ is not semiregular. We next show that $M_t \neq R[X]$. For this, let J be a finitely generated regular ideal contained in M. As N is a maximal t-ideal of $Q_0(R)[X]$, there is an element $t \in (JQ_0(R)[X])^{-1} \neq Q_0(R)[X]$. As in the proof of Theorem 5.6(b), there is a finitely generated semiregular ideal A of R such that $tJA \subseteq R[X]$. Hence $tA \subseteq J^{-1}$. But since t is not in $Q_0(R)[X]$, tA is not contained in $R[X]$. Thus $J^{-1} \neq R[X]$.

Let M' be a maximal t-ideal of $R[X]$ that contains M. Then, by Theorem 5.6, there is a maximal t-ideal N' of $Q_0(R)[X]$ such that $M' = N' \cap R[X]$ and $R[X]_{(M')} = Q_0(R)[X]_{(N')}$. All we need show is that $N \subseteq N'$. By way of contradiction, let $g \in N \backslash N'$. Then there is a finitely generated semiregular ideal A of R such that $gA \subseteq R[X]$. As $M = N \cap R[X]$, $gA \subseteq M$. Hence $gA \subseteq M' \subseteq N'$. Since N' is a prime ideal of $Q_0(R)[X]$ and g is not in N', N' contains A, a contradiction of the second statement in Theorem 5.6(a). Thus $N = N'$ and M is a maximal t-ideal of $R[X]$. $\square$

Obviously, if a maximal t-ideal M of $R[X]$ contains a polynomial with unit content, then $MR(X) = R(X)$. If R is an integral domain, then all of the other maximal t-ideals of $R[X]$ extend to maximal t-ideals of $R(X)$, and each maximal t-ideal of $R(X)$ is the extension of a maximal t-ideal of $R[X]$. This need not be the case when R is not a domain. In Example 8.2, we show that, even if a maximal t-ideal of $R[X]$ does not contain a polynomial with unit content, it may not extend to a maximal t-ideal of $R(X)$. In fact, the ring in this example is such that every maximal t-ideal of $R[X]$ is of type III while every one of $R(X)$ is of type II.

6. When $R(X)$ is a Krull ring. A simple consequence of the Dedekind-Mertens formula is that, if $r/f(X)$ is in $R(X)$ for some $r \in R$ and (regular) polynomial $f(X) \in R[X]$, then there is a finitely generated ideal B of R and an integer $n \geq 0$ such that $r \in C(f)B$ and $rC(f)^n = BC(f)^{n+1}$. For B, one can use the content of $g(X) \in R[X]$ where $g(X)$ is such that $r/f(X) = g(X)/u(X)$ for some $u(X) \in \mathcal{U}$. The converse holds when R is integrally closed in $Q_0(R)$, see [**32**, Theorem 8]. For an element $r \in R$ and finitely generated semiregular ideal A, we say that $r \in R$ is *well generated by* A if there is a finitely generated ideal B of R and an integer $n \geq 0$ such that $r \in AB$ and $rA^n = BA^{n+1}$.

For a finitely generated semiregular ideal A of a ring R with $A^{-1} = R$, we set $\mathcal{W}(A) = \{r \in R \mid$ some $s \in (N :_R r)$ is not well generated by

$A\}$. When R is reduced, $(N :_R r)$ is simply the annihilator of r so that $\mathcal{W}(A) = \{r \in R \mid \text{some } s \in \operatorname{Ann}(r) \text{ is not well generated by } A\}$.

We wish to characterize when $R(X)$ is a Krull ring completely in terms of properties which are satisfied by the elements and ideals of R. If R is not reduced, one of the main complications when dealing with $R(X)$ is that, even if the nilradical of R coincides with the nilradical of $Q_0(R)$, $R(X)$ need not contain the nilradical of $T(R[X])$. This can happen even when R is the integral closure of a Noetherian ring [**2**, Example]. Thus it is important to know conditions on the elements and ideals of R which are both necessary and sufficient for $R(X)$ to contain the nilradical of R. Such conditions are known. A direct proof of the following result can be found in [**33**, Theorem 11]. Essentially, it is a combination of Theorem 8 and Corollary 9 of [**32**].

Theorem 6.1 (cf., [**32**, Theorem 8 and Corollary 9]). *Let R be a nonreduced ring with nilradical N. If R is integrally closed in $Q_0(R)$, then the following are equivalent.*

(1) *$R(X)$ contains the nilradical of $T(R[X])$.*

(2) *For each nonzero nilpotent $n \in R$ and each finitely generated semiregular ideal A of R, there is a finitely generated ideal B of R and an integer $m \geq 1$ such that $n \in AB$ and $nA^m = BA^{m+1}$.*

(3) *For each nonzero nilpotent $n \in R$ and each finitely generated semiregular ideal A of R, there is a finitely generated ideal B of R and a finitely generated semiregular ideal C of R such that $n \in AB$ and $nC = BCA$, i.e., each nonzero nilpotent is well generated by each finitely generated semiregular ideal of R.*

(4) *For each nonzero nilpotent $n \in R$ and each finitely generated semiregular ideal A of R, $n \in A$ and $A/\operatorname{Ann}(n)$ generates a Q_0-invertible ideal in the Q_0-integral closure of $R/\operatorname{Ann}(n)$.*

We need three more theorems before giving our characterization of when $R(X)$ is a Krull ring.

Theorem 6.2. *Let R be a nonreduced ring, and let $f(X) \in R[X]\backslash\mathcal{U}$ be such that $C(f)^{-1} = R$. For a finitely generated ideal B, if $(R(X) : (f(X), B)) \neq R(X)$, then $(N :_R B)R(X)$ is not contained* *in $f(X)R(X)$. The converse holds if $R(X)$ contains the nilradical of $T(R[X])$.*

Proof. Throughout the proof we let $I = (N :_R B)$.

Assume $IR(X) \subset f(X)R(X)$, and let $\overline{R} = R/I$ and $\overline{B} = B\overline{R}$. Since N is a radical ideal, $\overline{B}$ is a semiregular ideal of $\overline{R}$. (Also as $f(X)$ does not have unit content, I does not equal R.) By Lemma 5.4 we must have $\overline{C(f)}^{-1} = \overline{R}$. From this we have $(\overline{R}(X) : (\bar{f}(X), \overline{B})) = \overline{R}(X)$ by Lemma 5.2. Now each element of $(R(X) : (f(X), B))$ can be written in the form $g(X/u(X)f(X)$ for some $g(X) \in R[X]$ and $u(X) \in \mathcal{U}$. The image of each such element in $T(\overline{R}(X))$ must be in $\overline{R}(X)$. Hence there are polynomials $h(X)$ and $v(X)$ with $v(X) \in \mathcal{U}$ such that $v(X)g(X) - h(X)u(X)f(X)$ is a polynomial with coefficients in I. As $IR(X) \subset f(X)R(X)$, $v(X)g(X)$ must be in $f(X)R(X)$. Hence $g(X)/f(X) \in R(X)$ and, therefore, $(R(X) : (f(X), B)) = R(X)$.

For the converse, assume $R(X)$ contains the nilradical of $T(R[X])$ and $IR(X)$ is not contained in $f(X)R(X)$. It follows that there is an element $r \in I \backslash f(X)R(X)$. Since $R(X)$ contains the nilradical of $T(R[X])$, $r/f(X)$ is in $(R(X) : (f(X), B))$ and not in $R(X)$. $\square$

For reduced rings, Theorem 6.2 can be stated more simply, as

Theorem 6.3. *Let R be a reduced ring and let $f(X) \in R[X] \backslash (\mathcal{U} \cup Z(R[X]))$ be such that $C(f)^{-1} = R$. For a finitely generated ideal B of R, $(R(X) : (f(X), B)) = R(X)$ if and only if* $\mathrm{Ann}\,(B)R(X)$ *is contained in $f(X)R(X)$.*

Theorem 6.4. *Let R be a ring which is Q_0-integrally closed, and let A be a finitely generated semiregular ideal for which $A^{-1} = R$. If Q is a t-prime of $R(X)$ that contains a polynomial $f(X)$ whose content is A, then $Q \cap R \subseteq \mathcal{W}(A)$.*

Proof. Since R is integrally closed in $Q_0(R)$, an element of $s \in R$ is well generated by A if and only if $s/f(X) \in R(X)$. Let $r \in Q \cap R$. Since Q is a t-prime of $R(X)$, $(R(X) : (f(X), r)) \neq R(X)$. It follows that the ideal $(N :_R r)R(X)$ is not contained in $f(X)R(X)$. Hence, some element of $(N :_R r)$ is not well generated by A, i.e., $Q \cap R \subseteq \mathcal{W}(A)$. $\square$

The next two theorems are primarily concerned with finitely generated ideals in $R[X]$ and $R(X)$, respectively, whose contents are t-invertible.

Theorem 6.5. *Let $I = (a_0(X), a_1(X), \ldots, a_n(X))$ be a finitely generated ideal of $R[X]$ and let $m = 1 + \max\{\deg a_i(X) \mid 0 \le i \le n\}$. Then the polynomial $a(X) = \sum a_i(X)X^{mi} \in I$ is such that $C(a) = C(I)$. Moreover, if $C(I)$ is t-invertible, then there is a polynomial $b(X)$ with coefficients in $C(I)^{-1}$ such that $C(ab)^{-1} = R$ and no maximal t-ideal of type I can contain II^{-1}.*

Proof. By the construction of $a(X)$, each coefficient of $a(X)$ is a coefficient of some $a_i(X)$, and, for each i, each coefficient of $a_i(X)$ is a coefficient of $a(X)$. Hence, $C(a) = C(I)$.

Assume $C(I)$ is t-invertible. Then there is a finite dense subset $B = \{b_0, b_1, \ldots, b_r\}$ of I^{-1} such that $(BC(I))^{-1} = R$. Set $p = 1 + \deg a(X)$ and $b(X) = \sum b_k X^{pk}$. As $p > \deg a(X)$, each coefficient of $a(X)b(X)$ is the product of a single coefficient of $a(X)$ and an element of B. It follows that $C(ab) = BC(I)$ and, therefore, $C(ab)^{-1} = R$. As $a(X)b(X)$ is in II^{-1}, it must be that no maximal t-ideal of type I can contain II^{-1}. $\square$

Since each finitely generated ideal of $R(X)$ can be generated by a finite set of polynomials, a similar result holds for finitely generated ideals of $R(X)$.

Theorem 6.6. *Let $I = (a_0(X), a_1(X), \ldots, a_n(X))$ be a finitely generated ideal of $R(X)$ where each $a_i(X)$ is in $R[X]$, and let $m = 1 + \max\{\deg a_i(X) \mid 0 \le i \le n\}$. Then the polynomial $a(X) = \sum a_i(X)X^{mi} \in I$ is such that $C(a) = C(I)$. Moreover, if $C(I)$ is t-invertible, then there is a polynomial $b(X)$ with coefficients in $C(I)^{-1}$ such that $C(ab)^{-1} = R$ and no maximal t-ideal of type I can contain II^{-1}.*

We are now ready to give our characterization of when $R(X)$ is a Krull ring.

Theorem 6.7. *The following are equivalent for a ring R.*

(1) *$R(X)$ is a Krull ring.*

(2) *R is a Q_0-Krull ring, $R(X)$ contains the nilradical of $T(R[X])$ and, for each finitely generated semiregular proper ideal A for which $A^{-1} = R$, the set $\mathcal{W}(A)$ is a finite union of minimal primes of R.*

(3) *R is a Q_0-Krull ring for which each nilpotent element is well generated by each finitely generated semiregular ideal and for each finitely generated semiregular proper ideal A for which $A^{-1} = R$, the set $\mathcal{W}(A)$ is a finite union of minimal primes of R.*

Proof. Note that (2) and (3) are equivalent simply by Theorem 6.1 and the definition of an element being well generated by a finitely generated semiregular ideal.

[(1)$\Rightarrow$(2) and (3)]. Assume that $R(X)$ is a Krull ring. We first show that R is a Q_0-Krull ring. By Theorem 4.2 it suffices to show that each semiregular ideal of R is t-invertible. For such an ideal I, $IR(X)$ is a regular ideal of $R(X)$ and $[IR(X)]^{-1} = I^{-1}R(X)$. Since $R(X)$ is a Krull ring, we have $[II^{-1}R(X)]_t = ([IR(X][IR(X)]^{-1})_t = R(X)$. Since each maximal t-ideal of R extends to a maximal t-ideal of $R(X)$, we must have $(II^{-1})_t = R$.

Since Krull rings are completely integrally closed, $R(X)$ must contain the nilradical of $T(R[X])$. All that remains is to show that $\mathcal{W}(A)$ is a finite union of minimal prime ideals for each finitely generated semiregular ideal A for which $A^{-1} = R$.

Let A be a finitely generated semiregular (proper) ideal of R for which $A^{-1} = R$. By Theorem 6.2, if $f(X) \in R[X]$ is such that $C(f) = A$, then each maximal t-ideal Q of $R(X)$ that contains $f(X)$ must be such that $Q \cap R$ is contained in $\mathcal{W}(A)$. Since $R(X)$ is a Krull ring, there can only be finitely many such t-primes and each must contract to a minimal prime ideal of R by Theorem 5.7. But, by Theorem 6.2 and Theorem 3.1, for each $r \in \mathcal{W}(A)$, there is a maximal t-ideal that contains both $f(X)$ and r. Hence, the set $\mathcal{W}(A)$ is a union of finitely many minimal primes of R.

