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FORMULAS FOR POWERS OF THE HYPERBOLIC TANGENT WITH AN APPLICATION TO HIGHER-ORDER TANGENT NUMBERS

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ABSTRACT. It is shown that the function $\tanh^{2n+1}(x)$ is a linear combination of even-order derivatives of $\tanh(x)$, while the function $1-\tanh^{2n+2}(x)$ is a linear combination of odd-order derivatives of $\tanh(x)$. These results are then used to express higher-order tangent numbers (coefficients in the Maclaurin series for $\tanh^n(x)$) as linear combinations of the ordinary tangent numbers (coefficients in the Maclaurin series for $\tanh(x)$).

1. Introduction. In a recently published book [**2**], the three sequences of polynomials $\{\delta_n\}_0^\infty$ [**2**, Chapter 10], $\{A_n\}_0^\infty$, $\{B_n\}_0^\infty$ [**2**, Chapters 13, 14] were introduced and studied. These sequences are defined by the following recurrences:

$$\delta_{n+2}(x) = x\,\delta_{n+1}(x) + n(n+1)\,\delta_n(x), \tag{1.1}$$

where $\delta_0(x)=1$, $\delta_1(x)=x$,

$$A_{n+2}(z) = \big(z+2(2n+3)^2\big)A_{n+1}(z) - 4(n+1)^2(2n+1)(2n+3)A_n(z), \tag{1.2}$$

where $A_0(z)=1$, $A_1(z)=z+2$, and

$$B_{n+2}(z) = \big(z+8(n+2)^2\big)B_{n+1}(z) - 4(n+1)(n+2)(2n+3)^2 B_n(z), \tag{1.3}$$

where $B_0(z)=1$, $B_1(z)=z+8$. The δ_n's are related to the A_n's and B_n's as follows:

$$\delta_{2n+1}(x) = xA_n(x^2), \quad n=0,1,2,\dots, \tag{1.4}$$

and

$$\delta_{2n+2}(x) = x^2B_n(x^2), \quad n=0,1,2,\dots. \tag{1.5}$$

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The δ_n's are called Mittag-Leffler polynomials.

The nonnegative integers $d_j^{(n)}$ [**1**, Chapter 11], where $n = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots n$, are the coefficients of $\delta_n(x)$. If $n = 0, 1, 2, \ldots$, then

$$\delta_n(x) = \sum_{j=0}^{n} d_j^{(n)} x^j. \tag{1.6}$$

Also,

$$A_n(z) = \sum_{j=0}^{n} d_{2j+1}^{(2n+1)} z^j \tag{1.7}$$

and

$$B_n(z) = \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} z^j. \tag{1.8}$$

The A_n's and B_n's are orthogonal, on $(-\infty, 0]$, sequences of polynomials. If

$$w_A(z) = \frac{1}{2} \operatorname{csch}\left(\frac{\pi}{2}\sqrt{|z|}\right), \quad z \in \mathbf{R}, \tag{1.9}$$

$$w_B(z) = \frac{1}{2} |z| \, w_A(z), \quad z \in \mathbf{R}, \tag{1.10}$$

and

$$(P, Q)_A = \int_{-\infty}^{0} P(z)\, Q(z)\, w_A(z)\, dz, \tag{1.11}$$

$$(P, Q)_B = \int_{-\infty}^{0} P(z)\, Q(z)\, w_B(z)\, dz, \tag{1.12}$$

for polynomials P and Q, then for $m, n = 0, 1, 2, \ldots$,

(1.13a)
$$(A_m, A_n)_A = (m+n)!\,(m+n+1)!\,\delta_{m,n}$$

and

(1.13b)
$$(B_m, B_n)_B = \frac{1}{2}(m+n+1)!\,(m+n+2)!\,\delta_{m,n}.$$

In Section 2 of this note, the powers of $\tanh(x)$ are expressed in terms of the derivative operators $A_n\left(D^2\right)$ and $DB_n\left(D^2\right)$ applied to $\tanh(x)$.

