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THE POINTWISE VIEW OF DETERMINACY: ARBOREAL FORCINGS, MEASURABILITY, AND WEAK MEASURABILITY

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ABSTRACT. We prove that for all standard arboreal forcing notions $\mathbf{P}$ there is a counterexample for the implication "If A is determined, then A is $\mathbf{P}$-measurable". Moreover, we investigate for which forcing notions this is extendible to "weakly $\mathbf{P}$-measurable".

1. Introduction. The use of coding techniques is ubiquitous in the theory of determinacy: we code countably many reals as one, finite sequences as natural numbers, and basic open sets as finite sequences. A consequence of this fact is that many proofs using determinacy do not work with the set under investigation but with some coded or decoded version. Since most of the literature on determinacy works with the assumption that a pointclass respected by the coding is determined, this is not a problem.

As soon as we do not talk about pointclasses anymore but move on to individual sets, we start getting into trouble:

For instance, if we ask whether a given determined set has nice properties, for example, the Baire property, we tend to get unpleasant answers: In general, determined sets can be as nasty as you want them to be, as determinacy is a very local property.

This paper is part of a project trying to understand the consequences of determinacy for individual sets better. The difference between pointwise and classwise views of determinacy plays a role in higher set theory, e.g., being homogeneously Suslin has pointwise consequences, while being determined usually has only classwise consequence, and the present author has used counterexamples like the ones constructed in this paper in [**12**] to show that the usual proof of Turing determinacy will not work under the assumption of imperfect information determinacy.

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The Baire property in the above example is just one of many regularity properties derived from forcing notions. The corresponding algebras and ideals have been investigated long before the advent of forcing in [**13**], and since then they have been the subject of a considerable amount of research. As a short reference list, we mention [**2–4, 9–11**].

Among these forcing notions, we tend to distinguish between so-called "topological forcings" with the property that the forcing conditions form a topology base, and the "non-topological forcings."

In this paper we shall look at the interplay between determinacy and the set algebras defined by those non-topological forcings. We shall show that, for any given forcing notion $\mathbf{P}$ (subject to certain non-triviality restraints), it is not the case that any determined set A lies in the algebra of measurability defined by $\mathbf{P}$. Furthermore, we look at the notion of weak $\mathbf{P}$-measurability, which is classwise equivalent to $\mathbf{P}$-measurability, and determine which forcing notions $\mathbf{P}$ allow us to construct counterexamples to pointwise implications from determinacy to weak $\mathbf{P}$-measurability by a Bernstein construction.

2. Definitions and standard constructions. Since this paper is about distinguishing properties of single sets from properties of classes of sets, let's introduce a way of expressing this difference.

If Φ and Ψ are predicates of sets of reals, we say that Φ *follows pointwise from* Ψ if for all sets A the implication "$\Psi(A) \rightarrow \Phi(A)$" holds. We say that Φ *follows classwise from* Ψ if for all boldface pointclasses $\mathbf{\Gamma}$ we have the following implication:

"if we have $\Psi(A)$ for all $A \in \mathbf{\Gamma}$, then we have $\Phi(A)$ for all $A \in \mathbf{\Gamma}$"

(where a boldface pointclass is a pointclass closed under continuous preimages, cf. [**14**, pp. 19 and 27]).

As we mentioned in the introduction, most of the work on determinacy and regularity properties focuses on classwise implications. We shall look at the fact that pointwise implications do not hold, i.e., we shall construct determined sets which don't have regularity properties.

Obviously, in order to construct sets which don't have regularity properties, we shall need the axiom of choice which will be assumed tacitly throughout this paper.

This paper uses the standard notation and the standard definitions of set theory of the reals, [**1**], descriptive set theory [**14**] and the theory of forcing [**8**]. The reals are identified either with Baire space ω^ω or with Cantor space 2^ω. In general, we look at spaces N^ω where $2 \leq N \leq \omega$.

2.1 Arboreal forcings. We consider (symmetric) regularity properties derived from arboreal forcing notions, as introduced in [**11**].

A forcing notion $\mathbf{P}$ is called *arboreal on* N if it is isomorphic to a family of pruned trees on N ordered by inclusion. This is equivalent to saying that the forcing conditions are closed sets in N^ω. For each tree P we let $[P]$ be the set of all branches through P. For a finite sequence $s \in \omega^{<\omega}$, we let $[s] := \{x\,;\, s \subseteq x\}$ be the set of all branches through the tree of all sequences compatible with s.

As usual in forcing, in most situations we shall assume that the forcing notions are *nonatomic*, i.e., that for each $P \in \mathbf{P}$ there are incompatible Q and R with $Q \leq P$ and $R \leq P$. (This corresponds to the generic reals not belonging to the ground model, cf. [**8**, Lemma VII.2.4].) Most of the forcings have an even stronger property: We call $\mathbf{P}$ *strongly nonatomic* if, for each $P \in \mathbf{P}$, there are Q and R with $Q \leq P$ and $R \leq P$ and $[Q] \cap [R] = \varnothing$.