[(2)⇒(1)]. Assume that $R(X)$ contains the nilradical of $T(R[X])$ and that R is a Q_0-Krull ring such that, for each finitely generated semiregular ideal A for which $A^{-1} = R$, the set $\mathcal{W}(A)$ is a finite union of minimal prime ideals of R. It is always the case that $R(X) = \cap\{R(X)_{(M)} \mid M \in t\text{Max}\,(R(X))\}$. We will show that, for each $M \in t\ \text{Max}\,(R(X))$, $(R(X)_{(M)}, (M)R(X)_{(M)})$ is a discrete rank one valuation pair and that each regular ideal of $R(X)$ is contained in at most finitely many such maximal t-ideals.

Throughout the proof we let M be a maximal t-ideal of $R(X)$ and $P = M \cap R$.

Assume M is of type I. Then each semiregular ideal which is contained in P is t-invertible since R is a Q_0-Krull ring. It follows that the prime at infinity of the valuation pair $(R_{\{P\}}, \{P\}R_{\{P\}})$ is not semiregular. Thus the valuation associated with $(R_{\{P\}}, \{P\}R_{\{P\}})$ may be extended to $R(X)$ [**15**, Theorem 3.3]. Moreover, the resulting valuation pair is $(R(X)_{(M)}, (M)R(X)_{(M)})$ which is discrete and rank one.

Now assume M is not of type I. It is relatively easy to show that M cannot be of type III. Since it is not of type I, each finitely generated ideal $B \subseteq P$ has a nonzero annihilator and there is a polynomial $f(X) \in M \cap R[X]$ which is a regular element and is such that $C(f)^{-1} = R$, Theorem 5.7. Hence the set $\mathcal{W}(C(f))$ is a finite union of minimal prime ideals of R. As P must be contained in $\mathcal{W}(C(f))$, P must be a minimal prime ideal of R. Thus M is of type II. But then $(R(X)_{(M)}, (M)R(X)_{(M)})$ is a discrete rank one valuation pair by Theorem 5.7.

It remains to show that each regular element of $R(X)$ is contained in at most finitely many maximal t-ideals. It suffices to show that if $f(X) \in R[X]$ is such that $C(f)$ is semiregular, then only finitely many maximal t-ideals contain $f(X)$. First note that for each minimal prime Q of R and each regular $g(X) \in R[X]$, there are at most finitely many uppers to zero of $(R/Q)(X)$ which contain the image of $g(X)$. Hence there are at most finitely many maximal t-ideals of $R(X)$ that both contract to Q and contain $g(X)$.

Let $f(X) \in R[X]$ be a regular element, and assume that M contains $f(X)$.

If M is of type II, then it does not contain $C(f)$. As R is a Q_0-Krull ring, $C(f)$ is t-invertible. Hence, by Theorem 6.5, there is a polynomial

$b(X)$ with coefficients in $C(f)^{-1}$ such that $C(bf)$ is semiregular and $C(bf)^{-1} = R$. As $C(f)b(X) \subseteq R(X)$, M must contain $b(X)f(X)$. By Theorem 6.4, P must be contained in $\mathcal{W}(C(bf))$. Since $\mathcal{W}(C(bf))$ is a finite union of minimal prime ideals of R, P must be one of these primes. It follows that there are at most finitely many maximal t-ideals of type II that contain $f(X)$.

By Theorem 5.7, if M is of type I, then $C(f)^{-1} \neq R$, $M = PR(X)$ and P must contain $C(f)$. As R is a Q_0-Krull ring, at most finitely many t-primes of R contain $C(f)$. Hence, only finitely many maximal t-ideals of type I can contain $f(X)$. □

For reduced rings we have the following.

Theorem 6.8. *Let R be a reduced ring. Then $R(X)$ is a Krull ring if and only if R is a Q_0-Krull ring and, for each finitely generated semiregular ideal proper A with $A^{-1} = R$, the set $\mathcal{W}(A)$ is a finite union of minimal primes of R.*

7. When $R[X]$ and $R(X)$ are Q_0-Prüfer v-multiplication rings. The first result of this section is related to Theorems 6.2–6.4 above. It provides a way to check for maximal t-ideals of type III in the ring $R(X)$ entirely in terms of the ideals of R.

Theorem 7.1. *Let P be a prime ideal of R, and let $a(X) \in R[X]\backslash(\mathcal{U} \cup Z(R[X]))$ be such that $C(a)^{-1} = R$. Then the following are equivalent provided R is integrally closed in $Q_0(R)$ and $R(X)$ contains the nilradical of $T(R[X])$.*

(1) $[(a(X), P)R(X)]_t = R(X)$.

(2) *There is a finitely generated ideal $B \subseteq P$ such that each element of $(N :_R B)$ is well generated by $C(a)$.*

(3) *There is a finitely generated ideal $B \subseteq P$ such that $(N :_R B) \subseteq a(X)R(X)$.*

Proof. Note that a necessary and sufficient condition to have $[(a(X), P)R(X)]_t = R(X)$ is for $(a(X), P)R(X)$ to contain a finitely generated regular ideal with trivial inverse. We may take such an ideal to be generated by $a(X)$ and some finitely generated subideal of P. The equivalence of the three statements now follows from Theorem 6.2. $\square$

By Theorem 7.1, what we need to consider when checking whether $R(X)$ has any maximal t-ideals of type III are prime ideals which are not semiregular and pairs of finitely generated ideals of R where one is semiregular with trivial inverse and the other is contained in the prime. Let A be a finitely generated semiregular ideal of R for which $A^{-1} = R$. Denote by $\mathcal{G}(A)$ the set of those finitely generated ideals B for which each element of $(N :_R B)$ is well-generated by A.

Theorem 7.2. *Let R be a ring. If R is integrally closed in its ring of finite fractions and each nilpotent element of R is well-generated by each finitely generated semiregular ideal, then $R(X)$ contains no maximal t-ideals of type* III *if and only if, for each finitely generated semiregular ideal proper A with $A^{-1} = R$, each prime ideal $P \in Spec(R)\backslash \mathrm{Min}\,(R)$ contains a member of $\mathcal{G}(A)$.*

Proof. Let A be a finitely generated semiregular ideal for which $A^{-1} = R$, and let B be a finitely generated ideal which is not contained in the set $\mathcal{G}(A)$. Then by Theorem 6.2, we have that $[(a(X), B)R(X)]_t \neq R(X)$ for each polynomial $a(X)$ for which $C(a) = A$. By Theorem 3.1, any prime which is minimal over such an ideal $[(a(X), B)R(X)]_t$ must be a t-prime of $R(X)$. As $A^{-1} = R$, each of these primes must contract to a prime of R which is not semiregular. If some such prime contracts to a prime that is not a minimal prime of R, then $R(X)$ has maximal t-ideals of type III. With this, the equivalence of the two statements is clear. $\square$

For reduced rings it is also the case that, if I is a finitely generated regular ideal of $R[X]$ (or $R(X)$), then no maximal t-ideal of type II can contain II^{-1}.

Theorem 7.3. *Let R be a reduced ring. Then the following hold.*

(a) *If I is a finitely generated regular ideal of $R[X]$, then no maximal t-ideal of type II contains II^{-1}.*

(b) *If I is a finitely generated regular ideal of $R(X)$, then no maximal t-ideal of type II contains II^{-1}.*

Proof. First we prove the statement in (a). Since $R[X]$ is a Marot ring, we may assume that I is generated by a finite set of regular polynomials $\{a_0(X), a_1(X), \dots, a_n(X)\}$. Let M be a maximal t-ideal of $R[X]$ of type II that contains I. Since R is a reduced ring and $P = M \cap R$ is a minimal prime of R, R_P is a field and, therefore, $R[X]_M$ is a discrete valuation domain of rank one. Hence $IR[X]_M$ is principal. Without loss of generality, we may assume $a_0(X)$ generates $IR[X]_M$. It follows that there are elements $s(X), r_1(X), r_2(X), \dots, r_n(X) \in R[X]$ with $s(X) \in R[X] \backslash M$ such that $r_i(X)a_0(X) = s(X)a_i(X)$ for each i. It follows that $s(X)/a_0(X) \in I^{-1}$. As $s(X)$ is not in M, II^{-1} is not contained in M.

All of the above statements are valid in $R(X)$ as well, hence no maximal t-ideal of type II in $R(X)$ can contain II^{-1}. $\square$

Recall that, for a ring R, R is said to be a Prüfer ring if, equivalently, each ring between R and $T(R)$ is integrally closed in $T(R)$, or each finitely generated regular ideal is invertible [**13**, Theorem 13]; R is a Q_0-Prüfer ring if each ring between R and $Q_0(R)$ is integrally closed in $Q_0(R)$ [**31**]. A ring R for which $R(X)$ is a Prüfer ring is referred to as a strongly Prüfer ring [**4**] (or as a strong Prüfer ring [**31**]). It is known that R is a strongly Prüfer ring if and only if each finitely generated semiregular ideal of R is Q_0-invertible [**31**]. Previously, Papick and Zafrullah independently established a similar result for PvMDs. Namely, an integral domain D is a PvMD if and only if $D[X]_{\mathcal{N}(D)}$ is a Prüfer domain for the set $\mathcal{N}(D) = \{f(X) \in D[X] \,|\, C(f)^{-1} = D\}$ [**40**, Theorem] and [**43**, Theorem 11]. Huckaba and Papick proved that, if R is an additively regular McCoy ring, then it is a PvMR if and only if $R[X]_{\mathcal{N}}$ is a Prüfer ring for the set $\mathcal{N} = \{f(X) \in R[X] \,|\, (R : C(f)) = R$, and $C(f)$ is a regular ideal of $R\}$ [**20**, Theorem 3.6]. In our next result we establish a similar characterization of Q_0-PvMRs. To this end we let $\mathcal{S}$ denote the set of those polynomials $f(X) \in R[X]$ which are regular and such that $C(f)^{-1} = R$.

Theorem 7.4. *The following are equivalent for a ring R.*

(1) *R is a Q_0-PvMR.*

(2) *R is integrally closed in $Q_0(R)$ and $Q_0(R)[X]_{\mathcal{S}} = T(R[X])$.*

(3) $R[X]_{\mathcal{S}}$ is a Prüfer ring.

Proof. We begin by showing the equivalence of (1) and (2).

[(1)⇒(2)]. Assume R is a Q_0-PvMR. Then, by Lemma 3.3, if J is a finitely generated semiregular ideal of R, then $\operatorname{Hom}(J, J) = R$. Thus, R is integrally closed by Theorem 3.6 of [**29**]. For an alternate proof simply use Theorems 3.1 and 4.3 and the fact that valuation pairs are always integrally closed.

Let $g(X) = g_n X^n + \cdots + g_1 X + g_0$ be a regular element of $R[X]$. Then $C(g)$ is in $\mathcal{F}(R)$. Hence, there is a finite dense subset $A = \{a_0, a_1, \ldots, a_m\}$ of $C(g)^{-1}$ such that $(AC(g))^{-1} = R$. Set $a(X) = \sum a_i X^{i(n+1)}$ and $f(X) = a(X)g(X)$. It is easy to see that $C(f) = AC(g)$. Thus, $f \in \mathcal{S}$. Moreover, $1/g(X) = a(X)/f(X) \in Q_0(R)[X]_{\mathcal{S}}$. Therefore $Q_0(R)[X]_{\mathcal{S}} = T(R[X])$.

[(2)⇒(1)]. Assume that R is integrally closed in $Q_0(R)$ and that $Q_0(R)[X]_{\mathcal{S}} = T(R[X])$. Let $J = (j_0, j_1, \ldots, j_n)$ be a finitely generated dense ideal of R and let $g(X) = \sum j_i X^i$. Since $Q_0(R)[X]_{\mathcal{S}} = T(R[X])$, there are polynomials $f(X) \in \mathcal{S}$ and $b(X) \in Q_0(R)[X]$ such that $1/g(X) = b(X)/f(X)$. Hence $f(X) = b(X)g(X)$. By the Dedekind-Mertens formula there is an integer k such that $C(g)^k C(f) = C(g)^k C(gb) = C(g)^{k+1} C(b) = C(g)^k (C(g)C(b))$. It follows that each element of $C(g)C(b)$ is integral over R. Thus $C(g)C(b)$ is contained in R. Since $C(f) \subseteq C(g)C(b)$, the ideal $J = C(g)$ is t-invertible.

[(2)⇒(3)]. Assume R is integrally closed in $Q_0(R)$ and that $Q_0(R)[X]_{\mathcal{S}} = T(R[X])$. We will show that each finitely generated regular ideal of $R[X]_{\mathcal{S}}$ is principal. Let $I = (g_1(X), g_2(X), \ldots, g_n(X))R[X]_{\mathcal{S}}$ be a finitely generated regular ideal of $R[X]_{\mathcal{S}}$ with each $g_i(X)$ in $R[X]$. As in Theorem 6.5, there is a polynomial $g(X) \in I$ with $C(g) = C(I)$. Hence, as in the proof of (2) implies (1), there is a polynomial $b(X) \in C(g)^{-1}R[X] = C(I)^{-1}R[X]$ such that $b(X)g(X) \in \mathcal{S}$. Thus $b(X)g(X)$ is a unit of $R[X]_{\mathcal{S}}$ and, for each coefficient j_i of $g(X)$, $j_i b(X) \in R[X]$. Therefore, $j_i \in g(X)R[X]_{\mathcal{S}}$ and $I = C(g)R[X]_{\mathcal{S}} = g(X)R[X]_{\mathcal{S}}$ is a principal ideal of $R[X]_{\mathcal{S}}$.

[(3)⇒(1)]. Assume $R[X]_{\mathcal{S}}$ is a Prüfer ring, and let $g(X) = g_nX^n + \cdots + g_1X + g_0$ be a regular element of $R[X]$. Then $C(g)R[X]_{\mathcal{S}}$ is invertible.