The tangent numbers $\{C_{2n+1}\}_0^\infty \subset \mathbf{Z}$ [**2**, Chapter 9] are defined by

$$\tanh(t) = -\sum_{n=0}^{\infty} C_{2n+1}\,\frac{t^{2n+1}}{(2n+1)!}. \tag{1.14}$$

Consequently, $C_1 = -1$, $C_3 = 2$, $C_5 = -16$, $C_7 = 272$ and $C_9 = -7936$. The higher-order tangent numbers $C_{2k+n}^{(n)}$, $k = 0,1,2,\ldots$ and $n = 1,2,3,\cdots$, are defined by

$$\tanh^n(t) = (-1)^n \sum_{k=0}^{\infty} C_{2k+n}^{(n)}\,\frac{t^{2k+n}}{(2k+n)!}, \tag{1.15}$$

so $C_{2k+1}^{(1)} = C_{2k+1}$. By [**1**, (9.7)],

$$\tanh^2(t) = \sum_{k=0}^{\infty} C_{2k+3}\,\frac{t^{2k+2}}{(2k+2)!},$$

so $C_{2k+2}^{(2)} = C_{2k+3}$.

Section 3 of this note expresses the $C_{2k+n}^{(n)}$'s as linear combinations of the ordinary tangent numbers C_{2j+1}, in which the coefficients are $d_j^{(n)}$'s divided by factorials.

2. The main results.

We begin by proving three lemmas.

Lemma 2.1. *Let $n \in \mathbf{N}_0$. Then*

$$\sum_{j=0}^{n} C_{2j+1} d_{2j+2}^{(2n+2)} = -(2n+1)!. \tag{2.1}$$

Proof. From [**1**, (14.3)] and (1.11), we find that

$$C_{2j+1} = -\int_{-\infty}^{0} x^j w_A(z)\,dz = -\big(z^j, 1\big)_A. \tag{2.2}$$

Then, using (1.8) and [**1**, (14.23)], we have

$$\begin{aligned}\sum_{j=0}^{n} C_{2j+1} d_{2j+2}^{(2n+2)} &= -\sum_{j=0}^{n} \big(z^j, 1\big)_A\, d_{2j+2}^{(2n+2)} = -\bigg(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} z^j, 1\bigg)_A \\ &= -\big(B_n, 1\big)_A = -(2n+1)!. \qquad \square\end{aligned}$$

Lemma 2.2. *For $n \in \mathbf{N}_0$,*

$$DB_n(D^2)\tanh(x)\Big|_{x=0} = (2n+1)!. \tag{2.3}$$

Proof. From (1.14),

$$D^{2j+1}\tanh(x)\Big|_{x=0} = -C_{2j+1}, \quad j \geq 0. \tag{2.4}$$

Thus, by (1.8) and (2.1),

$$\begin{aligned}DB_n(D^2)\tanh(x)\Big|_{x=0} &= \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} D^{2j+1} \\ \tanh(x)\Big|_{x=0} &= -\sum_{j=0}^{n} C_{2j+1} d_{2j+2}^{(2n+2)} = (2n+1)!. \qquad \square\end{aligned}$$

In the next lemma we utilize the Mittag-Leffler polynomials $\delta_n(x)$ [**1**, (10.2), Table 10.1].