If we talk about arboreal forcing notions, we always keep in mind the standard examples from [**1, 2, 5, 11**] and [**3**]: Cohen forcing $\mathbf{C}$, Hechler forcing $\mathbf{D}$, Eventually different forcing $\mathbf{E}$, Sacks forcing $\mathbf{S}$, Miller forcing $\mathbf{M}$, Laver forcing $\mathbf{L}$, Willowtree forcing $\mathbf{W}$, Silver forcing $\mathbf{V}$, Matet forcing $\mathbf{T}$, and Mathias forcing $\mathbf{R}$ (the definitions can be found in the above mentioned papers and monographs; for $\mathbf{E}$, check [**1**], for $\mathbf{W}$ and $\mathbf{T}$, check [**2, 11**]). Among these, $\mathbf{S}$, $\mathbf{V}$ and $\mathbf{W}$ are arboreal on 2; the others are arboreal on ω.

Let A be a subset of N^ω and $\mathbf{P}$ an arboreal forcing on N. We call A *weakly* $\mathbf{P}$*-measurable* if there is a $P \in \mathbf{P}$ such that either $[P] \subseteq A$ or $[P] \subseteq N^\omega \backslash A$. We call A $\mathbf{P}$*-measurable* if, for each $Q \in \mathbf{P}$, there is a $P \leq Q$ such that either $[P] \subseteq A$ or $[P] \subseteq N^\omega \backslash A$.

Both $\mathbf{P}$-measurability and weak $\mathbf{P}$-measurability are symmetric concepts: If A is (weakly) $\mathbf{P}$-measurable, then its complement is (weakly) $\mathbf{P}$-measurable as well. It is clear that $\mathbf{P}$-measurability implies weak $\mathbf{P}$-measurability pointwise. Brendle and the present author show in [**3**, Lemma 2.1] that the converse holds classwise.

Two remarks are in order:

(1) These notions of measurability do not necessarily generate a σ-algebra. For this, we need some sort of fusion property, cf. [**5**, Definition I.3.6], which holds for the mentioned standard examples. Since this is irrelevant for the investigation at hand, we shall not discuss it here.

(2) Some arboreal forcing notions $\mathbf{P}$ generate a topology, i.e., their conditions form a topology base. We call such a forcing notion *topological*, and the others *non-topological*. In this situation we have another candidate for a natural algebra defined by $\mathbf{P}$: the σ-algebra of sets having the Baire property in the topology generated by $\mathbf{P}$. More often than not, this algebra is more natural than the algebra of the $\mathbf{P}$-measurable sets: Consider Cohen forcing where $\mathbf{C}$-measurability translates into "for each open set there is an open subset contained in A or contained in the complement of A." Clearly, the set of rationals doesn't have this property. This means that the $\mathbf{C}$-measurable sets do not form a σ-algebra, and they don't include all F_σ sets.

In these cases (among our standard examples, this happens for $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{R}$) we define the corresponding regularity property to be the Baire property associated to the topology. We shall not deal with a situation like this in this paper.

2.2 Determinacy. If T is a tree on N and $t \in T$, we say that

- T *doesn't split at* t if t has exactly one immediate successor,
- T *fully splits at* t if, for all $n \in N$, $t^\frown\langle n\rangle \in T$.

A tree $\sigma \subseteq N^{<\omega}$ is called a *strategy for player* I *on* N if it doesn't split at any node of even length, and fully splits at every node of odd length. A tree $\tau \subseteq N^{<\omega}$ is called a *strategy for player* II *on* N if it doesn't split at any node of odd length, and fully splits at every node of even length.

A set A is called *determined* if it contains the branches of a strategy σ for player I (i.e., $[\sigma] \subseteq A$; in that case, we call σ a *winning strategy for player* I) or its complement contains the branches of a strategy τ for player II (i.e., $[\tau] \subseteq N^\omega \backslash A$; in that case, we call σ a *winning strategy for player* II).

A tree T on N will be called a *substrategic* I-*tree* (*substrategic* II-*tree*) if there is a strategy σ on N for player I (for player II) such that T is a perfect subtree of σ. We call T a substrategic tree if T is either a substrategic I-tree or a substrategic II-tree. The set of all substrategic trees, for a given N clear from the context, will be denoted by $\mathbf{ssT}$. The sets of substrategic I-trees (II-trees) will be denoted by $\mathbf{ssT}^{\mathrm{I}}$ ($\mathbf{ssT}^{\mathrm{II}}$). Note that every strategy is a substrategic tree.

If T is a substrategic I-tree there is a unique $N_T \in \omega$ such that, for all nonempty $t \in T$, we have $t(0) = N_T$. If T is a substrategic II-tree, then for all n there either is a unique number $M_{T,n}$ such that for all $t \in T$ with $\mathrm{lh}(t) \geq 2$ and $t(0) = n$ we have $t(1) = M_{T,n}$ or $\langle n \rangle \notin T$. In the latter case we let $M_{T,n}$ be undefined.