Let $b_0(X), b_1(X), \dots , b_n(X) \in (C(g)R[X]_{\mathcal{S}})^{-1}$ be such that the sum $\sum b_i(X)g_{n-i}$ equals 1. Since each $b_i(X)$ is in $(C(g)R[X]_{\mathcal{S}})^{-1}$, there is a polynomial $f(X) \in \mathcal{S}$ such that $f(X)b_i(X)g_j \in R[X]$ for each i and j. Thus, by Lemma 5.1, $f(X)b_i(X)$ is in $C(g)^{-1}R[X]$. Let $A = \sum C(f(X)b_i(X))$. Then $C(f) \subseteq AC(g) \subseteq R$. It follows that $C(g)$ is t-invertible and, therefore, R is a Q_0-PvMR. □

We have two more theorems to prove before we characterize when $R[X]$ is a PvMR.

Theorem 7.5. *Let R be a ring, and let M be a prime ideal of $R[X]$ which is maximal with respect to containing only zero divisors. Then the ideal $N = M + XR[X]$ is a maximal t-ideal of $R[X]$.*

Proof. Every polynomial ring is a McCoy ring, so each finitely generated ideal that is contained in M has a nonzero annihilator. As a polynomial is a zero divisor if and only if the annihilator of its content is nonzero, M must be the extension of a prime ideal of R which is not semiregular.

Write $M = PR[X]$, and let $N = M + XR[X] = PR[X] + XR[X]$. Let J be a regular finitely generated ideal contained in N, and let J_0 be the ideal of constant terms of J. Since J is finitely generated, so is J_0. Since N contains X, P contains J_0. It follows that $J \subseteq I = J_0R[X] + XR[X] \subseteq N$. To show that N is a t-ideal, it suffices to show that N contains I_v.

First note that, as P is not semiregular, $\mathrm{Ann}\,(J_0) \neq (0)$, and, for each nonzero $r \in \mathrm{Ann}\,(J_0)$, $(r/X)I \subseteq R[X]$. Hence $I^{-1} \neq R[X]$.

Now let $f \in I^{-1}\backslash R[X]$. Since X is in I, we may assume $f = r/X$ for some $r \in R$. But then the only way to have fJ_0 contained in $R[X]$ is if $r \in \mathrm{Ann}\,(J_0)$. Thus $I^{-1} = (1/X)\mathrm{Ann}\,(J_0)R[X] + R[X]$, and we have that $I_v \neq R[X]$.

It follows that $N \subseteq N_t = \cup I_v$ where the union is taken over all finitely generated ideals of N that contain X. Let $P' = N_t \cap R$. Since

N contains X, $N_t = P'R[X] + XR[X]$. Thus it suffices to show that $P' = P$. To this end, let B be a finitely generated ideal contained in P'. Since $N_t \neq R[X]$, $(BR[X] + XR[X])^{-1} \neq R[X]$. But this is possible only if $\operatorname{Ann}(B) \neq (0)$. Thus, $P'R[X]$ is contained in the set of zero divisors of $R[X]$. As $P'R[X]$ contains M and M is maximal with respect to containing only zero divisors, we must have $P' = P$ and consequently $N = N_t$.

That N is a maximal t-ideal of $R[X]$ follows from the same reasoning as that used to show that it is a t-ideal. □

Recall that a prime ideal of a ring R contains only zero divisors if and only if the extension of the prime to $T(R)$ is contained in the zero divisors of $T(R)$. Thus, $R[X]$ has no maximal t-ideals of type III if and only if $T(R[X])$ is zero dimensional. For reduced rings we may say even more.

Theorem 7.6. *The following are equivalent for a reduced ring R.*

(1) *The polynomial ring $R[X]$ has no maximal t-ideals of type III.*

(2) $T(R[X])$ *is von Neumann regular.*

(3) *The space* $\operatorname{Min}(R)$ *of minimal primes of R is compact in the Zariski topology.*

(4) $Q_0(R)$ *is von Neumann regular.*

Proof. Quentel established the equivalence of (2) and (3) in [**42**, Corollary 1]. He also proved that the total quotient ring of a reduced ring is von Neumann regular if and only if the ring is a McCoy ring whose space of minimal primes is compact in the Zariski topology [**42**, Proposition 9]. Thus, if $Q_0(R)$ is von Neumann regular, $\operatorname{Min}(Q_0(R))$ is compact. As $\operatorname{Min}(Q_0(R))$ is naturally isomorphic to both $\operatorname{Min}(Q_0(R)[X])$ and $\operatorname{Min}(T(R[X]))$, having $Q_0(R)$ von Neumann regular implies $T(R[X])$ is von Neumann regular.

Assume $T(R[X])$ is von Neumann regular. By [**27**, Corollary 5] and [**28**, Theorem 3], $Q_0(R)[X]$ is integrally closed in $T(R[x])$. But by the above, $\operatorname{Min}(Q_0(R))$ is compact and therefore the total quotient ring of $Q_0(R)$, namely, itself, must be von Neumann regular by [**1**, Theorem 2.1]. □

For an alternate proof that (2) implies (4) in Theorem 7.6, simply show that each idempotent of $T(R[X])$ must be in $Q_0(R)$, and from there show directly that for each element $f \in Q_0(R)$ there is unit $s \in Q_0(R)$ such that $f^2 s^{-1} = f$.

Theorem 7.7. *Let $R = Q_0(R)$ be a reduced ring. Then the following are equivalent.*

(1) *$R[X]$ is a PvMR.*

(2) *For each maximal t-ideal M and each finitely generated ideal A contained in M, $\mathrm{Ann}\,(A)R[X]_M = R[X]_M$.*

(3) *Each maximal t-ideal of $R[X]$ contracts to a minimal prime of R, i.e., each is of type II.*

(4) *$R[X]_M$ is a discrete rank one valuation domain for each maximal t-ideal M.*

(5) *R is von Neumann regular.*

Proof. Assume R is von Neumann regular. Then not only is $R[X]$ a PvMR, but it is a Prüfer ring as well [**11**]. Each maximal ideal of $R[X]$ is regular of height one and contracts to a minimal prime of R. Thus, for each maximal ideal M of $R[X]$, $R[X]_M$ is a localization of a PID. Namely, $R[X]_M$ is a localization of $R_P[X]$ where $P = M \cap R$. Thus (5) implies all four of (1)–(4).

The equivalence of (1) and (3) follows from Theorem 5.6.

We next show that (4) implies (5).

[(4)$\Rightarrow$(5)]. Since $R = Q_0(R)$, each regular element of R is a unit. Thus, it suffices to show that, for each nonzero zero divisor r, there is an element $b \in R$ such that $br^2 = r$.

By Theorem 5.6, if M is a prime of $R[X]$ which is maximal with respect to containing only zero divisors of $R[X]$, then $M = PR[X]$ where P is a prime ideal of R which is not semiregular and $N = XR[X] + M$ is a maximal t-ideal of $R[X]$. If (4) holds, then P must be a minimal prime of R. Thus, by Theorem 5.6, each maximal t-ideal must contract to a minimal prime of R since $R = Q_0(R)$. Hence $T(R[X])$ is zero dimensional and, therefore, $\mathrm{Min}\,(R[X])$ is compact [**42**, Proposition 9]. As $\mathrm{Min}\,(R)$ is naturally isomorphic to $\mathrm{Min}\,(R[X])$

we also have that $\mathrm{Min}(R)$ compact. For reduced rings, it is known that $\mathrm{Min}(R)$ being compact is equivalent to having for each nonzero zero divisor $r \in R$, a finite set $\{a_1, a_2, \ldots, a_n\} \subseteq \mathrm{Ann}(r)$ such that $\mathrm{Ann}(r, a_1, a_2, \ldots, a_n) = (0)$ [**42**, Proposition 4]. Consider the polynomial $a(X) = r + \sum a_i X^i$ and the quotient $r/a(X)$. Simple calculations show that $(r/a(X))^2 = r/a(X)$, $(r/a(X))r = r$ and $(r/a(X))a_i = 0$ for each i. Thus $r/a(X) = e$ is a finite fraction which is a nonzero idempotent such that $er = r$. Moreover, as $ea_i = 0$ for each i, e is not equal to 1. As $R = Q_0(R)$, e is in R. Now consider the quotient $e/a(X)$. Since $ea_i = 0$ for each i and (trivially) $er = er$, $b = e/a(X)$ is a finite fraction with the property that $br = e$ and $br^2 = er = r$. Therefore, R is von Neumann regular.

For the remainder of the proof, we let M be a maximal t-ideal of $R[X]$ and $P = M \cap R$. Since $R = Q_0(R)$, $(IR[X])^{-1} = I^{-1}R[X] = R[X]$ for each finitely generated semiregular ideal I. Thus, each finitely generated ideal contained in $M \cap R$ has a nonzero annihilator. It follows that $PR[X]$ is not semiregular. Also, since R is a reduced ring, if I is finitely generated, no minimal prime of R can contain both I and $\mathrm{Ann}(I)$.

[(2)⇒(3)]. If P is not a minimal prime of R, then there is a minimal prime $Q \subset P$ and an element $a \in P \setminus Q$. As each element of P is a zero divisor, Q must contain the annihilator of (a). Thus $\mathrm{Ann}(a)R[X]_M$ cannot equal $R[X]_M$.

[(3)⇒(2) and (4)]. If P is a minimal prime of R and A is a finitely generated ideal contained in P, then we have $\mathrm{Ann}(A)R_P = R_P$. Since $P = M \cap R$, $R[X]_M$ is (isomorphic to) a localization of $R_P[X]$. Hence $Ann(A)R[X]_M = R[X]_M$. Also, since R is reduced, R_P is a field. Thus $R[X]_M$ is a localization of a PID and, therefore, a rank one discrete valuation domain. So we have that (3) implies both (2) and (4). □

We are now ready to characterize when $R[X]$ is a PvMR in terms of statements (C), (C′), (D) and (D′).

Theorem 7.8. *The following are equivalent for a ring R.*

(1) $R[X]$ *is a PvMR.*

(2) $T(R)$ *is von Neumann regular and R is a Q_0-PvMR.*

(3) *$T(R)$ is von Neumann regular and R is a PvMR.*

(4) *$T(R)$ is von Neumann regular, and either $R = T(R)$ or $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a valuation pair for each maximal t-ideal P.*

(5) *$T(R)$ is von Neumann regular, and either $R = T(R)$ or $(R_{(P)}, (P)R_{(P)})$ is a valuation pair for each regular maximal t-ideal P.*

(6) *R is a reduced Q_0-PvMR and Min(R) is compact.*

(7) *R is a reduced Q_0-PvMR and $R[X]$ has no maximal t-ideals of type III.*

(8) *R is a reduced McCoy ring and a PvMR and $R[X]$ has no maximal t-ideals of type III.*

(9) *R is a reduced McCoy ring and a PvMR and* $\mathrm{Min}\,(R)$ *is compact.*

Proof. If $T(R)$ is von Neumann regular, then R is a McCoy ring so $Q_0(R) = T(R)$. Hence (2) and (3) are equivalent as are (4) and (5). That (2) implies (4) follows from Theorem 4.3. The equivalence of (6) and (7), and of (8) and (9) follows from Theorem 7.6. If R is a McCoy ring, $T(R) = Q_0(R)$, and R is a PvMR if and only if it is a Q_0-PvMR. Thus, both (8) and (9) imply both (6) and (7). For the converse, note that a Q_0-PvMR must be integrally closed in $Q_0(R)$. Thus if R is both reduced and a Q_0-PvMR, $R[X]$ will be integrally closed. If we add the assumption that $\mathrm{Min}\,(R)$ is compact, $R[X]$ can be integrally closed only if $T(R)$ is von Neumann regular [**1**, Theorem 2.1]. But this means that R must be a McCoy ring [**42**, Proposition 9]. Hence, statements (6)–(9) are equivalent. Each is equivalent to (2) (and (3)) since $T(R)$ is von Neumann regular if and only if R is a reduced McCoy ring with $\mathrm{Min}\,(R)$ compact.

[(1)$\Rightarrow$(2) and (3)]. Assume $R[X]$ is a PvMR. Then $R[X]$ is integrally closed and, therefore, R is a reduced ring. Moreover, if A is a finitely generated semiregular ideal of R, then $AR[X]$ is t-invertible. As $(AR[X])^{-1} = A^{-1}R[X]$, we must have $(AA^{-1})_t = R$. Hence, R is a Q_0-PvMR.

By Theorems 5.6 and 5.8, there is a one-to-one correspondence between the maximal t-ideals of $Q_0(R)[X]$ and the maximal t-ideals of $R[X]$ which are not of type I. Moreover, if M is maximal t-ideal of $R[X]$ which is not of type I, then $M = N \cap R[X]$ for some maximal t-ideal N of $Q_0(R)[X]$ and $R[X]_{(M)} = Q_0(R)[X]_{(N)}$. Thus $(R[X]_{(M)}, (M)R[X]_{(M)}) = (Q_0(R)[X]_{(N)}, (N)Q_0(R)[X]_{(N)})$ is a valuation pair of $T(R[X])$. By Theorem 5.6, both M and N must be of type II. Hence, by Theorem 7.7, $Q_0(R)$ must be a von Neumann regular ring. It follows that $\mathrm{Min}\,(R)$ is compact, in which case $R[X]$ integrally closed implies that $T(R)$ is von Neumann regular [**1**, Theorem 2.1].

[(2)⇒(1)]. Assume $T(R)$ is von Neumann regular and that R is a Q_0-PvMR. Then each prime of R that contains only zero divisors is a minimal prime of R. Hence $R[X]$ has no primes of type III. That $R[X]$ is a PvMR now follows from Theorems 6.5 and 7.3.