Lemma 2.3. *For $n \in \mathbf{N}$,*

$$D\tanh^n(x) = \frac{(-1)^{n-1}}{(n-1)!}\,\delta_n(D)\tanh(x). \tag{2.5}$$

Proof (Induction on n). The result is true for $n = 1, 2$. Assume for some $n \geq 3$ that the equation

$$(m-1)!\, D \tanh^m(x) = (-1)^{m-1} \delta_m(D) \tanh(x)$$

is true for $m = 1, 2, \ldots, n-1$. Then, by (1.1),

$$\begin{aligned}
&(-1)^{n-1}\delta_n(D) \tanh(x)\\
&\qquad = (-1)^{n-1}\Big\{D\, \delta_{n-1}(D) + (n-1)(n-2)\, \delta_{n-2}(D)\Big\} \tanh(x)\\
&\qquad = (-1)^{n-1}\Big\{(-1)^n (n-2)!\, D^2 \tanh^{n-1}(x)\\
&\qquad\quad + (-1)^{n-1}(n-1)!\, D \tanh^{n-2}(x)\Big\}\\
&\qquad = -(n-2)!\, D\Big\{D \tanh^{n-1}(x) - (n-1) \tanh^{n-2}(x)\Big\}\\
&\qquad = -(n-2)!\, D\Big\{(n-1) \tanh^{n-2}(x)\, [1 - \tanh^2(x)]\\
&\qquad\quad - (n-1) \tanh^{n-2}(x)\Big\}\\
&\qquad = (n-1)!\, D \tanh^n(x). \qquad \square
\end{aligned}$$

Theorem 2.4. *For* $n \in \mathbf{N}_0$,

$$\tanh^{2n+1}(x) = \frac{1}{(2n)!}\, A_n(D^2) \tanh(x) \tag{2.6}$$

and

$$\tanh^{2n+2}(x) = 1 - \frac{1}{(2n+1)!}\, DB_n(D^2) \tanh(x). \tag{2.7}$$

Proof. (2.6): Replacing n by $2n+1$ in (2.5), we have for $n \in \mathbf{N}_0$ that

$$(2n)!\, D \tanh^{2n+1}(x) = \delta_{2n+1}(D) \tanh(x) = DA_n(D^2) \tanh(x),$$

using (1.4). Integrating this equation, we find that

$$(2n)! \tanh^{2n+1}(x) = A_n(D^2) \tanh(x) + K_n.$$

Setting $x = 0$ gives $K_n = 0$, since $D^{2k}\tanh(x)$ is an odd function for $k \in \mathbf{N}_0$.

(2.7): Replacing n by $2n+2$ in (2.5), we find for $n \in \mathbf{N}_0$ that

$$(2n+1)!D\tanh^{2n+2}(x) = -\delta_{2n+2}(D)\tanh(x) = -D^2 B_n(D^2)\tanh(x),$$

using (1.5). Integrating this equation gives

$$(2n+1)!\tanh^{2n+2}(x) = -DB_n(D^2)\tanh(x) + L_n.$$

Setting $x = 0$ and using (2.3), we find that $L_n = (2n+1)!$. □

TABLE 1. $\tanh^n(x)$, $1 \le n \le 6$.

n	$\tanh^n(x)$
1	$\tanh(x)$
2	$1 - D\tanh(x)$
3	$(1/2)\,(D^2+2)\tanh(x)$
4	$1 - (1/6)\,(D^3 + 8\,D)\tanh(x)$
5	$(1/24)\,(D^4 + 20\,D^2 + 24)\tanh(x)$
6	$1 - (1/120)\,(D^5 + 40\,D^3 + 184\,D)\tanh(x)$

3. Higher-order tangent numbers. The Maclaurin series for $\tanh^n(\mathbf{x})$.

Definition 3.1. The real numbers $\left\{C^{(n)}_{2k+n} : (k,n) \in \mathbf{N}_0 \times \mathbf{N}\right\}$ are defined by

$$\tanh^n(x) = (-1)^n \sum_{k=0}^{\infty} C^{(n)}_{2k+n} \frac{x^{2k+n}}{(2k+n)!}. \tag{3.1}$$

We next give a recursion formula for the $C^{(n)}_{2k+n}$'s and show that these numbers are integers.