It is important to note that the numbers N_T and $M_{T,n}$ are not changed by enlarging or shrinking the tree:

Observation 2.1. *If $T, T^* \in \mathbf{ssT}^{\mathrm{I}}$ and $T \subseteq T^*$, then $N_T = N_{T^*}$. If $T, T^* \in \mathbf{ssT}^{\mathrm{II}}$ and $T \subseteq T^*$, then $M_{T,n} = M_{T^*,n}$ whenever $M_{T,n}$ is defined.*

Proof. Obvious. $\square$

A finite sequence $s \in \omega^{<\omega}$ is called *compatible* with a tree T if $s \in T$. If s is compatible with T, we call $T{\uparrow}s := \{t \in T\ ;\ t \subseteq s \text{ or } s \subseteq t\}$ the *truncation of T at s*. All truncations of strategies for player I (player II) are substrategic I-trees (substrategic II-trees).

If T is a substrategic I-tree and $C \subseteq \omega^\omega$, we call C *isolated from T* if all elements of C differ from T in the first digit, i.e., if for all $x \in C$, we have $x(0) \neq N_T$. If T is a substrategic II-tree and $C \subseteq \omega^\omega$, we call C *isolated from T* if all elements of C differ from T in the second digit, i.e., if for all $x \in C$, we have that $M_{T,x(0)}$ is defined and $x(1) \neq M_{T,x(0)}$. We introduce the notation $\mathrm{Isol}(C, T)$ for "C is isolated from T."

Observation 2.2. *The following are easy combinatorial properties of substrategic trees*:

(i) *If* $T, T^* \in \mathbf{ssT}$, *S is isolated from T and $T^* \supseteq T$, then S is isolated from T^* as well.*

(ii) *If T and S are substrategic trees and $T \not\subseteq S$, then $[T]\backslash[S]$ contains a perfect set.*

Proof. Part (i) follows from Observation 2.1 and the fact that being isolated from T only depends in N_T or $\{M_{T,n}\,;\, n \in \omega\}$.

As for (ii), since T is perfect, $T{\uparrow}s$ is a perfect tree for $s \in T\backslash S$, hence $[T{\uparrow}s]$ is a perfect subset of $[T]\backslash[S]$. □

The question whether determinacy implies **P**-measurability classwise has been dealt with in [**11**] where the present author shows that there are games corresponding to the forcing notions whose determinacy gives the appropriate measurability. In most cases, these games use real moves, so the determinacy in the sense defined above may not be enough to get the wanted classwise implications. For a notion of **P**-measurability, having an integer game associated with it seems to be a strong property as the analysis of Zapletal [**15**] suggests. This is connected to the famous open problem whether the Axiom of Determinacy AD implies the Ramsey property for all sets of reals [**6**, Question 27.18].

3. Measurability. If our question is whether determinacy implies measurability pointwise, we get the expected answer: "In all interesting cases, no."

We need a technical lemma which is just an abstract version of Bernstein's theorem:

Lemma 3.1 (General Bernstein construction). *Let κ be any infinite cardinal, A any set and $\langle S_\alpha\ ;\ \alpha < \kappa\rangle$ any sequence of sets such that for each α, the set $S_\alpha\backslash A$ has cardinality κ or greater. Then there are disjoint sets X and Y with*

(1) $X \cap A = \varnothing$, $Y \cap A = \varnothing$,

(2) *for each α, $X \cap S_\alpha \neq \varnothing$, and*

(3) *for each α, $Y \cap S_\alpha \neq \varnothing$.*

X and Y are called Bernstein components *for A and $\langle S_\alpha \; ; \; \alpha < \kappa\rangle$.*

Proof. The sets X and Y are constructed by recursion. In each step β, designate two elements x_β and y_β. Write $X_\alpha := \{x_\beta \; ; \; \beta < \alpha\}$ and $Y_\alpha := \{y_\beta \; ; \; \beta < \alpha\}$. Both of these sets have α many elements (in particular, less than κ many). Consequently, the set $S_\alpha \backslash (A \cup X_\alpha \cup Y_\alpha)$ still has cardinality κ, and we are allowed to pick two new elements x_α and y_α.

Finally, set

$$X := \bigcup_{\alpha<\kappa} X_\alpha \quad \text{and} \quad Y := \bigcup_{\alpha<\kappa} Y_\alpha. \qquad \square$$

Theorem 3.2. *If $\mathbf{P}$ is an arboreal forcing such that $[P]$ has cardinality continuum for all $P \in \mathbf{P}$, then determinacy does not imply $\mathbf{P}$-measurability pointwise.*

Proof. We *claim* the following:

There are a strategy σ and a $\mathbf{P}$-condition Q such that, for all $P \leq Q$, the set $[P]\backslash[\sigma]$ has cardinality continuum.

