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JACOBI AND MODULAR FORMS ON SYMMETRIC DOMAINS

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ABSTRACT. We prove that there is an isomorphism between the space of Jacobi forms on a symmetric domain associated to an equivariant holomorphic map and the space of a certain vector-valued modular form of half-integral weight on the given symmetric domain. We also construct Eisenstein series for Jacobi forms on symmetric domains and express those in terms of theta functions.

1. Introduction. Jacobi forms on the Poincaré upper half plane or more generally on Siegel upper half spaces generalize classical theta functions, and they arise naturally as coefficients of Siegel modular forms, cf., [**3, 17**]. As is expected, they play an important role in number theory, and various arithmetic aspects of such Jacobi forms have been investigated in numerous papers, see, e.g., [**5, 7, 11, 14**]. On the geometric side, they are closely linked to elliptic surfaces over modular curves or families of abelian varieties parametrized by Siegel modular varieties, see [**8, 15**]. Jacobi forms are also related to elliptic genera of complex manifolds, and they appear as partition functions of super symmetric sigma models whose target spaces are Calabi-Yau manifolds, see [**4**].

The Poincaré upper half plane and Siegel upper half spaces are special cases of (Hermitian) symmetric domains, and Jacobi forms can also be defined on symmetric domains. For example, Borcherds studied Jacobi forms on symmetric domains associated to orthogonal groups, cf., [**1, 2**]. We are interested in symmetric domains which allow equivariant holomorphic maps into Siegel upper half spaces. Let $\mathcal{D}$ be a symmetric domain associated to a semi-simple Lie group G of Hermitian type, and let $\mathcal{H}_n$ be the Siegel upper half space of degree n. We assume that there is a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ that is equivariant with respect to a homomorphism $\rho : G \to \mathrm{Sp}\,(n)$ of Lie groups. Then we can construct

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Jacobi forms on $\mathcal{D}$ by essentially pulling back the usual ones on $\mathcal{H}_n$ via τ, cf., [**10**]. If Γ is a torsion-free discrete subgroup of G, a family of abelian varieties parametrized by the locally symmetric space $\Gamma\backslash\mathcal{D}$ can be constructed. In fact, this family is given by a fiber bundle over $\Gamma\backslash\mathcal{D}$ whose fibers are abelian varieties, and the total space of this bundle is known as a Kuga fiber variety. Jacobi forms on $\mathcal{D}$ of this type can be identified with sections of a line bundle over a Kuga fiber variety, cf., [**9**], and they can be used to prove the algebraicity of Kuga fiber varieties, see [**12**, Section IV.8].

In addition to their occurrence as Fourier coefficients of Siegel modular forms described above, Jacobi forms on Siegel upper half spaces are also closely connected with modular forms in other ways. One such connection can be provided via an isomorphism between the space of Jacobi forms and the space of certain vector-valued modular forms of half-integral weight, see, e.g., [**5, 13, 16**]. In this paper we establish such an isomorphism for Jacobi forms on symmetric domains associated to equivariant holomorphic maps. We also construct Eisenstein series for such Jacobi forms and express those in terms of theta functions.

2. Jacobi forms on Siegel upper half spaces. In this section we review some of the properties of Jacobi forms on Siegel upper half spaces as well as associated Jacobi groups. We also discuss theta functions on Siegel upper half spaces and their relations with such Jacobi forms.

We fix a positive integer n, and denote by H the Heisenberg group associated to $\mathbf{R}^n$. Thus H consists of the triples $[u, v, \lambda]$ with $u, v \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ whose multiplication operation is given by

$$[u, v, \lambda] \cdot [u', v', \lambda'] = [u + u', v + v', \lambda + \lambda' + uv'^t - vu'^t]$$

for $\lambda, \lambda' \in \mathbf{R}$ and $u, v, u', v' \in \mathbf{R}^n$ considered as row vectors. Then the symplectic group $\mathrm{Sp}\,(n, \mathbf{R})$ acts on H on the right by

$$[u, v, \lambda] \cdot M = [(u, v)M, \lambda]$$

for $M \in \mathrm{Sp}\,(n, \mathbf{R})$ and $[u, v, \lambda] \in H$, where $(u, v)M$ is the matrix product of the row vector $(u, v) \in \mathbf{R}^{2n}$ and the matrix M. The associated Jacobi group G_n^J is the semi-direct product $\mathrm{Sp}\,(n, \mathbf{R}) \ltimes H$ with respect to this action, so that its multiplication operation is given by