Proposition 3.2. (1) *For each* $(k,n) \in \mathbf{N}_0 \times \mathbf{N}$, *we have*

$$C^{(n+1)}_{2k+n+1} = \sum_{j=0}^{k} \binom{2k+n+1}{2j+1} C_{2j+1} C^{(n)}_{2(k-j)+n}. \tag{3.2}$$

(2) *Each* $C^{(n)}_{2k+n}$ *is an integer.*

Proof. (1) From (3.1) and (1.14), we have that

$$\begin{aligned}(-1)^{n+1} \sum_{k=0}^{\infty} C^{(n+1)}_{2k+n+1} \frac{x^{2k+n+1}}{(2k+n+1)!} &= \tanh^{n+1}(x) = \tanh(x) \tanh^n(x) \\ &= (-1)^{n+1} \Big\{ \sum_{i=0}^{\infty} C_{2i+1} \frac{x^{2i+1}}{(2i+1)!} \Big\} \Big\{ \sum_{k=0}^{\infty} C^{(n)}_{2k+n} \frac{x^{2k+n}}{(2k+n)!} \Big\}.\end{aligned}$$

Setting $y = x^2$ and canceling $(-1)^{n+1} x^{n+1}$, we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty} C^{(n+1)}_{2k+n+1} \frac{y^k}{(2k+n+1)!} \\ &\quad = \Big\{ \sum_{i=0}^{\infty} C_{2i+1} \frac{y^i}{(2i+1)!} \Big\} \Big\{ \sum_{k=0}^{\infty} C^{(n)}_{2k+n} \frac{y^k}{(2k+n)!} \Big\} \\ &\quad = \sum_{k=0}^{\infty} \Big\{ \sum_{j=0}^{k} \frac{C_{2j+1}}{(2j+1)!} \frac{C^{(n)}_{2(k-j)+n}}{[2(k-j)+n]!} \Big\} y^k \\ &\quad = \sum_{k=0}^{\infty} \Big\{ \sum_{j=0}^{k} \binom{2k+n+1}{2j+1} C_{2j+1} C^{(n)}_{2(k-j)+n} \Big\} \frac{y^k}{(2k+n+1)!},\end{aligned}$$

which implies the result.

(2) For $k \in \mathbf{N}_0$, the number $C^{(1)}_{2k+1}$ is the integer C_{2k+1}. Using induction on n, we see from (3.2) that each $C^{(n)}_{2k+n}$ is an integer. $\square$

Our next goal is Theorem 3.7, where we give a formula for $C^{(n)}_{2k+n}$ as a linear combination of the ordinary tangent numbers C_{2j+1}. The coefficients in this linear combination are the $d^{(n)}_j$'s divided by factorials.

Lemma 3.3. *Let $(k, n) \in \mathbf{N}_0 \times \mathbf{N}$ be such that $k < n$. Then*

$$\sum_{j=0}^{n} d_{2j+1}^{(2n+1)} C_{2(j+k)+1} = 0. \tag{3.3}$$

Proof. From (2.2), we see that $C_{2(j+k)+1} = -(z^{j+k}, 1)_A$. Then using (1.7) and (1.11), we obtain

$$\begin{aligned}
\sum_{j=0}^{n} C_{2(j+k)+1} d_{2j+1}^{(2n+1)} &= -\sum_{j=0}^{n} \left(z^{j+k}, 1\right)_A d_{2j+1}^{(2n+1)} \\
&= -\left(z^k \sum_{j=0}^{n} d_{2j+1}^{(2n+1)} z^j, 1\right)_A \\
&= -(z^k A_n(z), 1)_A = -(z^k, A_n(z))_A.
\end{aligned}$$

By [**1**, (14.20)], it follows that $(z^k, A_n(z))_A = 0$. $\square$

Proposition 3.4. *Let $k, n \in \mathbf{N}_0$. Then*

$$C_{2(k+n)+1}^{(2n+1)} = \frac{1}{(2n)!} \sum_{j=0}^{n} d_{2j+1}^{(2n+1)} C_{2(j+k+n)+1}. \tag{3.4}$$