Proof of Claim. Suppose that τ and $Q \in \mathbf{P}$ are such that the cardinality of $[Q]\backslash[\tau]$ is strictly less than $2^{\aleph_0}$ (if these don't exist, there is nothing to be shown). This means that $[Q]\cap[\tau]$ must have cardinality $2^{\aleph_0}$ because $[Q] = ([Q] \cap [\tau]) \cup ([Q]\backslash[\tau])$ and $[Q]$ has cardinality continuum.

Now take any strategy σ with $[\tau] \cap [\sigma] = \varnothing$. Clearly, for any $P \leq Q$, $[P] \cap [\sigma] \subseteq [Q]\backslash[\tau]$, so $[P]\backslash[\sigma]$ must have cardinality continuum for cardinality reasons. $\square$

We fix σ and Q as in the *Claim.*

Let $\langle P_\alpha \; ; \; \alpha < 2^{\aleph_0}\rangle$ be an enumeration of the set $\{P \; ; \; P \leq Q\}$. We can apply Lemma 3.1 to $[\sigma]$ and $\langle [P_\alpha] \; ; \; \alpha < 2^{\aleph_0}\rangle$ in order to get Bernstein components X and Y.

Now let $B := [\sigma] \cup X$. Then Q witnesses that neither B nor $N^\omega \backslash B$ is **P**-measurable: every condition below Q has a branch through both B and its complement.

But if σ was a strategy for player I, then B is determined. If, on the other hand, σ was a strategy for player II, the complement of B is determined. □

Corollary 3.3. *For all mentioned non-topological forcing notions* **P** (**S**, **M**, **L**, **W**, **V**, **T**), *determinacy does not imply* **P**-*measurability pointwise.*

Proof. All mentioned forcings are strongly nonatomic and all strongly nonatomic forcings have perfect sets as conditions. Thus the result follows from Theorem 3.2. □

4. Weak measurability. The situation for weak **P**-measurability is much more complicated and there are not always counterexamples to pointwise implications.

So we have to be careful, and our of means of classifying forcings is a measure called fatness. We shall be able to show that we have definitive answers for fatness zero and uncountable fatness and that all classical examples of arboreal forcing notions have one of these two values.

The situation of other values of fatness is more subtle, and we shall give examples of (not too natural) forcing notions that show that for finite or countable fatness the relationship between determinacy and weak measurability is unclear.

Fix any $\kappa \leq 2^{\aleph_0}$. We call a forcing notion **P** that is arboreal on N *κ-fat* if there is a strategy σ on N such that for all $P \in \mathbf{P}$ the set $[P] \backslash [\sigma]$ has cardinality at least κ.

We shall now define the notion of fatness:

• We say that **P** has *fatness* n for some natural number n if **P** is n-fat but not $n+1$-fat.

• We say that **P** has *fatness* $< \aleph_0$ if **P** is n-fat for all natural numbers n but not $\aleph_0$-fat.

• We say that **P** has *countable fatness* if **P** is $\aleph_0$-fat but not $\aleph_1$-fat.

- We say that $\mathbf{P}$ has *uncountable fatness* if $\mathbf{P}$ is $\aleph_1$-fat.

The conspicuous gap (we don't distinguish between different uncountable values of fatness) is easily explained:

Proposition 4.1. *If* $\mathbf{P}$ *has uncountable fatness, then it is* $2^{\aleph_0}$*-fat.*

Proof. Suppose $\mathbf{P}$ is $\aleph_1$-fat as witnessed by a strategy σ. Then the sets $[P]\backslash[\sigma]$ are uncountable for all $P \in \mathbf{P}$. But both $[P]$ and $[\sigma]$ are closed sets, so their difference is Borel. By Hausdorff's theorem [**7**, Theorem (13.6)], every uncountable Borel set contains a perfect set, so all of these sets are actually of cardinality continuum. Consequently, σ witnesses that $\mathbf{P}$ is $2^{\aleph_0}$-fat. □

As we anticipated, forcings with fatness zero or uncountable fatness can be handled quite easily. Before we prove this, let us note that all natural forcing notions seem to have either fatness zero or uncountable fatness:

Observation 4.3. *Sacks forcing* $\mathbf{S}$ *and Miller forcing* $\mathbf{M}$ *have fatness zero. Laver forcing* $\mathbf{L}$*, Willowtree forcing* $\mathbf{W}$*, Silver forcing* $\mathbf{V}$ *and Matet forcing* $\mathbf{T}$ *have uncountable fatness. As an aside, note that all mentioned topological forcings,* $\mathbf{C}$*,* $\mathbf{D}$*,* $\mathbf{E}$*, and* $\mathbf{R}$ *have uncountable fatness as well.*

Proof. For $\mathbf{S}$ and $\mathbf{M}$, this is clear, since each strategy on 2 is a Sacks condition and each strategy on ω is a Miller condition.

Uncountable fatness for the other forcings has to be checked by picking the strategy cleverly that does the job. (In the case of Matet forcing this depends on the definition: Matet forcing has fatness continuum only in its strictly increasing variant.)