[(4)⇒(2)]. If $T(R)$ is von Neumann regular, then each semiregular ideal of R is regular. Thus, if $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a valuation pair, then the prime at infinity is not semiregular. Hence, R is a Q_0-PvMR by Theorem 4.3. $\square$

In [**42**], Quentel constructs a reduced ring R which is its own total quotient ring but is not von Neumann regular even though $\mathrm{Min}\,(R)$ is compact. From [**42**, Proposition 9], it follows that R is not a McCoy ring, and from [**1**, Theorem 2.1] the polynomial ring $R[X]$ is not integrally closed. However, the total quotient ring $T(R[X])$ is von Neumann regular, so $R[X]$ does not have any maximal t-ideals of type III, but $R[X]$ is not a PvMR since, among other things, it fails to be integrally closed in $T(R[X])$.

We are now in a position to establish necessary and sufficient conditions in order that $R(X)$ be a Prüfer v-multiplication ring. Unlike the case for polynomial rings, it is possible for $R(X)$ to be a PvMR without having $T(R)$ von Neumann regular. The trivial case is when $R = Q_0(R)$ and is a McCoy ring (but $\mathrm{Min}\,(R)$ is not compact), but nontrivial examples exist. We construct a nontrivial example below in Example 8.13. We begin by restricting to reduced rings.

Theorem 7.9. *Let R be a reduced ring. Then the following are equivalent.*

(1) *$R(X)$ is a PvMR.*

(2) *R is a Q_0-PvMR and $R(X)$ has no maximal t-ideals of type III.*

(3) *R is a Q_0-PvMR and, for each finitely generated semiregular ideal A with $A^{-1} = R$, each prime ideal $P \in \mathrm{Spec}\,(R)\backslash\mathrm{Min}\,(R)$ contains a member of $\mathcal{G}(A)$.*

(4) *$R(X)$ has no maximal t-ideals of type III and either $R = Q_0(R)$ or, for each maximal t-ideal P, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair for which the corresponding prime at infinity is not semiregular.*

(5) *For each finitely generated semiregular ideal A with $A^{-1} = R$, each prime ideal $P \in \mathrm{Spec}\,(R)\backslash\mathrm{Min}\,(R)$ contains a member of $\mathcal{G}(A)$, and either $R = Q_0(R)$ or for each maximal t-ideal P, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair for which the corresponding prime at infinity is not semiregular.*

Proof. The proof given for (1) implies (3) in Theorem 7.8 can be adapted easily from the polynomial ring setting to the setting at hand to show that (1) implies (2). From Theorem 4.3, it is easy to see that (2) and (4) are equivalent. The equivalence of (2) and (3) and of (4) and (5) is from Theorem 7.2. All that remains is to show (2) ((3), (4) or (5)) implies (1). But, by Theorems 6.6 and 7.3, if R is a Q_0-PvMR with no maximal t-ideals of type III, then each finitely generated regular ideal of $R(X)$ is t-invertible. $\square$

We have two corollaries; the first follows from combining Theorems 7.8 and 7.9, and the second can be thought of as a corollary to the first.

Corollary 7.10. *The following are equivalent for a ring R.*

(1) *R is a PvMR and $T(R)$ is von Neumann regular.*

(2) *$R[X]$ is a PvMR.*

(3) *$R(X)$ is a PvMR and $T(R)$ is von Neumann regular.*

Corollary 7.11. *Let D be an integral domain. Then D is a PvMD if and only if $D(X)$ is a PvMD.*

For nonreduced rings we must add some assumptions to assure that the nilradical of $T(R[X])$ is contained in $R(X)$.

Theorem 7.12. *Let R be ring. Then the following are equivalent.*

(1) *$R(X)$ is a PvMR.*

(2) *R is a Q_0-PvMR, and $R(X)$ both contains the nilradical of $T(R[X])$ and has no maximal t-ideals of type III.*

(3) *R is a Q_0-PvMR such that each nilpotent element is well-generated by each finitely generated semiregular ideal and, for each finitely generated semiregular ideal A with $A^{-1} = R$, each prime ideal $P \in \operatorname{Spec}(R) \backslash \operatorname{Min}(R)$ contains a member of $\mathcal{G}(A)$.*

(4) *$R(X)$ contains the nilradical of $T(R[X])$ and has no maximal t-ideals of type III, and either $R = Q_0(R)$ or, for each maximal t-ideal P, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair for which the corresponding prime at infinity is not semiregular.*

(5) *Each nilpotent element of R is well generated by each finitely generated semiregular ideal, and, for each finitely generated semiregular ideal A with $A^{-1} = R$, each prime ideal $P \in \operatorname{Spec}(R) \backslash \operatorname{Min}(R)$ contains a member of $\mathcal{G}(A)$, and either $R = Q_0(R)$ or for each maximal t-ideal P, $(R_{\{P\}}, \{P\}R_{\{P\}})$ is a Q_0-valuation pair for which the corresponding prime at infinity is not semiregular.*

Proof. As with the characterization of when $R(X)$ is a Krull ring, (2) and (3) are easily seen to be equivalent by Theorems 6.1 and 7.2 and the definition of what it means for an element to be well generated by a finitely generated semiregular ideal. The same goes for (4) and (5).

[(1)⇒(2) and (3)]. Assume $R(X)$ is a PvMR. Then $R(X)$ is integrally closed, so $R(X)$ must contain the nilradical of $T(R[X])$. Let A be a finitely generated semiregular ideal of R. By Lemma 5.1, $(AR(X))^{-1} = A^{-1}R(X)$. Thus, A must be a t-invertible ideal of R. Hence, R is a Q_0-PvMR. It remains to show that $R(X)$ has no maximal t-ideals of type III. But, as in the case where R is assumed to be reduced, the proof given for (1) implies (3) in Theorem 7.7 carries over to the case when R may have nonzero nilpotents. Hence $R(X)$ has no maximal t-ideals of type III.

[(2)⇔(4)]. The equivalence of (2) and (4) follows from Theorem 4.3. And thus we have that (2), (3), (4) and (5) are equivalent.

[(2)⇒(1)]. Assume R is a Q_0-PvMR, and $R(X)$ contains the nilradical of $T(R[X])$ and has no maximal t-ideals of type III. Let NT denote the nilradical of $T(R[X])$, and let I be a finitely generated regular ideal of $R(X)$. As I is regular, it will contain NT. In fact, for each regular element $a(X) \in I$, $a(X)NT = NT$. Since $R(X)$ is a Marot ring and a localization of $R[X]$, $I = (a_0(X), a_1(X), \dots, a_n(X))$ where each $a_i(X)$ is both a polynomial and a regular element of $R(X)$. By Theorem 6.6, no maximal t-ideal of type I contains II^{-1}. We must show the same is true for each maximal t-ideal of type II. Let M be a maximal t-ideal of type II which contains I, and let $P = M \cap R$. It follows that $R(X)_M$ is a localization of $R_P[X]$. Moreover, $MR(X)_M \cap R_P = PR_P$, the nilradical and only maximal ideal of R_P. Let $M' = M \cap R[X]$. As PR_P is both the nilradical and only maximal ideal of R_P, a polynomial in $R_P[X]$ either has unit content or is a zero divisor with content contained in PR_P. Consider the ring $R(X)_M/PR(X)_M$. This ring is naturally isomorphic to the ring $R_P[X]_{M'}/PR_P[X]_{M'}$. As PR_P is the maximal ideal of R_P, $R_P[X]_{M'}/PR_P[X]_{M'}$ is a discrete valuation domain of rank one. It follows that $IR(X)_M/PR(X)_M$ is principal. We may assume it is generated by the image of $a_0(X)$. Then for each i, there is a $t_i \in R(X)_M$ such that $t_i a_0(X) - a_i(X) \in PR(X)_M$. Hence there is an element $s \in R(X)\backslash M$ such that $st_i a_0(X) - sa_i(X) \in PR(X)$ and $st_i \in R(X)$ for each i. Since P is a minimal prime ideal of R, for each finitely generated ideal $A \subseteq P$ there is an element $r \in R\backslash P$ such that rA is contained in the nilradical of R. Since $M \cap R = P$, such an element is not contained in M. Hence we may further assume that the element s is such that each $st_i a_0(X) - sa_i(X)$ is nilpotent. As $R(X)$ contains NT and $a_0(X)$ is a regular element of $R(X)$, $s/a_0(X)$ is in I^{-1} and $s \in II^{-1}\backslash M$. It follows that I is t-invertible and that $R(X)$ is a PvMR. □

In Example 8.1, we present an example of a nonreduced Q_0-PvMR R such that $R(X)$ has no type III maximal t-ideals but does not contain the nilradical of $T(R[X])$.

8. Examples. Our first two examples are formed using the technique of idealization of a module [**32**]. The first is similar to Example 5 in [**32**], and the second is similar to Example 4.4 in [**15**], see also, [**24**, Exercise 6, p. 62]. Our first example is of a nonreduced Q_0-PvMR such that the corresponding Nagata ring has no maximal

t-ideals of type III, yet cannot be a PvMR since it does not contain the nilradical of its total quotient ring.

Example 8.1. Let $D = K[Y, Z]$ and let $B = K(Y, Z)$ where K is a field. Let $R = D(+)B$ be the ring formed by idealization, i.e., R is the ring formed from $D \times B$ with addition and multiplication defined by $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$. Then the following hold.

(a) Identifying B with the set $\{(0, b) \mid b \in B\}$, we have that $Z(R) = B$. Thus the total quotient ring of R can be identified with the ring $K(Y, Z)(+)K(Y, Z)$.

(b) Each semiregular ideal of R is regular, so $Q_0(R) = T(R)$.

(c) For each nonzero ideal J of D, $JR = J(+)B$ and $(JR)^{-1} = J^{-1}(+)B$. These are the only regular ideals of R.

(d) R is a Q_0-PvMR and $R(X)$ has no type III maximal t-ideals.

(e) $R(X)$ is not a PvMR since it does not contain the nilradical of $T(R[X])$. In particular, the element $(0, 1)/((YX + Z), 0)$ is a nilpotent element which is not contained in $R(X)$.

Proof. Since D is an integral domain, the zero divisors of R consist of those elements (r, b) such that there is a nonzero element $c \in B$ for which $rc = 0$ [**15**, Proposition 4.3]. Thus $Z(R) = B$ and $T(R)$ can be identified with $K(Y, Z)(+)K(Y, Z)$. We also have $(0, a)(0, b) = (0, 0)$ for each $a, b \in B$. Hence each semiregular ideal is regular and we have $Q_0(R) = T(R)$. Also each prime ideal of R is of the form $P(+)B$ where P is a prime of D [**15**, Proposition 4.2].

For each nonzero element $d \in D$, $d^{-1}K(Y, Z) = K(Y, Z)$. Hence each regular ideal of R is of the form $JR = J(+)B$ where J is a nonzero ideal of D. The inverse of such an ideal JR is simply $J^{-1}(+)B$ where J^{-1} is the inverse of J as an ideal of D [**18**, Theorem 25.10]. Hence, the maximal t-ideals are the prime ideals of the form $PR = P(+)B$ where P is a height one prime of D. Since $D = K[Y, Z]$, each such prime is principal. Thus $(R_{(P)}, (P)R_{(P)})$ is a discrete rank one valuation ring. It follows that R is a Q_0-PvMR. As the only zero divisors are the nilpotent elements, $R(X)$ has no maximal t-ideals of type III. As the annihilator of each nonzero nilpotent element is the ideal B, the only finitely generated ideals which well-generate a nonzero nilpotent element are the invertible ideals of D. As D is integrally closed and not a Prüfer domain, $R(X)$ does not contain the nilradical of $T(R[X])$. Hence $R(X)$ is not a PvMR. $\square$

The ring constructed in our second example is also nonreduced. It is trivially a Q_0-PvMR since it coincides with its ring of finite fractions. The associated Nagata ring is a PvMR which is not a Prüfer ring. Another rather unusual property of this ring is that while the Nagata ring has no maximal t-ideals of type III, the corresponding polynomial ring has no maximal t-ideals of type II.

Example 8.2. Let $D = K[Y, Z]$, and let $\mathcal{P}$ be the set of nonzero principal primes of D. For each $P_\alpha \in \mathcal{P}$, let K_α denote the quotient field of D/P_α. Let $R = D(+)B$ be the idealization of the D-module $B = \sum K_\alpha$. For each $b \in B$, let $\mathcal{P}_b$ denote the set of those $P_\alpha \in \mathcal{P}$ such that the P_α-component of b is not zero. Then the following hold.

(a) $R = Q_0(R)$ but it is not a McCoy ring.

(b) $R(X)$ contains the nilradical of $T(R[X])$.

(c) $R(X)$ has no maximal t-ideals of type III and $R[X]$ has no maximal t-ideals of type II.

(d) $R(X)$ is a PvMR but not a Prüfer ring. In particular, the ideal $(Y, Z)R(X)$ is regular but not invertible.

Proof. Each semiregular ideal of R is of the form $J(+)B$ where J is an ideal of D which is contained in no height one prime [**32**, Theorem 3]. Moreover, $(J(+)B)^{-1} = J^{-1}(+)B = D(+)B$ since D is a Krull domain. It follows that $R = Q_0(R)$.