Proof. By (3.1), (2.6), (1.7), and (2.4), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} C_{2(k+n)+1}^{(2n+1)} \frac{x^{2(k+n)+1}}{[2(k+n)+1]!} \\
&\qquad = -\tanh^{2n+1}(x) = -\frac{1}{(2n)!} A_n(D^2) \tanh(x) \\
&\qquad = \frac{1}{(2n)!} \left(\sum_{j=0}^{n} d_{2j+1}^{(2n+1)} D^{2j} \right) \left(\sum_{k=0}^{\infty} C_{2k+1} \frac{x^{2k+1}}{(2k+1)!} \right) \\
&\qquad = \frac{1}{(2n)!} \sum_{j=0}^{n} d_{2j+1}^{(2n+1)} \sum_{k=j}^{\infty} C_{2k+1} \frac{x^{2(k-j)+1}}{[2(k-j)+1]!}
\end{aligned}$$

$$= \frac{1}{(2n)!} \sum_{j=0}^{n} d_{2j+1}^{(2n+1)} \sum_{k=0}^{\infty} C_{2(j+k)+1} \frac{x^{2k+1}}{(2k+1)!}$$
$$= \frac{1}{(2n)!} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} d_{2j+1}^{(2n+1)} C_{2(j+k)+1} \right) \frac{x^{2k+1}}{(2k+1)!}.$$

But the first n terms on the right side are zero by (3.3), which gives the equation

$$\sum_{k=0}^{\infty} C_{2(k+n)+1}^{(2n+1)} \frac{x^{2(k+n)+1}}{[2(k+n)+1]!}$$
$$= \frac{1}{(2n)!} \sum_{k=n}^{\infty} \left(\sum_{j=0}^{n} d_{2j+1}^{(2n+1)} C_{2(j+k)+1} \right) \frac{x^{2k+1}}{(2k+1)!}$$
$$= \frac{1}{(2n)!} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} d_{2j+1}^{(2n+1)} C_{2(j+k+n)+1} \right) \frac{x^{2(k+n)+1}}{[2(k+n)+1]!},$$

from which the result follows. □

Lemma 3.5. *Let $k, n \in \mathbf{N}$ be such that $k \leq n$. Then*

$$\sum_{j=0}^{n} C_{2(j+k)+1}\, d_{2j+2}^{(2n+2)} = 0. \tag{3.5}$$

Proof. By (2.2), we have that $C_{2(j+k)+1} = -(z^{j+k}, 1)_A$. Then, using (1.8), we obtain

$$\sum_{j=0}^{n} C_{2(j+k)+1} d_{2j+2}^{(2n+2)} = -\sum_{j=0}^{n} (z^{j+k}, 1)_A\, d_{2j+2}^{(2n+2)}$$
$$= -\left(z^k \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} z^j, 1 \right)_A$$
$$= -(z^k B_n(z), 1)_A = -(z^k, B_n(z))_A.$$

By [**1**, (14.23)], we have $(z^k, B_n(z))_A = 0$. □

Proposition 3.6. *Let $k, n \in \mathbf{N}_0$. Then*

$$C_{2(k+n+1)}^{(2n+2)} = \frac{1}{(2n+1)!} \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} C_{2(j+k+n)+3}. \tag{3.6}$$