As a simple example, let us discuss Laver forcing $\mathbf{L}$. In this case, it doesn't matter what strategy we pick: Fix a strategy σ. We shall show that for each Laver tree L, the set $[L]\backslash[\sigma]$ has uncountably many elements. There is $s \in L \cap \sigma$ such that s has exactly one immediate successor in σ (if not, $[L] \cap [\sigma] = \varnothing$ and there's nothing to be shown). But then let s^* be any immediate extension of s which is not in σ but in L (there are infinitely many of those). $[L\uparrow s^*]$ is a perfect set that's disjoint from $[\sigma]$, and hence $[L]\backslash[\sigma] \supseteq [L\uparrow s^*]$.

Since we didn't give the definitions of the forcings, it doesn't make sense to go into details here. We leave it to the reader to play around with those combinatorial objects (which is not too hard). □

The most common values zero and continuum are easy to handle:

Proposition 4.4. *If* $\mathbf{P}$ *has fatness zero, then weak* $\mathbf{P}$*-measurability follows pointwise from determinacy.*

Proof. Having fatness zero means in particular that $\mathbf{P}$ is not 1-fat, so for all strategies σ there is a $\mathbf{P}$-condition P_σ whose branches are completely contained in $[\sigma]$.

Thus if $[\sigma] \subseteq A$, then $[P_\sigma] \subseteq A$, and if $[\tau] \subseteq N^\omega\backslash A$, then $[P_\tau] \subseteq N^\omega\backslash A$. Hence any determined A is weakly $\mathbf{P}$-measurable. □

Theorem 4.5. *If* $\mathbf{P}$ *has uncountable fatness, weak* $\mathbf{P}$*-measurability does not follow pointwise from determinacy.*

Proof. We have to construct a determined set which is not weakly $\mathbf{P}$-measurable:

By Proposition 4.1, $\mathbf{P}$ is $2^{\aleph_0}$-fat, i.e., there is a strategy σ with the property that for all $P \in \mathbf{P}$, $[P]\backslash[\sigma]$ still contains $2^{\aleph_0}$ reals. We now apply Lemma 3.1 to $[\sigma]$ and the sequence $\langle [P_\alpha]\ ;\ \alpha < 2^{\aleph_0}\rangle$ of $\mathbf{P}$-conditions. Let X and Y be Bernstein components.

Finally, let $B := [\sigma] \cup X$. Clearly, B cannot contain the branches through an element of $\mathbf{P}$ as witnessed by Y. But X witnesses that the complement of B cannot contain the branches through an element of $\mathbf{P}$. Consequently, neither B nor its complement are weakly $\mathbf{P}$-measurable. But, if σ is a strategy for player I, then B is determined, and if σ was a strategy for player II, the complement of B is determined. □

Corollary 4.6. *Weak Sacks measurability and weak Miller measurability follow pointwise from determinacy. This is not true for weak Laver, Willowtree, Silver or Matet measurability.*

Proof. This is now clear from Observation 4.3, Proposition 4.4 and Theorem 4.5. ◻

5. Finite and countable fatness: Pointwise implications. Proposition 4.4 and Theorem 4.5 settle the cases of fatness zero and of uncountable fatness.

This leaves the cases of fatness n, for $n > 0$, of fatness $< \aleph_0$ and of countable fatness. As we mentioned, these cases are less interesting, since all canonical examples have either fatness 0 or uncountable fatness. It turns out that we get nonstructure results in all three mentioned cases, and typical forcings with these values of fatness tend to be somewhat odd. It would be interesting to ask whether there are any natural examples of forcings ("natural" meaning "having any applications") with these values of fatness, or, in contrast, whether we can find a natural property of forcings (that all forcings used in applications share) that implies that the fatness of all forcings with this property is either zero or uncountable. We can show that strong nonatomicity excludes two of the problematic values for fatness:

Proposition 5.1. *If* $\mathbf{P}$ *is strongly nonatomic and* $\mathbf{P}$ *is not* $\aleph_0$*-fat, then* $\mathbf{P}$ *has fatness zero.*

Proof. Take some n such that $\mathbf{P}$ is n-fat and a strategy σ witnessing this. Towards a contradiction, assume that $n > 0$. Since $\mathbf{P}$ is not $\aleph_0$-fat, there must be some $P \in \mathbf{P}$ such that $S := [P]\backslash[\sigma]$ is finite, say it has k elements.

We can use the strong nonatomicity to split P into subconditions $\{P_i\,;\, i < k+1\}$ such that the sets $[P_i]$ are pairwise disjoint. Since σ was witnessing that $\mathbf{P}$ is n-fat, each $S \cap [P_i]$ must contain at least n elements. So, $\mathrm{Card}(S) \geq n \cdot (k+1)$, and since $n > 0$, this contradicts $\mathrm{Card}(S) = k$. ◻

Let us look at canonical examples for forcings of finite and countable fatness. All trees will be trees on ω from now on.