$$\begin{aligned}(M, [u, v, \lambda]) \cdot (M', [u', v', \lambda'])\\ &= (MM', ([u, v, \lambda] \cdot M') \cdot [u', v', \lambda'])\\ &= (MM', [\hat{u} + u', \hat{v} + v', \lambda + \lambda' + \hat{u}v'^t - \hat{v}u'^t]),\end{aligned}$$

where $\hat{u}, \hat{v} \in \mathbf{R}^n$ with $(\hat{u}, \hat{v}) = (u, v)M'$.

Denoting by $\mathcal{H}_n$ the Siegel upper half space of degree n, the symplectic group $\mathrm{Sp}\,(n, \mathbf{R})$ operates on $\mathcal{H}_n$ as usual by

$$M\langle\zeta\rangle = (A\zeta + B)(C\zeta + D)^{-1}$$

for all $\zeta \in \mathcal{H}_n$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}\,(n, \mathbf{R})$, and the Jacobi group G_n^J acts on $\mathcal{H}_n \times \mathbf{C}^n$ by

$$(M, [u, v, \lambda]) \cdot (\zeta, w) = (M\langle\zeta\rangle, (w + u\zeta + v)(C\zeta + D)^{-1}) \tag{2.1}$$

for $M \in \mathrm{Sp}\,(n, \mathbf{R})$, $[u, v, \lambda] \in H$ and $(\zeta, w) \in \mathcal{H}_n \times \mathbf{C}^n$. Let $\Gamma_n = \mathrm{Sp}\,(n, \mathbf{Z})$, and consider the associated discrete subgroup

$$\Gamma_n^J = \{(M, [u, v, \lambda]) \in G_n^J \mid M \in \Gamma_n;\ u, v \in \mathbf{Z}^n;\ \lambda \in \mathbf{Z}\} \tag{2.2}$$

of G_n^J. Given positive integers ℓ and m, we define the map $J_{\ell,m} : G_n^J \times (\mathcal{H}_n \times \mathbf{C}^n) \to \mathbf{C}$ by

$$\begin{aligned}J_{\ell,m}&((M, [u, v, \lambda]), (\zeta, w))\\ &= \det(C\zeta + D)^\ell \mathbf{e}^m((w + u\zeta + v)(C\zeta + D)^{-1}C(w + u\zeta + v)^t\\ &\quad - u\zeta u^t - 2uw^t - uv^t - \lambda)\end{aligned} \tag{2.3}$$

for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}\,(n, \mathbf{R})$, $[u, v, \lambda] \in H$ and $(\zeta, w) \in \mathcal{H}_n \times \mathbf{C}^n$, where $\mathbf{e}^m(*) = e^{2\pi m i(*)}$. Then $J_{\ell,m}$ is an automorphy factor of G_n^J, which means that it satisfies the relation

$$J_{\ell,m}(\tilde{M}\tilde{M}', (\zeta, w)) = J_{\ell,m}(\tilde{M}, \tilde{M}' \cdot (\zeta, w)) \cdot J_{\ell,m}(\tilde{M}', (\zeta, w)) \tag{2.4}$$

for $\tilde{M}, \tilde{M}' \in G_n^J$ and $(\zeta, w) \in \mathcal{H}_n \times \mathbf{C}^n$. Given a function $F : \mathcal{H}_n \times \mathbf{C}^n \to \mathbf{C}$, we set

$$(F \mid_{\ell,m} \widetilde{M})(\zeta, w) = J_{\ell,m}(\widetilde{M}, (\zeta, w))^{-1} F(\widetilde{M} \cdot (\zeta, w)) \tag{2.5}$$

for all $\widetilde{M} \in G_n^J$ and $(\zeta, w) \in \mathcal{H}_n \times \mathbf{C}^n$.

Definition 2.1. A *Jacobi form of weight ℓ and index m for Γ_n^J* is a holomorphic function $F : \mathcal{H}_n \times \mathbf{C}^n \to \mathbf{C}$ satisfying the following conditions:

(i) $F \mid_{\ell,m} \widetilde{M} = F$ for all $\widetilde{M} \in \Gamma_n^J$.