Proof. By (3.1), (2.7), (1.8), and (1.14), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} C_{2(k+n+1)}^{(2n+2)} \frac{x^{2(k+n+1)}}{[2(k+n+1)]!} \\
&\quad = \tanh^{2n+2}(x) = 1 - \frac{1}{(2n+1)!} DB_n(D^2)\tanh(x) \\
&\quad = 1 + \frac{1}{(2n+1)!} \left(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} D^{2j+1} \right) \left(\sum_{k=0}^{\infty} C_{2k+1} \frac{x^{2k+1}}{(2k+1)!} \right) \\
&\quad = 1 + \frac{1}{(2n+1)!} \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} \sum_{k=j}^{\infty} C_{2k+1} \frac{x^{2(k-j)}}{[2(k-j)]!} \\
&\quad = 1 + \frac{1}{(2n+1)!} \sum_{j=0}^{n} d_{2j+2}^{(2n+2)} \sum_{k=0}^{\infty} C_{2(j+k)+1} \frac{x^{2k}}{(2k)!} \\
&\quad = 1 + \frac{1}{(2n+1)!} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} C_{2(j+k)+1} \right) \frac{x^{2k}}{(2k)!} \\
&\quad = \frac{1}{(2n+1)!} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} C_{2(j+k)+1} \right) \frac{x^{2k}}{(2k)!},
\end{aligned}$$

where the 1 cancels with the $k = 0$ term in the sum using (2.1). Then by (3.5),

$$\begin{aligned}
\tanh^{2n+2}(x) &= \frac{1}{(2n+1)!} \sum_{k=n+1}^{\infty} \left(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} C_{2(j+k)+1} \right) \frac{x^{2k}}{(2k)!} \\
&= \frac{1}{(2n+1)!} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} d_{2j+2}^{(2n+2)} C_{2(j+k+n)+3} \right) \frac{x^{2(k+n+1)}}{[2(k+n+1)]!}. \qquad \square
\end{aligned}$$

Theorem 3.7. *Let $n \in \mathbf{N}$ and $\varepsilon_n = \big(3+(-1)^n\big)/2$. Then*

$$C^{(n)}_{2k+n} = \frac{1}{(n-1)!} \sum_{j=0}^{[(n-1)/2]} d^{(n)}_{2j+\varepsilon_n} C_{2(j+k)+n+\varepsilon_n-1}. \tag{3.7}$$

Proof. This result combines Propositions 3.4 and 3.6. □

TABLE 2. $C^{(n)}_{2k+n}$, $1 \le n \le 6$.

n	$C^{(n)}_{2k+n}$
1	C_{2k+1}
2	C_{2k+3}
3	$(2\,C_{2k+3} + C_{2k+5})/2$
4	$(8\,C_{2k+5} + C_{2k+7})/6$
5	$(24\,C_{2k+5} + 20\,C_{2k+7} + C_{2k+9})/24$
6	$(184\,C_{2k+7} + 40\,C_{2k+9} + C_{2k+11})/120$

Remark 3.8. For each $n \in \mathbf{N}$,

$$C^{(n)}_{n} = (-1)^n\, n!, \tag{3.8}$$

$$C^{(n)}_{n+2} = (-1)^{n+1}\, \frac{n(n+2)!}{3}, \tag{3.9}$$

$$C^{(n)}_{n+4} = (-1)^n\, \frac{n(5n+17)(n+4)!}{90}. \tag{3.10}$$

Proof. From (1.14) and (3.1), we have

$$\left(x - \frac{1}{3}\,x^3 + \frac{2}{15}\,x^5 - \cdots\right)^n = \tanh^n(x)$$
$$= (-1)^n \left(C^{(n)}_{n}\, \frac{x^n}{n!} + C^{(n)}_{n+2}\, \frac{x^{n+2}}{(n+2)!} + C^{(n)}_{n+4}\, \frac{x^{n+4}}{(n+4)!} + \cdots \right). \qquad \square$$

The next, and last, objective is to show that $n!$ divides the integer $C^{(n)}_{2k+n}$ for $k, n \in \mathbf{N}_0$.

Definition 3.9. $C_{2k}^{(0)} = \delta_{k,0}$, for $k \in \mathbf{N}_0$.