For each nonzero nilpotent b of R, the annihilator of b is the ideal $J(+)B$ where J is the (finite) intersection of the principal prime ideals $P_\alpha \in \mathcal{P}_b$. As the sum is finite and each D/P_α is a one-dimensional Noetherian domain, the total quotient ring of $R/\mathrm{Ann}\,(b)$ is naturally isomorphic to the ring $\sum K_\alpha$ where the sum is done over the ideals P_α in $\mathcal{P}_b$ and, under this isomorphism, the integral closure of $R/\mathrm{Ann}\,(b)$ is a finite direct sum of integrally closed one-dimensional Noetherian domains. Hence, each semiregular ideal of $R/\mathrm{Ann}\,(b)$ is regular and extends to an invertible ideal in the integral closure of $R/\mathrm{Ann}\,(b)$. That $R(X)$ contains the nilradical of $T(R[X])$ now follows from Theorem 6.1, i.e., [**33**, Theorem 11], or [**32**, Corollary 9]. Each prime ideal of R is of the form $P(+)B$ where P is a prime ideal of D [**15**, Proposition 4.2]. If P is a maximal ideal of D, then $P(+)B$ is semiregular. Otherwise, $P(+)B$ has a nonzero annihilator. We first show that $R[X]$ has no maximal t-ideals of type II.

As with R, the prime ideals of $R[X]$ are all of the form $Q(+)B[X]$ where Q is a prime ideal of $D[X]$. Since $R = Q_0(R)$, $R[X]$ has no maximal t-ideals of type I. Since B is the unique minimal prime ideal of R, to show that $R[X]$ has no maximal t-ideals of type II it suffices to show that if Q is an upper to zero of $D[X]$, then $Q(+)B[X]$ is not a maximal t-ideal of $R[X]$. Let Q be an upper to zero of $D[X]$. Since $D = K[Y, Z]$, there is an irreducible polynomial $f \in D[X]$ with $C_D(f)^{-1} = D$ such that $Q = fD[X]$. Let P_α be a height one prime of D such that f is not a constant modulo P_α. Since D is a UFD, $P_\alpha = (p)$ is a principal prime ideal. Consider the ideal $(f, p)R[X]$. Let $b \in B\backslash\{0\}$ be such that $pb = 0$. Since f is not a constant modulo P_α, b/f is not in $K_\alpha[X]$. Thus $b/f \in (f, p)^{-1}\backslash R[X]$. It follows that $Q(+)B$ is not a maximal t-ideal of $R[X]$.

The nilradical of $T(R[X])$ can be identified with $\sum K_\alpha(X)$ which we will denote by $B(X)$. As with $R[X]$, $R(X)$ has no maximal t-ideals of type I. For each finitely generated regular ideal J of $R(X)$, there is a finitely generated ideal I of $D(X)$ such that $J = IR(X) = I(+)B(X)$. Moreover, I cannot be contained in the extension of a height one prime of D and $J^{-1} = I^{-1}(+)B(X)$, see, for example, [**18**, Theorem 25.10]. It follows that $J_v = I_v(+)B(X)$. Hence each maximal t-ideal of $R(X)$ is the extension of a maximal t-ideal of $D(X)$. As D is a Krull domain, the maximal t-ideals of $D(X)$ are all height one primes. In $R(X)$, the extension of a height one prime of D has a nonzero annihilator. Hence each maximal t-ideal of $R(X)$ is the extension of an upper to zero of $D(X)$, i.e., each maximal t-ideal of $R(X)$ is of type II. That $R(X)$ is a PvMR follows from Theorem 7.9. □

To construct the last two examples of this paper we use a variation on the technique used to construct rings of the form $A + B$, see [**18**, Section 26]. Before doing these examples, we do several others that use the more standard version of this construction (although not in the form given in [**18**], instead see, for example, [**29**] or [**30**]). We start with the most general variation. The basic idea is to begin with a ring D and a nonempty subset $\mathcal{P}$ of $\operatorname{Spec} D$. Next let $\mathcal{A} = \{i = (\alpha, n) \mid P_\alpha \in \mathcal{P}$ and $n \geq 1\}$, and then, for each $i = (\alpha, n) \in \mathcal{A}$, let K_i denote the quotient field of D/P_α. Now let I be an ideal of D which is contained in the intersection of the P_αs (this is the beginning of the "variation" from [**18**]). Then $B = \sum K_i$ is a D/I-module and we can form a ring $R = (D/I) + B$ from the direct sum of D/I and B by defining multiplication as $(r, a)(s, b) = (rs, rb + sa + ab)$. Given a ring D, a nonempty subset $\mathcal{P}$ of $\operatorname{Spec} D$ and an ideal I which is contained in each $P_\alpha \in \mathcal{P}$, we say that $R = (D/I) + B$ is the *$A + B$ ring corresponding to D, $\mathcal{P}$ and I*. When $I = (0)$, we will simply say that $R = D + B$ is the *$A + B$ ring corresponding to D and $\mathcal{P}$*. In the event that $I = \cap P_\alpha$, then the $A + B$ ring R is isomorphic to the ring $A' + B$ where A' denotes the canonical image of D in the direct product $\prod K_i$ (and $A' + B$ is simply the sum of A' and B as subrings of $\prod K_i$). In [**18**], it is the ring $A' + B$ that is constructed. In most of the examples constructed in [**18**], the ideal I is the zero ideal (due to having $\cap P_\alpha = (0)$). In all but our next to last example, we will take I to be the zero ideal. In our last example we will be concerned with a subring of R rather than R.

For an element (r, a) of R, we let $(r)_i$ denote the image of r in K_i and let $(a)_i$ denote the ith component of a. Also, to simplify the notation, we may use the "r" in (r, a) to denote both an element of D and the image of that element in D/I. For each $i \in \mathcal{A}$ we let e_i denote the element of B for which $(e_i)_i = 1$ and all other components are 0. To simplify the proof that the rings presented in our examples have the various properties attributed to them, we give the following theorem which establishes some of the basic properties of $A + B$ rings.

Theorem 8.3. *Let D be a ring, $\mathcal{P} = \{P_\alpha\}$ be a nonempty subset of* $\operatorname{Spec} D$ *and I an ideal of D which is contained in $\cap P_\alpha$. Let $R = (D/I) + B$ be the $A + B$ ring corresponding to D, $\mathcal{P}$ and I. Then the following hold.*

(a) *A finitely generated ideal $J = ((r_1, a_1), (r_2, a_2), \dots, (r_m, a_m))$ is semiregular if and only if no P_α/I contains the ideal $J' = (r_1, r_2, \dots, r_m)D/I$, $J = J'R$ and J' is a semiregular ideal of D/I.*

(b) *Let* $S = D/I \backslash (Z(D/I) \cup (\cup P_\alpha/I))$. *Then* $T(R)$ *can be identified with the ring* $(D/I)_S + B$.

(c) $Q_0(R)$ *can be identified with the ring* $E + B$ *where* $E = \cup\{\mathrm{Hom}\,(J', D/I) \mid J'$ *is an ideal of* D/I *for which* $J'R$ *is semiregular*$\}$.

(d) *Let* $J = ((r_1, a_1), (r_2, a_2), \ldots, (r_n, a_n))$ *be a finitely generated semiregular ideal of* R, *and let* $J' = (r_1, r_2, \ldots, r_n)D/I$. *Then* J^{-1} *can be identified with* $J'^{-1} + B$. *In particular,* $J_t \neq R$ *if and only if* $J'_t \neq D/I$, J *is* Q_0*-invertible if and only if* J' *is* Q_0*-invertible, and* J *is* t*-invertible if and only if* J' *is* t*-invertible.*

(e) *If* J *is a semiregular ideal of* R, *then there is a semiregular ideal* J' *of* D/I *for which* $J = J'R = J' + B$ *with* J' *contained in no prime of the set* $\mathcal{P}$. *Moreover,* $J^{-1} = J'^{-1} + B$ *and* J *is* t*-invertible if and only if* J' *is a* t*-invertible ideal of* D/I.

(f) *An ideal* J *of* R *is regular if and only if there is a regular ideal* J' *of* D/I *such that* $J = J'R = J' + B$ *with some element of* J' *not contained in the union of the primes* $\cup P_\alpha/I$. *Moreover,* J *is invertible if and only if it is regular and* J' *is invertible as an ideal of* D/I.

Proof. Statements (a), (b) and (c) are from Theorem 19 of [**33**]. As in the proof given there, we note that for each $i = (\alpha, n)$, K_i is naturally isomorphic to the quotient field $\overline{K}_i$ of $(D/I)/(P_\alpha/I)$. Thus B is naturally isomorphic to $\overline{B} = \sum \overline{K}_i$. It follows that $(D/I) + B$ is naturally isomorphic to $(D/I) + \overline{B}$. Hence in establishing the remaining statements we may assume $I = (0)$.

The statements in (e) follow easily from (d) and the earlier results on t-invertibility, namely, Lemma 3.3. Those in (f) follow easily from (b) and (d).

For the proof of (d), let $J = J'R$ be a finitely generated semiregular ideal of R, and let $f \in \mathrm{Hom}\,(J', D)$ and $b \in B$. For each j let $fr_j = s_j$ and fix an $i = (\alpha, n)$. Since J' is not contained in P_α, we may assume that some r_j, say r_1, survives in D/P_α. Since P_α is prime and $r_1 s_k = r_k s_1$ for each k, $s_1 \in P_\alpha$ implies each $s_k \in P_\alpha$. Moreover, if $r_j \in P_\alpha$, then $s_j \in P_\alpha$. Hence, we can set $fe_i = (s_1/r_1)e_i$ and then extend f to a map on all of B. It follows that $(f, b)(r, a) = (fr, fa + rb + ab)$ is in R for each (r, a) in J. Thus, (f, b) is an R-module homomorphism from J into R.

Now let g be an R-module homomorphism from J to R. Since $B \subset J$, $g(0, e_i) = (0, e_i)(g(0, e_i))$ for each i. It follows that $gB \subset B$. For each j, let $g(r_j, 0) = (s_j, c_j)$. Then the function f defined by $fr_j = s_j$ is a D-module homomorphism from J' to D. Hence $h = g - (f, 0)$ is an R-module homomorphism which maps J into B. Let $(0, b_j) = h(r_j, 0)$. Since J' is finitely generated, there are only finitely many i for which $(b_j)_i \neq 0$ for some j. Moreover, $(b_j)_i \neq 0$ implies $(r_j)_i \neq 0$. Since $r_j b_k = r_k b_j$ for each j and k, there is an element $b \in B$ such that $(b)_i = 0$ whenever $(b_j)_i = 0$ for each j and $(b)_i = (b_j/r_j)_i$ for some nonzero $(b_j)_i$. Then h is equivalent to multiplication by $(0, b)$ and g can be identified with multiplication by (f, b).

The various statements about J and J' with regard to having nontrivial ts, or being either Q_0-invertible or t-invertible follow from having $J^{-1} = J'^{-1} + B$. $\square$

Theorem 8.4. *Let D be a domain, let $\mathcal{P} = \{P_\alpha\}$ be a nonempty subset of* $\operatorname{Spec} D$, *and let $I = (0)$. Let R be the $A{+}B$ ring corresponding to D and $\mathcal{P}$. Then the following hold.*

(a) *For each $i \in \mathcal{A}$, the set $M_i = \{(r, b) \in R \mid (r)_i = -(b)_i\}$ is a principal ideal generated by the idempotent $(1, -e_i)$ and is both a maximal and a minimal prime of R.*

(b) *If P' is a prime ideal of R, then either $P = M_i$ for some $i \in \mathcal{A}$ or $P' = P + B$ where P is a prime ideal of D.*

(c) *B is a minimal prime of R and $BT(R[X]) = B(X) = \sum K_i(X)$ is a minimal prime of $R(X)$.*

(d) *If $f(X) \in D[X]$ is such that $(f(X), 0)$ is a regular element of $R[X]$ and J is an ideal of $R(X)$ generated by $(f(X), 0)$ and a nonzero element $(r, 0) \in R$, then $J^{-1} \neq R(X)$ if and only if $C_D(f)^{-1} \neq D$.*

(e) *$R(X)$ has no maximal t-ideals of type III. Moreover, if M is a maximal t-ideal of type II, then $M \cap R = B$.*

(f) *If A is a finitely generated semiregular proper ideal of R with $A^{-1} = R$, then $\mathcal{W}(A) = B$.*

Proof. [Proof of (a)]. Fix $i = (\alpha, n) \in \mathcal{A}$. Note that the element $(1, -e_i)$ is in M_i and $(1, 0)$ is not, so M_i is a proper ideal that contains $(1, -e_i)R$. For equality note that, for $(r, b) \in M_i$, $(r, b) =$

$(r,0)(1,-e_i)+(0,b-be_i)(1,-e_i)$. To see that M_i is a maximal ideal of R, let $(s,a)\in R\backslash M_i$. Then $(s)_i\neq -(a)_i$. Thus $(s,a)(0,e_i)=(0,se_i+ae_i)$ is not zero, but for each $j\neq i$, $(se_i+ae_i)_j=0$. Since B is a direct sum of fields, there is an element $c\in B$ such that $c(se_i+ae_i)=e_i$. Hence, M_i is a maximal ideal of R. As each element of M_i is annihilated by the element $(0,e_i)$, R_{M_i} is isomorphic to K_i. Therefore, M_i is also a minimal prime of R.

[Proof of (b)]. Simple calculations show that $P+B$ is a prime ideal of R for each prime ideal P of D. For each $i\in\mathcal{A}$, the only prime that contains the idempotent $(1,-e_i)$ is M_i. Hence, if Q is a prime ideal which is not one of the M_is, it must contain each of the elements $(0,e_i)$. Hence, Q contains B, and, therefore, is of the form $P+B$ for some prime ideal P of D.

[Proof of (c)]. Obviously, R/B is isomorphic to D. Thus, B is a minimal prime of R.