For the case of forcings with fatness $< \aleph_0$ we need to fix a partition of the substrategic I-trees by defining $\chi(T) := N_T + 1$. If T is a substrategic II-tree, we set $\chi(T) = 1$. Note that this function has the property that if $T \subseteq T^*$, then $\chi(T) = \chi(T^*)$ by Observation 2.1.

We define

$$\begin{aligned} \mathbf{G}_n &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{Isol}(S,T), \mathrm{Card}(S) = n\}, \\ \mathbf{G}_{<\aleph_0} &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{Isol}(S,T), \mathrm{Card}(S) = \chi(T)\}, \\ \mathbf{G}_{\aleph_0} &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{Isol}(S,T), \mathrm{Card}(S) = \aleph_0, \mathrm{rk}_{\mathrm{CB}}(S) = 0\}. \end{aligned}$$

In this definition, $\mathrm{rk}_{\mathrm{CB}}$ denotes the Cantor-Bendixson rank function. $\mathrm{rk}_{\mathrm{CB}}(S) = 0$ means that S has no limit points. In particular, S is closed.

Theorem 5.2. *$\mathbf{G}_n$ is a nonatomic forcing notion of fatness n. Determinacy implies weak $\mathbf{G}_n$-measurability pointwise.*

Proof. (1) Let's first show the nonatomicity: We take any $P := [T] \cup S$ for $T \in \mathbf{ssT}$. Since T is a perfect tree, we find proper perfect subtrees T_0 and T_1. Now $P_0 := [T_0] \cup S$ and $P_1 := [T_1] \cup S$ witness nonatomicity.

(2) It is clear that $\mathbf{G}_n$ is not $n+1$-fat, since for every strategy σ and every set S isolated from σ and with n distinct elements, the set $P := [\sigma] \cup S$ is a condition, and $S = P \backslash [\sigma]$ has n elements.

(3) For every strategy σ and every $P \in \mathbf{G}_n$, the set $P \backslash [\sigma]$ has at least n elements: Let $P = [T] \cup S$ where $T \in \mathbf{ssT}$.

If T is a subtree of σ, then S is isolated from σ by Observation 2.2 (i). So, $P \backslash [\sigma] = S$.

If T is not a subtree of σ, then by Observation 2.2 (ii), $[T] \backslash [\sigma] \subseteq P \backslash [\sigma]$ contains a perfect set.

Thus, $\mathbf{G}_n$ is n-fat.

(4) Suppose that A is determined but not weakly $\mathbf{G}_n$-measurable.

Case A. There is a strategy σ for player I such that $[\sigma] \subseteq A$. With any set of pairwise distinct elements $\{x_1, \ldots, x_n\}$ such that $x_i(0) \neq N_\sigma$, we can construct a $\mathbf{G}_n$ condition $P := [\sigma] \cup \{x_1, \ldots, x_n\}$. Thus (we are assuming that A doesn't contain a condition) A cannot contain n such elements. But this means that there is an infinite set of natural numbers M such that $A \cap [\langle m \rangle] = \varnothing$ for all $m \in M$.

Now pick $n+1$ distinct elements $\{k_0, \ldots, k_n\}$ from M. Take any strategy σ^* for player I with $N_{\sigma^*} = k_0$ and any elements $\{x_1, \ldots, x_n\}$ with $x_i(0) = k_i$ (for $1 \leq i \leq n$). Clearly, $[\sigma^*] \cup \{x_1, \ldots, x_n\}$ is a condition and it's contained in $\omega^\omega \backslash A$. Contradiction.

Case B. We have a strategy τ for player II such that $[\tau] \subseteq \omega^\omega \backslash A$. This case is very similar to Case A:

We look at all sequences of length two. Of these infinitely many are not in τ. Any element of ω^ω starting with one of these is isolated from τ, so $\omega^\omega \backslash A$ cannot contain n pairwise distinct elements starting with one of those. That means there are infinitely many sequences s of length two such that $\omega^\omega \backslash A \cap [s] = \varnothing$. Take $n+1$ sequences $\{s_0, \ldots, s_n\}$ like this, find a substrategic tree T such that $[T] \subseteq [s_0]$ and elements $\{x_1, \ldots, x_n\}$ with $s_i \subseteq x_i$ (for $1 \leq i \leq n$). Then $[T] \cup \{x_1, \ldots x_n\}$ is a condition that's contained in A. Contradiction. □

Theorem 5.3. $\mathbf{G}_{<\aleph_0}$ *is a nonatomic forcing notion of fatness* $< \aleph_0$. *Determinacy implies weak* $\mathbf{G}_{<\aleph_0}$*-measurability pointwise.*

Proof. The proof is close to the proof of Theorem 5.2, and we use the numbering (1) to (4) from there. The proof of nonatomicity (1) is the same.