Lemma 3.10. *Let* $(k,n) \in \mathbf{N}_0 \times \mathbf{N}$. *Then*

$$C_{2(k+1)+n}^{(n)} = n\Big(C_{2k+n+1}^{(n+1)} - C_{2(k+1)+n-1}^{(n-1)}\Big). \tag{3.11}$$

Proof. For $n = 1$, the equation is true, since from Table 2 the equation $C_{2k+3}^{(1)} = C_{2k+2}^{(2)}$ is $C_{2k+3} = C_{2k+3}$. For $n \geq 2$, differentiating each side of (3.1) yields

$$\begin{aligned}
&(-1)^n \sum_{k=0}^{\infty} C_{2k+n}^{(n)} \frac{x^{2k+n-1}}{(2k+n-1)!}\\
&\quad = n \tanh^{n-1}(x)\Big(1 - \tanh^2(x)\Big)\\
&\quad = n\Big(\tanh^{n-1}(x) - \tanh^{n+1}(x)\Big)\\
&\quad = (-1)^n n\bigg\{-\sum_{k=0}^{\infty} C_{2k+n-1}^{(n-1)} \frac{x^{2k+n-1}}{(2k+n-1)!} + \sum_{k=0}^{\infty} C_{2k+n+1}^{(n+1)} \frac{x^{2k+n+1}}{(2k+n+1)!}\bigg\}.
\end{aligned}$$

Canceling $(-1)^n x^{n-1}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} C_{2k+n}^{(n)} \frac{x^{2k}}{(2k+n-1)!} = -n \sum_{k=0}^{\infty} C_{2k+n-1}^{(n-1)} \frac{x^{2k}}{(2k+n-1)!}\\
+ n \sum_{k=0}^{\infty} C_{2k+n+1}^{(n+1)} \frac{x^{2k+2}}{(2k+n+1)!}.
\end{aligned}$$

By (3.8), $C_n^{(n)} = -nC_{n-1}^{(n-1)}$, $n \in \mathbf{N} \setminus \{1\}$, so canceling two of the $k = 0$ terms yields

$$\begin{aligned}
\sum_{k=1}^{\infty} C_{2k+n}^{(n)} \frac{x^{2k}}{(2k+n-1)!} = -n \sum_{k=1}^{\infty} C_{2k+n-1}^{(n-1)} \frac{x^{2k}}{(2k+n-1)!}\\
+ n \sum_{k=0}^{\infty} C_{2k+n+1}^{(n+1)} \frac{x^{2k+2}}{(2k+n+1)!},
\end{aligned}$$

or

$$\sum_{k=0}^{\infty} C^{(n)}_{2(k+1)+n} \frac{x^{2k}}{(2k+n+1)!} = -n \sum_{k=0}^{\infty} C^{(n-1)}_{2(k+1)+n-1} \frac{x^{2k}}{(2k+n+1)!} + n \sum_{k=0}^{\infty} C^{(n+1)}_{2k+n+1} \frac{x^{2k}}{(2k+n+1)!}.$$

Equating the coefficients of $x^{2k}/(2k+n+1)!$ yields (3.11). □

Definition 3.11. For $k, n \in \mathbf{N}_0$

$$\gamma^{(n)}_{2k+n} = \frac{1}{n!} C^{(n)}_{2k+n}. \tag{3.12}$$

We now must show that $\gamma^{(n)}_{2k+n}$ is an integer for $k, n \in \mathbf{N}_0$.

Remark 3.12. (1) $\gamma^{(0)}_{2k} = \delta_{k,0}$, $k \in \mathbf{N}_0$,
(2) $\gamma^{(n)}_{n} = (-1)^n$, $n \in \mathbf{N}$,

(3) $\gamma^{(n)}_{2+n} = (-1)^{n+1} n(n+1)(n+2)/3$, $n \in \mathbf{N}$,

(4) $\{\gamma^{(0)}_{2k} : k \in \mathbf{N}_0\} \cup \{\gamma^{(n)}_{n} : n \in \mathbf{N}\} \cup \{\gamma^{(n)}_{2+n} : n \in \mathbf{N}\} \subset \mathbf{Z}$.