Since B is the direct sum of the fields K_i, to show that $BR(X)=BT(R[X])=\sum K_i(X)$, it suffices to prove that $(0,e_i/f_i(X))\in R(X)$ for each $i\in\mathcal{A}$ and each nonzero $f_i(X)\in D_i[X]$. Fix $i=(\alpha,n)\in\mathcal{A}$ and let $f_i(X)$ be a nonzero polynomial in $D_i[X]$. Then there is a polynomial $f(X)\in D[X]$ whose image in $D_i[X]$ is $f_i(X)$. Consider the element $u(X)=(Xf(X)+1,-e_i)\in R[X]$. The content of $u(X)$ is the ideal of R generated by $(1,-e_i)$ and the content of $f(X)$ in D. Since $f_i(X)$ is nonzero, some coefficient of $f(X)$ is not in the prime ideal P_α. Thus $(0,e_i)\in C(f)R$. It follows that $u(X)$ has unit content in R. As $(1,-e_i)(0,e_i)=(0,0)$, $(0,e_i/f_i(X))=(0,e_iX)/u(X)\in R(X)$.

[Proof of (d)]. Let $J=(f(X),r)$ be an ideal of $D[X]$ for which $r\in D$ is nonzero and $f(X)\in D[X]$ is such that $(f(X),0)$ is a regular element of $R[X]$. Since r is not zero (and D is a domain), if $J^{-1}\neq D[X]$, then $C_D(f)^{-1}\neq D$. It follows that $C(JR(X))^{-1}\neq R(X)$. Thus $(JR(X))^{-1}\neq R(X)$.

Conversely, assume $(JR(X))^{-1}\neq R(X)$, and let $t\in (JR(X))^{-1}\backslash R(X)$. We may assume t has the form $(a(X),0)/(f(X),0)$ where $a(X)\in D(X)$ since $BT(R[X])\subset R(X)$. As J is finitely generated, we may further assume that $a(X)$ is a polynomial in $D[X]$. Since $r\in J$ is not zero, there is a polynomial $s(X)\in D[X]$ such that $a(X)/f(X)=s(X)/r$. That $C_D(f)^{-1}\neq D$ now follows from Lemma 5.2.

[Proof of (e)]. Let M be a maximal t-ideal of $R(X)$, and let $Q = M \cap R$. For each $i \in \mathcal{A}$, the ideal M_i is both a maximal and a minimal prime ideal of R. Hence $M_iR(X)$ is both a maximal and a minimal prime ideal of $R(X)$. Thus Q must be of the form $P + B$ for some prime ideal P of D. If $P \neq (0)$, then by (d) and Theorem 5.7, we must have that Q is a maximal t-ideal of R and $M = QR(X)$. If $P = (0)$, then M is of type II.

[Proof of (f)]. Let A be a finitely generated semiregular ideal of R with $A^{-1} = R$. Then by Theorem 8.3(e), there is a finitely generated ideal J of D that is contained in no prime of the set $\mathcal{P}$ for which $A = JR = J + B$. By the proof of (c), each polynomial with content A divides each element of B in $R(X)$. Thus, A well generates each element of B. Since $I = (0)$, all other elements of R have annihilators contained in B, so no other elements can be in $\mathcal{W}(A)$. On the other hand, each nonzero element of B can be annihilated by a finite intersection of ideals of the form M_i as described in (a). The only way A could contain such an intersection is if $J = D$. Hence, $\mathcal{W}(A) = B$. $\square$

Note that in the proof that $BR(X) = BT(R[X]) = \sum K_i(X)$ we did not use the assumptions that D was an integral domain and that I was zero. Hence we may derive the following corollary to the proof of that part of statement (c).

Corollary 8.5. *Let D be a ring, $\mathcal{P} = \{P_\alpha\}$ a nonempty subset of* Spec D *and I an ideal of D which is contained in $\cap P_\alpha$. Let $R = (D/I) + B$ be the $A + B$ ring corresponding to D, $\mathcal{P}$ and I. Then $BR(X) = BT(R[X]) = \sum K_i(X)$ and $fR(X) \supset BR(X)$ for each regular element f of $R(X)$.*

An easy consequence of Theorems 8.3 and 8.4 is the following result.

Theorem 8.6. *Let D be a domain, let $\mathcal{P} = \{P_\alpha\}$ be a nonempty subset of* Spec D*, and let $I = (0)$. Let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then the following hold.*

(a) *If D is a Prüfer domain, then R is a strongly Prüfer ring.*

(b) *If D is a Krull domain, then R is both a Krull ring and a Q_0-Krull ring.*

(c) *If D is a PvMD, then R is both a PvMR and a Q_0-PvMR.*

(d) *R is a Q_0-PvMR if and only if $R(X)$ is a PvMR.*

(e) *R is a Q_0-Krull ring if and only if $R(X)$ is a Krull ring.*

Proof. The first three statements follow from statements (d), (e) and (f) of Theorem 8.3. Statement (d) follows from the fact that $R(X)$ has no type III maximal t-ideals (Theorem 8.4(e)), and statement (e) follows from statements (e) and (f) of Theorem 8.4. $\square$

The converse of each of the statements (a), (b) and (c) in Theorem 8.6 is false. A trivial counterexample to all of them can be constructed by simply starting with a domain D that is not integrally closed. Then take $\mathcal{P}$ to be the set of maximal ideals of D. The $A + B$ ring R corresponding to D and $\mathcal{P}$ is a reduced total quotient ring which is also a McCoy ring. Thus, $R = Q_0(R)$, and it is a strongly Prüfer ring, both a Krull ring and a Q_0-Krull ring and both a PvMR and a Q_0-PvMR. But D is none of these since it is not integrally closed.

Our first example of a reduced ring is both a Krull ring and a Q_0-Krull ring with infinitely minimal primes, so the corresponding polynomial ring cannot be a Krull ring, but it is a PvMR.

Example 8.7. Let $D = K[Y]$, and let $\mathcal{P} = \{(0)\}$ (thus the only choice for I is (0)). Let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then the following hold.

(a) R is a strongly Prüfer ring, a PvMR, a Q_0-PvMR, a Krull ring and a Q_0-Krull ring.

(b) As R has infinitely many minimal primes, $R[X]$ is not a Krull ring.

(c) $T(R) = K(Y) + B$ is von Neumann regular.

(d) $R[X]$ is a PvMR.

(e) $R(X)$ is a Prüfer ring and a Krull ring.

Our second example of a reduced ring is of a nontrivial Krull ring for which the associated Nagata ring is not a Krull ring even though it has no maximal t-ideals of type III and $\mathcal{W}(A)$ is a finite union of minimal prime ideals for each finitely generated semiregular ideal A. What fails is that the ring in question is not a Q_0-Krull ring since it is not even integrally closed in its ring of finite fractions.

Example 8.8. Let $D = K[U, Y, YZ, Z^2, Z^3]$, and let $\mathcal{P}$ be the set of height one primes of D except for $P = UD$. Let $R = D + B$. Then R is a Krull ring but $R(X)$ is not even though $R(X)$ has no maximal t-ideals of type III and, for each finitely generated semiregular ideal A of R, $\mathcal{W}(A)$ is a finite union of minimal primes of R. Since no height one prime of D contains both Y and Z, the ideal $(Y, YZ, Z^2, Z^3)R$ is semiregular. Moreover, the ring of finite fractions over R can be identified with the ring $K[U, Y, Z] + B$. Hence R is not integrally closed in $Q_0(R)$, and, therefore, it is not a Q_0-Krull ring.

Proof. The only elements of D which are not contained in at least one prime from the set $\mathcal{P}$ are the constants and the constant multiples of powers of U. Hence $T(R) = D[1/U] + B = R[1/U]$. For each nonzero $f \in D[1/U]$, there is a unique integer m and element $g \in D \backslash P$ such that $f = U^m g$. For $t = (f, b) \in T(R)$, set $\nu(t) = m$ and for $s = (0, b)$, set $\nu(s) = \infty$. Straightforward calculations show that $\nu : T(R) \to \mathbf{Z} \cup \{\infty\}$ is a (discrete rank one) valuation with (R, PR) the corresponding valuation pair.

The ideal $(Y, YZ, Z^2, Z^3)R$ is semiregular. Thus, Z can be realized as an element of $Q_0(R)$ using the finite fraction form $g(X)/f(X)$ where $g(X) = YZX^3 + YZ^2X^2 + Z^3X + Z^4$ and $f(X) = YX^3 + YZX^2 + Z^2X + Z^3$. Obviously, Z is integral over R, hence R is not integrally closed in $Q_0(R)$ and, therefore, $R(X)$ cannot be a Krull ring. $\square$

The next two examples are somewhat unusual in that the total quotient rings and the rings of finite fractions coincide, yet the rings in question are Krull rings which are not Q_0-Krull rings and are PvMRs which are not Q_0-PvMRs. Thus, the associated Nagata rings are neither Krull rings nor PvMRs. The second of these two also satisfies a.c.c. on semiregular divisorial ideals but contains a semiregular divisorial ideal which is contained in infinitely many maximal t-ideals. It also has the property that each regular ideal is principal, so it is a Prüfer ring.

Example 8.9. Let $D = \mathbf{Z} + (Y,Z)\mathbf{Z}[(1/2),Y,Z] + YZ\mathbf{Q}[Y,Z]$, and let $\mathcal{P}$ be the set of primes of D which do not contain both Y and Z. Let $R = D + B$ be the associated ring of the form $A + B$ associated with D, $\mathcal{P}$ and (0). Then the following hold.

(a) The maximal ideal $M = (2)R$ is the only regular prime ideal of R and $Q_0(R) = T(R)$.

(b) $R = D+B$ is a discrete rank one Q_0-valuation ring with invertible maximal ideal M, but the prime at infinity is semiregular. Thus R is both a Prüfer ring and a Krull ring but it is neither a Q_0-Krull ring nor a Q_0-PvMR.

(c) While there are infinitely many maximal ideals which are semiregular, M is the only maximal t-ideal.

Proof. No prime in $\mathcal{P}$ contains 2 since 2 divides both Y and Z in D. On the other hand, all other prime numbers as well as all irreducible polynomials are contained in at least one member of $\mathcal{P}$. Thus, the only regular elements of R are those of the form $(2^n, b)$ for some nonnegative integer n and some element $b \in B$. Hence, $T(R)$ can be identified with $\mathbf{Z}[1/2] + (Y,Z)\mathbf{Z}[(1/2),Y,Z] + YZ\mathbf{Q}[Y,Z] + B$. Each semiregular ideal of R must contain a power of $(Y,Z)R$. Each such ideal has inverse equal to $T(R)$ since the only rational multiples of each of Y^n and Z^n which are contained in R are those whose denominators are powers of 2. Hence $T(R) = Q_0(R)$. If $(r,b) \in T(R)\backslash R$, then the constant term of r is a rational number whose reduced form has a positive power of 2 in the denominator, say 2^n. Multiplying by $(2^n, 0)$ gives an element of R which is not in MR. It follows that (R, MR) is a discrete rank one valuation pair and discrete rank one Q_0-valuation pair. The prime at infinity is the ideal $(Y,Z)\mathbf{Z}[(1/2),Y,Z] + YZ\mathbf{Q}[Y,Z]$ which is semiregular. The semiregular ideal $(Y,Z)R$ is not t-invertible. Hence, R is neither a Q_0-Krull ring nor a Q_0-PvMR, but it is both a Krull ring and a PvMR even though $T(R)$ coincides with $Q_0(R)$. $\square$

Example 8.10. Let $D = \mathbf{Z} + (Y, Z)\mathbf{Q}[[Y, Z]]$, and let $\mathcal{P}$ be the set of height one primes of D. Let $R = B+B$ be the $A+B$ ring corresponding to D and $\mathcal{P}$. Then the following hold.

(a) $T(R) = Q_0(R)$ can be identified with $\mathbf{Q}[[Y, Z]] + B$.

(b) Each regular ideal of R is principal.

(c) R is a Krull ring, a Prüfer ring and a PvMR.

(d) For $P = (Y, Z)\mathbf{Q}[[Y, Z]]$, $P + B = PR$ is a semiregular divisorial ideal with inverse $T(R)$, and PR is the only semiregular divisorial ideal which is not regular.

(e) R satisfies a.c.c. on semiregular divisorial ideals, but PR is contained in infinitely many maximal t-ideals.

(f) The semiregular ideal $(Y, Z)R$ is not t-invertible, so R is neither a Q_0-Krull ring nor a Q_0-PvMR, but it is a Q_0-Prüfer ring (but not a strongly Prüfer ring).

Proof. First each power series with constant term ± 1 is a unit in D and each element of P is a zero divisor of R. So the regular nonunits are those power series with a constant nonzero integer with at least one prime divisor. Each such power series can be factored as its constant term times a unit of D. The semiregular ideals of R are those of the form $I + B$ where I contains a power of P. As P is a common ideal of D and $\mathbf{Q}[Y, Z]]$, $(D : I) = \mathbf{Q}[[Y, Z]]$. Hence $T(R) = Q_0(R)$ and each regular ideal of R is principal of the form nR where n is a positive integer greater than 1. From this we have that R is both a Krull ring and a Prüfer ring, and that PR is the only semiregular divisorial ideal which is not regular and it has inverse equal to $T(R)$. Now an ascending chain of semiregular divisorial ideals either starts with PR with the second member of the chain a principal regular ideal nR for some positive integer n, or it starts with nR (for some positive integer n). Either way the chain must terminate. Hence, R satisfies a.c.c. on semiregular divisorial ideals. As $(Y, Z)R$ is neither Q_0-invertible nor t-invertible, R is not a strongly Prüfer ring and is neither a Q_0-Krull ring nor a Q_0-PvMR. However, it is a Q_0-Prüfer ring since it is a Prüfer ring whose total quotient ring coincides with its ring of finite fractions. $\square$

The ring in Example 8.11 is a nontrivial reduced Q_0-Krull ring which is not a Q_0-Prüfer ring. The associated Nagata ring is a nontrivial Krull ring but is not a Prüfer ring.