(2) It is clear that $\mathbf{G}_{<\aleph_0}$ is not $\aleph_0$-fat. If we take any strategy σ and a set S of $\chi(\sigma)$ many points isolated from σ, then $P := [\sigma] \cup S$ is a condition, and $S = P \backslash [\sigma]$ has $\chi(\sigma)$, i.e., finitely, many points.

(3) We shall now show that $\mathbf{G}_{<\aleph_0}$ is n-fat for every natural number n. We fix a strategy σ for player I with $\chi(\sigma) = n$. Then the proof of (3) in Theorem 5.2 shows that σ witnesses that $\mathbf{G}_{<\aleph_0}$ is n-fat.

(4) Again, we use the analogous part from the proof of Theorem 5.2. In Case A, we have a strategy σ with $[\sigma] \subseteq A$ and find an infinite set M of natural numbers such that $[\langle m \rangle] \cap A = \varnothing$. Let m_0 be the least such number, and $\{m_1, \ldots, m_{m_0+1}\}$ a set of $m_0 + 1$ pairwise different numbers from M (all different from m_0). If we pick any strategy σ^* for player I such that $N_{\sigma^*} = m_0$, then for any choice of $x_i \in [\langle m_i \rangle]$ (for $1 \leq i \leq m_0 + 1$), the set $[\sigma^*] \cup \{x_1, \ldots, x_{m_0+1}\}$ is a condition that lies completely in the complement of A.

In Case B, we have an infinite set of sequences s of length two such that $\omega^\omega \backslash A \cap [s] = \varnothing$. We pick two of them, say s_0 and s_1, find a substrategic II-tree T such that $[T] \subseteq [s_0]$ and pick $x \in [s_1]$. Then $[T] \cup \{x\}$ is a condition that's contained in A. Contradiction. □

Theorem 5.4. $\mathbf{G}_{\aleph_0}$ *is a strongly nonatomic forcing notion of countable fatness. Determinacy implies weak* $\mathbf{G}_{\aleph_0}$*-measurability pointwise.*

Proof. Again the proof is along the lines of the proof of Theorem 5.2.

(1) In this case the forcing is strongly nonatomic, since we can split up every infinite internally isolated set into two infinite internally isolated disjoint sets. (This is where we need $\mathrm{rk}_{\mathrm{CB}}(S) = 0$. Note that we are essentially using that $\mathbf{G}_{\aleph_0}$ lives on ω–in the space N^ω for $N < \omega$ there is no infinite set with $\mathrm{rk}_{\mathrm{CB}}(S) = 0$ due to König's Lemma and the pigeon hole principle.)

(2) The forcing is clearly not $\aleph_1$-fat, since for each strategy σ there is a countable internally isolated set S that is isolated from σ, hence $P := [\sigma] \cup S$ is a condition, and $S = P \backslash [\sigma]$ is countable.

(3) The proof that $\mathbf{G}_{\aleph_0}$ is $\aleph_0$-fat follows exactly the corresponding proof for $\mathbf{G}_n$ in Theorem 5.2.

(4) Proving that weak $\mathbf{G}_{\aleph_0}$-measurability follows pointwise from determinacy also is exactly as in the proof of Theorem 5.2. If we assume that A is determined but not weakly $\mathbf{G}_{\aleph_0}$-measurable, A must be disjoint from infinitely many basic open sets generated by sequences of length one or two (depending on which player wins the game on A). Use these to construct a condition that witnesses weak $\mathbf{G}_{\aleph_0}$-measurability. □

6. Finite and countable fatness: Pointwise non-implications. We shall now modify the canonical examples of Section 5 to get examples for pointwise non-implications. It is interesting to see that our examples do not use the Bernstein construction and thus do not use the Axiom of Choice.

In addition to the isolation properties defined in Section 2.2 we define a notion of being *properly isolated*:

Each tree T has a *leftmost bit*:

$$\ell_T := \min\{t(0)\,;\, t \in T\}.$$

For substrategic I-trees T, the leftmost bit ℓ_T is equal to N_T.

Let T be a substrategic tree and S be a countable or finite set. We write $S = \{x_i\,;\, i \in \xi\}$ where $\xi \leq \omega$. We call S *properly isolated from* T (in symbols: $\mathrm{PIsol}(S,T)$) if S is isolated from T and $x_i(0) = \ell_T + i + 1$, for $i \in \xi$. Note that $\mathrm{PIsol}(S,T)$ implies that $\mathrm{rk}_{\mathrm{CB}}(S) = 0$.