Proof. By Definition 3.9, (3.8), and (3.9). □

Lemma 3.13. *For each* $(k, n) \in \mathbf{N}_0 \times \mathbf{N}$,

$$\gamma^{(n)}_{2(k+1)+n} = n(n+1)\, \gamma^{(n+1)}_{2k+n+1} - \gamma^{(n-1)}_{2(k+1)+n-1}. \tag{3.13}$$

Proof. Use Definition 3.11 and (3.11). □

Proposition 3.14. *For* $k, n \in \mathbf{N}_0$,

$$\gamma^{(n)}_{2k+n} \in \mathbf{Z}. \tag{3.14}$$

Proof (Double induction on k then n). We begin with induction on $k \in \mathbf{N}_0$. For $k = 0$ and $n \in \mathbf{N}_0$, the result is true, since $\gamma_n^{(n)} = (-1)^n$ by Remark 3.12 (1) and (2). Assume now for some $k \geq 0$ that $\gamma_{2k+n}^{(n)} \in \mathbf{Z}$ for each $n \in \mathbf{N}_0$. We then show that $\gamma_{2(k+1)+n}^{(n)} \in \mathbf{Z}$ for $n \in \mathbf{N}_0$.

For $n = 0$, $\gamma_{2(k+1)}^{(0)} = 0$ by Remark 3.12 (1). Also, for $n = 1$ by (3.13), we find that

$$\gamma_{2(k+1)+1}^{(1)} = 2\,\gamma_{2k+2}^{(2)} - \gamma_{2(k+1)}^{(0)} \in \mathbf{Z},$$

since the first term on the right is an integer by the induction assumption on k and the second term is zero by Remark 3.12 (1).

Now assume for the given k that $\gamma_{2(k+1)+n-1}^{(n-1)} \in \mathbf{Z}$ for some $n \geq 2$. Then it follows that $\gamma_{2(k+1)+n}^{(n)} \in \mathbf{Z}$, since on the right side of (3.13), the first term is an integer by the induction assumption on k ($\gamma_{2k+n}^{(n)} \in \mathbf{Z}$) and the second term is an integer by the induction assumption on n ($\gamma_{2(k+1)+n-1}^{(n-1)} \in \mathbf{Z}$). $\square$

Remark 3.15. For $(k, n) \in \mathbf{N}_0 \times \mathbf{N}$,

$$\operatorname{sgn}(\gamma_{2k+n}^{(n)}) = (-1)^{k+n}. \tag{3.15}$$

Proof. From the familiar expansion

$$\tan(x) = \sum_{k=0}^{\infty} |C_{2k+1}|\, \frac{x^{2k+1}}{(2k+1)!},$$

where $C_{2k+1} \neq 0$, it follows that

$$\frac{1}{n!}\, \tan^n(x) = \sum_{k=0}^{\infty} T(2k+n, n)\, \frac{x^{2k+n}}{(2k+n)!}.$$

The coefficient $T(2k+n, n)$ is clearly positive for $(k, n) \in \mathbf{N}_0 \times \mathbf{N}$. Also, the equation $\tan(x) = -i\tanh(ix)$ implies that

$$\frac{1}{n!}\, \tanh^n(x) = \sum_{k=0}^{\infty} (-1)^k T(2k+n, n)\, \frac{x^{2k+n}}{(2k+n)!}.$$

By (3.1) and (3.12), we have the equation

$$\frac{1}{n!}\,\tanh^n(x) = (-1)^n \sum_{k=0}^{\infty} \gamma_{2k+n}^{(n)}\,\frac{x^{2k+n}}{(2k+n)!},$$

so $\gamma_{2k+n}^{(n)} = (-1)^{k+n} T(2k+n,n)$. $\square$

Numerical values of $|\gamma_{2k+n}^{(n)}|$ $(= T(2k+n,n))$ are tabulated in [**1**, p. 259].

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