Example 8.11. Let $E = D[Z]$ where D is a Dedekind domain with a maximal ideal $N = (a, b)$ for which no power of N is principal. Let $\mathcal{P}$ denote the set of prime ideals of E which contain neither Z nor NE. Form the ring $R = E + B$. Then the following hold.

(a) $T(R) = E[1/Z] + B$ and $Q_0(R) = T(N)[1/Z] + B$ where $T(N)$ $(= \cup(D : N^m))$ is the Nagata transform of the ideal N.

(b) R is Q_0-Krull ring which is not Q_0-Prüfer.

(c) $R(X)$ is a Krull ring but not a Prüfer ring.

(d) $R[X]$ is not a PvMR.

Proof. For each (maximal) ideal $P_\alpha \in \mathcal{P}$, the ideal $P_\alpha R[X] + XR[X]$ is a maximal t-ideal of $R[X]$ which is not the extension of a maximal t-ideal of R and does not contract to a minimal prime of R. Thus $R[X]$ is not a PvMR. Since E is a Krull domain, R is a Q_0-Krull ring and $R(X)$ is a Krull ring by Theorem 8.6. This also makes R a Q_0-PvMR and a PvMR. The ideal of E generated by z and N is not invertible. Thus R is neither a Prüfer ring nor a Q_0-Prüfer ring. $\square$

Our next example is of a reduced ring R where R is a discrete rank one valuation ring of $T(R)$, yet is a rank two Q_0-valuation ring. Moreover, the prime at infinity under both valuations is semiregular.

Example 8.12. As in Example 8.11, let D be a Dedekind with a maximal ideal $N = (a, b)$ such that no power of N is principal. Let K be the quotient field of D, and let Y, W and Z be indeterminates over K. Let $E = D[Y, W, Z] + N[Z, Z^{-1}] + YD[1/a][Y, Z, Z^{-1}] + WD[1/b][W, Z, Z^{-1}] + YWK[Z, Z^{-1}]$, and let $\mathcal{P}$ be the set of primes of E which do not contain both Y and W. Let $R = E + B$ be the $A + B$ ring formed from E and $\mathcal{P}$. Then the following hold.

(a) The total quotient ring can be identified with $E[Z^{-1}] + B$ and the ring of finite fractions can be identified with $T(N)[Y, W, Z, Z^{-1}]+$

$E[Z^{-1}] + B$ where $T(N)$ is the ideal transform of N in the quotient field of D, i.e., $T(N) = \cup(D : N^n)$.

(b) (R, ZR) is a discrete rank one valuation pair of $T(R)$ with prime at infinity the ideal $N[Z, Z^{-1}]R$.

(c) (R, ZR) is rank two Q_0-valuation pair with semiregular prime at infinity $JR + B$ where $J = (Y, W)T(N)[Y, W, Z, Z^{-1}]$.

(d) R is a Krull ring and a PvMR which is neither a Q_0-Krull ring nor a Q_0-PvMR.

Proof. First we show that the only elements of E which are not contained in at least one member of $\mathcal{P}$ are those of the form uZ^n where u is a unit of D and n is a nonnegative integer. Since Z divides both Y and W in E, no prime in $\mathcal{P}$ can contain the set $\{uZ^n \mid n \geq 0$ and u is a unit of $D\}$. Suppose $f \in E \backslash D$ is not in the set above. Then there is a minimal integer n such that $fZ^n \in K[Y, W, Z] \backslash \{uZ^m \mid u \in K\}$. Pick an irreducible factor g of fZ^n which is not in $\{uZ^m \mid u \in K\}$. Then $P_g = gK[Y, W, Z, Z^{-1}] \cap E$ is a prime ideal of E which contains f but does not contain both Y and W. Now let c be a nonzero nonunit of D. Then there is a maximal ideal M of D such that $c \in M$ and $M \neq N$. Thus at least one of a and b is not in M, say a is not in M. Note that $tYW \in E$ for each nonzero $t \in K$, thus every prime ideal of E that contains nonzero constants from D must contain either Y or W. Since we have assumed a is not in M, we will construct a prime that contains M and W but not Y. To do so, simply contract the maximal ideal $M + WK(Z, Y)[W]$ of $D + WK(Z, Y)[W]$ to E. It follows that $T(R) = E[Z^{-1}] + B$ by Theorem 8.3.

The characterization of $Q_0(R)$ is a bit more difficult. It is fairly easy to see that $Q_0(R)$ should contain $T(N)[Y, W, Z, Z^{-1}] + E[Z^{-1}] + B$. Obviously it must contain $E[Z^{-1}] + B$. To verify that it also contains $T(N)[Y, W, Z, Z^{-1}]$ simply consider the finitely generated semiregular ideal $(Y, W)R$. Let $t \in T$ and let n be an integer. Then there is an positive integer m such that both ta^m and tb^m are in D. Hence, $t = r/a^m = s/b^m$ for some $r, s \in D$. Then $tZ^nY = rz^nY/a^m \in E$ and $tZ^nW = sZ^nW/b^m \in E$. It follows that $Q_0(R)$ contains $T(N)[Y, W, Z, Z^{-1}] + E[Z^{-1}] + B$.

By Theorem 8.3, each finitely generated semiregular ideal of R is of the form AR where A is a finitely generated ideal of E which is not contained in any of the primes of the set $\mathcal{P}$. For such an ideal AR, it must be that the minimal primes of A as an ideal of E all contain both Y and W. Hence the radical of A as an ideal of E must contain both Y and W, whence A contains Y^k and W^k for some positive integer k. Thus by Theorem 8.3, to complete the proof that $Q_0(R) = T(N)[Y,W,Z,Z^{-1}] + E[Z^{-1}] + B$, we simply need to show that $((Y^k,W^k)E)^{-1} = T(N)[Y,W,Z,Z^{-1}] + E[Z^{-1}]$ for each positive integer k. Note that since $E \subset K[Y,W,Z,Z^{-1}]$ and $((Y^k,W^k)K[Y,W,Z,Z^{-1}])^{-1} = K[Y,W,Z,Z^{-1}]$, we know that $((Y^k,W^k)E)^{-1}$ is contained in $K[Y,W,Z,Z^{-1}]$. Let $f \in ((Y^k,W^k) \times E)^{-1}$. As we may multiply fY^k and fW^k by any positive power of Z and stay in E, we will assume for now that f is an element of $K[Y,W,Z]$. As such, the constant term must be in $D[1/a] \cap D[1/b] = T(N)$ as desired. So we may assume the constant term of f is zero. As E contains $YWK[Y,W,Z,Z^{-1}]$, we may assume that none of terms of f have positive powers of both Y and W. Thus we may write f in the form $Za(Z) + Yb(Y,Z) + Wc(W,Z)$ with $a(Z) \in K[Z]$, $b(Y,Z) \in K[Y,Z]$ and $c(W,Z) \in K[W,Z]$. Multiply this expression by Y^k and W^k. Obviously, $Y^kWc(W,Z)$ and $W^kYb(Y,Z)$ are in E, no matter what $c(W,Z)$ and $b(Y,Z)$ are. Where we obtain useful information is in having the products $Y^kZa(Z) + Y^{k+1}b(Y,Z)$ and $W^kZa(Z) + W^{k+1}c(W,Z)$ in E. Since the power of Y in each term of $Y^{k+1}b(Y,Z)$ is at least $k+1$, we must have $Y^kZa(Z) \in E$. Similarly, $W^kZa(Z) \in E$. It follows that the coefficients of $a(Z)$ are in $D[1/a] \cap D[1/b] = T(N)$ as desired. To have $Y^{k+1}b(Y,Z)$ in E we must have started with $Yb(Y,Z)$ in E. Similarly, we must have started with $Wc(W,Z)$ in E. Hence, we have $f \in T(N)[Y,W,Z]+E$. Note that the products fY^k and fW^k stay in $YD[1/a][Y,Z,Z^{-1}] + WD[1/b][W,Z,Z^{-1}] + YWK[Y,W,Z,Z^{-1}]$. Any power of Z times any element of this ideal stays in this ideal. Thus $((Y^k,W^k)E)^{-1} = T(N)[Y,W,Z,Z^{-1}] + E[Z^{-1}]$.

That R is a discrete rank one valuation ring follows quite easily from the fact that $T(R) = E[Z^{-1}] + B$. Since each power of Z divides all elements of N and terms containing a positive power of either Y or W, if $f \in E[Z^{-1}] \backslash E$, then f must have at least one term of the form r/Z^n where $r \in D\backslash N$ and n a positive integer. It can of course have only finitely many such terms, we simply pick the one with the largest value of n and multiply f by Z^n to obtain an element of E which is not in ZE. From this it follows that (R, ZR) is a discrete rank one valuation pair. As $N[Z, Z^{-1}]R$ is a maximal ideal of $T(R)$ and is contained in ZR, it is the prime at infinity for the valuation.

Now consider the pair (R, ZR) with regard to $Q_0(R)$. Let $(t, b) \in Q_0(R)\backslash R$. We must have $t \in T(N)[Y, W, Z, Z^{-1}] + E[Z^{-1}]\backslash E$. If $t \in E[Z^{-1}]$ we know there is an element of ZE that will multiply t into $E\backslash ZE$. So we may assume t is not in $E[Z^{-1}]$. As we did above in establishing the elements of $Q_0(R)$, we may multiply by some nonnegative power of Z, say Z^m, to obtain a polynomial in Y, W and Z. Call this polynomial s. Write s in the form $a(Z) + Yb(Y, Z) + Wc(W, Z) + YWd(Y, W, Z)$. Note that $Yb(Y, Z) + Wc(W, Z) + YWd(Y, W, Z)$ is in E, so we must have $a(Z) \in T(N)[Z]\backslash D[Z]$. Since N^k is an invertible ideal of D for each positive integer k, no element of $T(N)\backslash D$ can multiply all of the elements of N^k back into N^k. Let k be the smallest positive integer such that $C(a)$ is contained in $(D : N^k)$. Then $C(a)N^k$ is contained in D but not contained in N^k. In fact, as k was minimal, some element of $C(a)N^k$ must be in $D\backslash N$ for otherwise we could multiply by $(D : N)$ to have $C(a)N^{k_1} = C(a)N^k(D : N) \subseteq D$. Write $a(Z) = a_0 + \cdots + a_nZ^n$, and let a_i be the first coefficient of $a(Z)$ for which $a_ir \in D\backslash N$ for some $r \in N^k$. Multiply s by rZ^{-i}. By the construction of E, all integer powers of Z multiply $Yb(Y, Z) + Wc(W, Z) + YWd(Y, W, Z)$ into ZE. Also $rZ^{-i}(a_0 + a_1Z + \cdots + a_{i-1}Z^{i-1})$ is in $N[Z, Z^{-1}] \subseteq ZE$ and $rZ^{-i}(a_{i+1}Z^{i+1} + \cdots + a_nZ^n)$ is in ZE. On the other hand, $rZ^{-i}a_iZ^i = ra_i \in E\backslash ZE$. It follows that $rZ^{m-i}t \in E\backslash ZE$. Therefore (R, ZR) is the valuation pair of $Q_0(R)$. It has rank two since there are no primes between $N[Z, Z^{-1}]E$ and $YD[1/a][Y, Z, Z^{-1}] + WD[1/b][W, Z, Z^{-1}] + YWK[Y, W, Z, Z^{-1}]$ and the latter generates a common prime ideal of R and $Q_0(R)$. The prime at infinity is semiregular, so R is not a Q_0-PvMR. □

In Example 8.13 we construct a nontrivial reduced Q_0-PvMR which is neither a Q_0-Krull ring nor a Q_0-Prüfer ring. Moreover, the ring, its total quotient ring and its ring of finite fractions are all distinct. Since the base ring is an integral domain and the ideal I is taken to be (0), the corresponding Nagata ring will be a PvMR by Theorem 8.6. The ring is constructed in a similar manner to the ring built in Example 8.11. A little background information is in order before presenting this example. Recall that the set $\mathrm{Int}(\mathbf{Z}) = \{f \in \mathbf{Q}[Y] \mid f(n) \in \mathbf{Z}$ for each $n \in \mathbf{Z}\}$ is known as the ring of integer valued polynomials on $\mathbf{Z}$.

The ring has many nice properties, including being a two-dimensional Prüfer domain which is not Bezout [**7**, p. 127]. Also, the spectrum of $\mathrm{Int}\,(\mathbf{Z})$ is known, [**6**] and [**8**]. There are height one primes of the form $P_f = f\mathbf{Q}[Y] \cap \mathrm{Int}\,(\mathbf{Z})$ where f is an irreducible polynomial. The other primes are all maximal ideals, each is (distinctly) denoted as $\mathcal{M}_{p,\alpha}$ where p is a prime integer and α is an element in the ring of p-adic integers, $\widehat{\mathbf{Z}_p}$. Each such ideal consists of those polynomials $g \in \mathrm{Int}\,(\mathbf{Z})$ for which $g(\alpha) \in p\widehat{\mathbf{Z}_p}$. If α is algebraic over $\mathbf{Z}$, $\mathcal{M}_{p,\alpha}$ is a height two maximal ideal. On the other hand, if α is transcendental over $\mathbf{Z}$, then $\mathcal{M}_{p,\alpha}$ is a height one maximal ideal, see [**7**, Proposition V.2.7] for details.