We define

$$\begin{aligned}
\mathbf{G}^*_n &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{PIsol}(S,T), \mathrm{Card}(S) = n\},\\
\mathbf{G}^*_{<\aleph_0} &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{PIsol}(S,T), \mathrm{Card}(S) = \chi(T)\},\\
\mathbf{G}^*_{\aleph_0} &:= \{[T] \cup S\,;\, T \in \mathbf{ssT}, \mathrm{PIsol}(S,T), \mathrm{Card}(S) = \aleph_0\}.
\end{aligned}$$

Proposition 6.1. *$\mathbf{G}^*_n$, $\mathbf{G}^*_{<\aleph_0}$, and $\mathbf{G}^*_{\aleph_0}$ are nonatomic.*

Proof. This is exactly as in (1) of the proofs of Theorems 5.2, 5.3 and 5.4. Note that we do not get strong nonatomicity as in Theorem 5.4 (1), because the requirement of proper isolation prevents us from splitting up the set S. □

Currently, we have no example of a strongly nonatomic forcing $\mathbf{P}$ with countable fatness such that determinacy doesn't imply weak $\mathbf{P}$-measurability pointwise. (For fatness n and $< \aleph_0$, Propositions 5.1 and 4.4 tell us that such a thing can't exist.)

Proposition 6.2. *$\mathbf{G}^*_n$ has fatness n, $\mathbf{G}^*_{<\aleph_0}$ has fatness $< \aleph_0$ and $\mathbf{G}^*_{\aleph_0}$ has countable fatness.*

Proof. The proof of the upper bound is identical to the proof of number (2), in Theorems 5.2, 5.3 and 5.4: We can easily supplement any strategy σ with a set S meeting the requirements of the definition to produce a condition $[\sigma] \cup S$ witnessing that the forcings are not $n+1$-fat, $\aleph_0$-fat, or $\aleph_1$-fat, respectively.

For the lower bound take any strategy σ for player I. We shall show that σ witnesses that $\mathbf{G}^*_n$ is n-fat (the proof for the other two forcings is similar):

If $P = [T] \cup S$ is a $\mathbf{G}^*_n$-condition, $P \backslash [\sigma]$ is uncountable if T is not a subtree of σ as in step (3) of the proof of Theorem 5.2. If $T \subseteq \sigma$, then by Observation 2.1, $N_T = N_\sigma$, but for substrategic I-trees, we have $\ell_T = N_T$. Thus, S is also properly isolated from σ, and in particular, it is disjoint from σ, so $P \backslash [\sigma] = S$. □

As our counterexample, we simply let

$$A := \bigcup \{ [\langle n \rangle] \, ; \, n \text{ is even} \}.$$

Obviously, A is determined since player I controls the game with payoff A with his first move.

Proposition 6.3. *Let $n > 0$. Then A is a determined set which is not weakly $\mathbf{G}^*_n$-, $\mathbf{G}^*_{<\aleph_0}$-, or $\mathbf{G}^*_{\aleph_0}$-measurable.*

Proof. Any condition P (for all of the three forcings) has the property that there are $x, y \in P$ with $y(0) = x(0) + 1$. (This is where we need $n > 0$; note that the definition of χ makes sure that the additional set S is never empty.) But clearly, neither A nor $\omega^\omega \backslash A$ can contain such a set. □

Corollary 6.4. *For $\mathbf{P} \in \{\mathbf{G}^*_n, \mathbf{G}^*_{<\aleph_0}, \mathbf{G}^*_{\aleph_0}\}$, determinacy does not imply weak $\mathbf{P}$-measurability pointwise.*

Proof. Obvious from Proposition 6.3. □

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REFERENCES

1. Tomek Bartoszyński and Haim Judah, *Set theory, On the structure of the real line*, A.K. Peters, Wellesley, MA 1995.

2. Jörg Brendle, *Strolling through paradise*, Fund. Math. **148** (1995), 1–25.

3. Jörg Brendle and Benedikt Löwe, *Solovay-type characterizations for forcing algebras*, J. Symbolic Logic **64** (1999), 1307–1323.

4. Erik Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163–165.

5. Thomas Jech, *Multiple forcing*, Cambridge Tracts in Math., vol. 88, Cambridge, 1986.

6. Akihiro Kanamori, *The higher infinite, Large cardinals in set theory from their beginnings*, in *Perspectives in mathematical logic*, Springer-Verlag, Berlin, 1994.

7. Alexander S. Kechris, *Classical descriptive set theory*, Grad. Texts in Math., vol. 156, Springer, New York, 1994.

8. Kenneth Kunen, *Set theory, An introduction to independence proofs*, Stud. Logic Found. Math., vol. 102, North-Holland, Amsterdam, 1980.

9. Grzegorz Łabędzki, *A topology generated by eventually different functions*, Acta Univ. Carolin.: Math. Phys. **37** (1996), 37–53.

10. Grzegorz Łabędzki and Miroslav Repický, *Hechler reals*, J. Symbolic Logic **60** (1995), 444–458.

11. Benedikt Löwe, *Uniform unfolding and analytic measurability*, Arch. Math. Logic **37** (1998), 505–520.

12. ———, *Turing cones and set theory of the reals*, Arch. Math. Logic **40** (2001), 651–664.

13. Edward Marczewski (Szpilrajn), *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.

14. Yiannis N. Moschovakis, *Descriptive set theory*, Stud. Logic Found. Math., vol. 100, North-Holland, Amsterdam, 1980.

15. Jindřich Zapletal, *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc. **167**, 2004.

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