Example 8.13. Let $D = \mathrm{Int}\,(\mathbf{Z})[Z]$, and let $\mathcal{P}$ be the union of the set of height one primes $\{P_f[Z] \mid f \in (2,Y)\mathrm{Int}\,(\mathbf{Z})\}$ and the set of maximal ideals $\{\mathcal{M}_{2,\alpha}[Z] \mid \alpha \in \widehat{\mathbf{Z}_2}\backslash 2\widehat{\mathbf{Z}_2}\}$. Let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then the following hold.

(a) $R \neq T(R) \neq Q_0(R)$.

(b) R is a Q_0-PvMR which is neither Q_0-Prüfer nor Q_0-Krull.

(c) $R(X)$ is a PvMR which is neither a Krull ring nor a Prüfer ring.

Proof. None of the primes in $\mathcal{P}$ contain Z, so Z is a regular element of R which is not a unit. The same can be said for each odd prime. Hence, $R \neq T(R)$. The ideal $(2,Y)R$ is semiregular but not regular. As $(2,Y)D$ is invertible, $(2,Y)R$ is Q_0-invertible. As it is not regular, its inverse is not contained in $T(R)$ and therefore $T(R) \neq Q_0(R)$. Since $\mathrm{Int}\,(\mathbf{Z})$ is a Prüfer domain, D is a PvMD. Thus R is a Q_0-PvMR. The maximal ideal $\mathcal{M}_{3,1/2}$ has height 2 and contains the prime ideal P_{2Y-1}, [**7**, Proposition V.2.7], which is obviously comaximal with each prime that contains 2. Hence both $\mathcal{M}_{3,1/2}$ and P_{2Y-1} generate regular prime ideals of R. As $\mathrm{Int}\,(\mathbf{Z})_{\mathcal{M}_{3,1/2}}$ is a two-dimensional valuation domain, $(R_{(\mathcal{M}_{3,1/2}R)}, (\mathcal{M}_{3,1/2})R_{(\mathcal{M}_{3,1/2}R)})$ is a valuation pair of $T(R)$ which is not rank one and $(R_{\{\mathcal{M}_{3,1/2}R\}}, \{\mathcal{M}_{3,1/2}\}R_{\{\mathcal{M}_{3,1/2}R\}})$ is a valuation pair of $Q_0(R)$ which is not rank one. Thus, R is neither a Krull ring nor a Q_0-Krull ring. As $(2,Y,Z)D$ has inverse equal to D, the regular ideal $(2,Y,Z)R$ has inverse equal to R. Thus R is neither a Prüfer ring nor a Q_0-Prüfer ring. That $R(X)$ is a PvMR follows from Theorem 8.6. $\square$

We need a more complicated example to show that a reduced Q_0-PvMR need not be such that the associated Nagata ring is a PvMR. Since we are again going to employ the $A + B$ construction to build such a ring, we are finally at a point where we need to either start with a ring which is not an integral domain or take the ideal I to be nonzero. In our next example, we will start with a domain D and an ideal $I \neq (0)$ which is a radical ideal that is not prime. The resulting ring R will be a Q_0-PvMR for which the associated Nagata ring is not a PvMR.

Example 8.14. Let $D_1 = K_0[U, V, W, Y]/(UV - YW)$, and let K_1 be the quotient field of D_1. Let $\mathcal{I} = \{f_\alpha\}$ be the set of monic irreducible polynomials in $K_1[T]$. For each $f_\alpha \in \mathcal{I}$ and each positive integer n, let $j = (\alpha, n)$, and then let $\mathcal{Z} = \{Z_j\}$ be a set of indeterminates. Let $D = D_1 + (T, \mathcal{Z})K_1[T, \mathcal{Z}]$. Let I be the ideal generated by the set $\{Z_i Z_j \mid i \neq j\} \cup \{f_\alpha Z_j \mid j = (\alpha, n) \text{ for some } n \geq 1\}$, and let $S = D/I$. Let $\mathcal{P}$ be the set of prime ideals of D that contain I but do not contain both U and W. Finally, let $R = S + B$ be the $A + B$ ring corresponding to D, $\mathcal{P}$ and I. Then R is a Q_0-PvMR and a Q_0-Krull ring, but $R(X)$ is neither a PvMR nor a Krull ring.

Proof. By Theorem 8.3, each semiregular ideal of R is of the form $J'R$ where $J' = J/I$ is a semiregular ideal of S which is contained in no prime of the set $\mathcal{P}$. Thus J must be an ideal of D that contains I with both U and W in $\sqrt{J}$. Hence, J' is a regular ideal of S. Since each nonzero ideal A of D_1 is such that $A(T, \mathcal{Z})K_1[T, \mathcal{Z}] = (T, \mathcal{Z})K_1[T, \mathcal{Z}]$, $J = AD$ for some ideal A of D_1 whose radical in D_1 contains both U and W. As D_1 is Noetherian, A is finitely generated. Moreover, each nonzero element of A divides each element of $(T, \mathcal{Z})K_1[T, \mathcal{Z}]$. Thus, if $(S : J') \neq S$, then we must also have $(D_1 :_{K_1} A) \neq D_1$. Since D_1 is a Krull domain and $\sqrt{A}$ contains both U and W, $(D_1 :_{K_1} A) \neq D_1$ implies $\sqrt{A}$ is the (height one) prime ideal $N = (U, W)D_1$. It follows that $Q_0(R) = F/I + B$ where $F = T(N) + (T, \mathcal{Z})K_1[T\mathcal{Z}]$ and $T(N) = \cup(D_1 : N^n)$ is the ideal transform of N in K_1.

Since D_1 is a Noetherian Krull domain, each nonzero ideal of D_1 is t-invertible and finitely generated (as an ideal of D_1). Thus in R, each semiregular ideal is of the form $(AD/I) + B$ where A is a finitely generated ideal of D_1 whose radical contains U and W. For each such ideal A, $(A(D_1 : A))_t = D_1$. It follows that if C is a semiregular ideal of R, then $(CC^{-1})_t = R$. Hence R is both a Q_0-PvMR and a Q_0-Krull ring.

By Theorem 7.9, to show that $R(X)$ is not a PvMR, it suffices to show that it contains at least one maximal t-ideal of type III. Let $g(X) = UX + W$ and let Q_g denote the upper to zero $g(X)K_1[X] \cap D_1[X]$. Note that this ideal also contains the polynomial $YX + V$ since in K_1, $Y/U = V/W \in (D_1 : (U, W))$. But, since $(U, W)D_1$ is not an invertible ideal of D_1, Q_g does not contain a polynomial with unit content in D_1. It follows that $C(Q_g) = (U, V, W, Y)D_1$ so $(D_1 : C(Q_g)) = D_1$ and $Q_gD(X) \neq D(X)$. Thus no maximal t-ideal of type I contains Q_g. Let M be the ideal of $R(X)$ generated by (the images of) Q_g and $P = (T, \mathcal{Z})K_1[T, \mathcal{Z}]$. We will show that M_t is a proper t-ideal of $R(X)$. It is in fact a maximal t-ideal and equal to M, but all we need is that each maximal t-ideal that contains M is of type III. To this end, first note that since $(UX + W, 0)$ is a regular element of $R(X)$, $BR(X)$ is contained in M by Corollary 8.5. As Q_g is an upper to zero of $D_1[X]$, $M \cap R = P/I + B$. That $P/I + B$ is not semiregular follows from the fact that P contains neither U nor W. Also, it is not a minimal prime of R since, for example, P properly contains the prime ideal of $K_1[T, \mathcal{Z}]$ generated by $\mathcal{Z}$.

To show that M_t is a proper t-ideal, we must show that, for each finitely generated regular ideal J contained in M, $J^{-1} \neq R(X)$. Since the v-operation preserves order, it does no harm to assume J contains the regular element $(UX + W, 0)$. We start by considering an ideal J generated by $(UX+W, 0)$ and elements $(p_1, b_1), (p_2, b_2), \dots, (p_n, b_n) \in P/I + B$. Since $(UX + W, 0)$ is a regular element of $R(X)$, each of the elements $(0, b_k)/(UX + W, 0)$ is in $R(X)$. Thus, we may assume each b_k equals zero. As there are only finitely many p_ks and each is a polynomial in $(T, \mathcal{Z})K_1[T, \mathcal{Z}]$, there is an element $Z_j \in \mathcal{Z}$ such that $Z_jp_k \in I$ for each k. It follows that $(Z_j, 0)/(UX + W, 0)$ is in $(R(X) : J)\backslash R(X)$. Now note that if $p(X)$ is a polynomial in Q_g, then $p(X) = (UX + W)h(X)$ for some polynomial in $K_1[X]$. Thus $p(X)[Z_j/(UX + W)] = Z_jh(X) \in Z_jK_1[T, \mathcal{Z}][X] \subset D[X]$. It follows that each finitely generated regular ideal contained in M has a nontrivial inverse, i.e., $M_t \neq R(X)$. Since P is not a minimal prime of R, no maximal t-ideal of type II contains M. Hence $R(X)$ has maximal t-ideals of type III and $R(X)$ is not a PvMR. $\square$

In all of our previous examples of reduced Q_0-PvMRs where the associated Nagata ring is a PvMR, the set $\mathcal{W}(A)$ turned out to be a single minimal prime ideal of R, namely the ideal B. Our last example shows that, unlike what occurs when $R(X)$ is a Krull ring, $\mathcal{W}(A)$ can be an infinite union of minimal primes and still have $R(X)$ a PvMR. The ring in this example is a reduced PvMR with total quotient ring von Neumann regular, but it is not a Krull ring.

Example 8.15. Let $D = K[Y, Z]$, and let $\mathcal{P} = \{(0)\}$. Let $R = D+B$ be the $A+B$ ring corresponding to D and $\mathcal{P}$, and let $S = D+C$ where $C = \sum D_i$ is the sum of infinitely many copies of D. Then the following hold.

(a) As in Example 8.7, $T(S) = K(Y, Z)+B$ is von Neumann regular.

(b) S is a PvMR and a Q_0-PvMR, but not a Krull ring.

(c) $S(X)$ is a PvMR.

(d) For $A = (Y, Z)S$, $\mathcal{W}(A)$ is an infinite union of minimal primes.

Proof. Let $F = K(Y, Z)$, and let F_d and D_d denote the image of F and D, respectively, using the canonical embedding of each along the diagonal of $\prod F_i$. For this example it is convenient to consider S as the sum of C and D_d and $T(S)$ as the sum of B and F_d. In this view each element of $T(S)$ can be written uniquely as a sum $r + b$ where $r \in F_d$ and $b \in B$. Let A be a finitely generated semiregular (hence regular) ideal of S and write $A = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$ with each $a_j \in D_d$ and $b_j \in C$. Since $T(S)$ is von Neumann regular, S is a Marot ring so we may assume that, for each i and j, the ith component of $a_j + b_j$ is not zero. By the construction, there are at most finitely many i where $(b_j)_i$ is not zero. Since D is a Noetherian Krull domain, each (finitely generated) nonzero ideal of D is t-invertible. Let $i_1, i_2, \ldots, i_m$ be the is for which some $(b_j)_i$ is not zero, and let $t = s + c \in (S : A)$. Then $s \in (D : (a_1, a_2, \ldots, a_n))$ and c is such that $c_i = 0$ for those i not in the set $\{i_1, i_2, \ldots, i_m\}$ and $(s+c)_{i_j}(a_k+b_k)_{i_j} \in D_{i_j}$ otherwise. As $D_i = D$ for each i and the image of A in D_i is a nonzero ideal, A is t-invertible. Thus S is both a PvMR and a Q_0-PvMR. We also have that $S(X)$ is a PvMR by Corollary 7.10.

S is not Krull ring since there are infinite ascending chains of regular principal ideals. To build such a chain, start with r, a nonzero nonunit of D. Then, let $r_n = r_d + f_n$ where r_d is the image of r in D_d and f_n is the element of C for which $(f_n)_i = 0$ for $i > n$ and $(f_n)_i = 1 - r$ for $i \leq n$. Fix n, and, for $m < n$, let $s_m = 1_d + g_m$ where $(g_m)_i = r - 1$ for $m < i \leq n$ and $(g_m)_i = 0$ otherwise. Clearly, $r_m = r_n s_m$. Thus, $r_m \in r_n S$. On the other hand, r_n is not in $r_m S$. Thus, $\{r_m S\}$ is an infinite ascending chain of regular principal ideals.

It remains to show that for $A = (Y, Z)S$, $\mathcal{W}(A)$ is an infinite union of minimal prime ideals. Since each ideal $M_i = \{r \in S \mid (r)_i = 0\}$ is a minimal prime of S, it suffices to show that there are no (nonzero) elements of S that are well generated by A. Let $r = a + b$ be a zero divisor of S. If $a = 0$, then $\operatorname{Ann}(r)$ is the finite intersection $\cap M_{i_j}$ where i_j is such that $(b)_{i_j} \neq 0$. We have $S/\operatorname{Ann}(r)$ naturally isomorphic to the finite direct sum $\sum D_{i_j}$, a finite sum of Krull domains where the image of (Y, Z) is not invertible. Thus r is not well generated by A. If $a \neq 0$, then there are only finitely many i for which $(r)_i = 0$. The annihilator of r will simply be generated by the idempotent e where $(e)_i = 0$ when $(r)_i \neq 0$ and $(e)_i = 1$ when $(r)_i = 0$. Moding out by $\operatorname{Ann}(r)$ will simply give us a ring isomorphic to S with the image of r a regular element and the image of A still not invertible, so again r is not well generated by A. It follows that $\mathcal{W}(A)$ is the infinite union of the ideals M_i. $\square$

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