(ii) For each $M \in \Gamma_n = \mathrm{Sp}\,(n, \mathbf{Z})$ the function $F \mid_{\ell,m} (M, [0,0,0])$ has a Fourier expansion of the form

$$(F \mid_{\ell,m} (M, [0,0,0]))(\zeta, w) = \sum_{T,r} a(T,r) \cdot \mathbf{e}(\mathrm{Tr}\,(T\zeta)/\nu) \cdot \mathbf{e}(rw^t)$$

for some $\nu \in \mathbf{Z}$, where T runs over $n \times n$ symmetric half integral matrices and r over the elements of $\mathbf{Z}^n$, and $a(T,r) \neq 0$ only if the matrix $\left(\begin{smallmatrix} T/\nu & r/2 \\ r^t/2 & m \end{smallmatrix}\right)$ is positive semi-definite.

Remark 2.1. It is well-known that Koecher's principle holds for Jacobi forms, which means that the condition (ii) in Definition 2.1 is automatically satisfied for $n > 1$, assuming that the condition (i) holds.

Given rational vectors $a, b \in \mathbf{Q}^n$, we denote by $\theta_{a,b}$ the associated theta function on the Siegel upper half space $\mathcal{H}_n$ given by

$$\theta_{a,b}(\zeta, w) = \sum_{q \in \mathbf{Z}^n} \mathbf{e}\left(\frac{1}{2}(q+a)\zeta(q+a)^t + (q+a)(w+b)^t\right) \tag{2.6}$$

for all $\zeta \in \mathcal{H}_n$ and $w \in \mathbf{C}^n$. For fixed $\zeta \in \mathcal{H}_n$ and a positive integer m, let $R_m(\zeta)$ be the complex vector space of all holomorphic functions $h : \mathbf{C}^n \to \mathbf{C}$ satisfying

$$h(w + \lambda\zeta + \mu) = \mathbf{e}^m\left(-\frac{1}{2}\lambda\zeta\lambda^t - \lambda w^t\right)h(w)$$

for all $\lambda, \mu \in \mathbf{Z}^n$ and $w \in \mathbf{C}^n$. We denote by Ξ_m the subset of $\mathbf{Q}^n$ given by

$$\Xi_m = \{r = (r_1, \ldots, r_n) \in \mathbf{Q}^n \mid 0 \leq r_i < 1,\ 2mr_i \in \mathbf{Z} \text{ for } 1 \leq i \leq n\}. \tag{2.7}$$

Thus Ξ_m is in fact a set of representatives of the abelian group $(1/2m)\mathbf{Z}^n$ modulo $\mathbf{Z}^n$.

Proposition 2.1. *Given $\zeta \in \mathcal{H}_n$ and a positive integer m, the set*

$$\{\theta_{r,0}(2m\zeta, 2mw) \mid r \in \Xi_m\}$$

of functions $w \mapsto \theta_{r,0}(2m\zeta, 2mw)$ on $\mathbf{C}^n$ with $r \in \Xi_m$ is a basis for $R_m(\zeta)$. In particular, the dimension of $R_m(\zeta)$ is $(2m)^n$.

Proof. This follows from the remarks given on page 187 in [**6**]. ◻

Proposition 2.2. *If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $r \in \Xi_m$, then we have*

$$\begin{aligned} \theta_{r,0}(2mM\zeta, 2mw(C\zeta+D)^{-1}) &= \det(C\zeta+D)^{1/2}\mathbf{e}^m(w(C\zeta+D)^{-1}Cw^t) \\ &\quad \times \sum_{s\in\Xi_m} u_{rs}(M)\theta_{s,0}(2m\zeta, 2mw) \end{aligned} \tag{2.8}$$

for all $(\zeta, w) \in \mathcal{H}_n \times \mathbf{C}^n$, where the matrix $(u_{rs}(M))$ is a constant unitary $(2m)^n \times (2m)^n$ matrix depending on the choice of $\det(C\zeta + D)^{1/2}$.

Proof. This follows from Theorem 6 in [**6**, Section II.5], see also [**16**, Proposition 2.5]. ◻

3. Jacobi forms on symmetric domains. In this section we consider Jacobi forms as well as modular forms on symmetric domains which allow equivariant holomorphic maps into Siegel upper half spaces. We then establish a correspondence between such Jacobi and modular forms.

Let G be a semi-simple Lie group of Hermitian type. Thus, if K is a maximal compact subgroup of G, the associated symmetric space $\mathcal{D} = G/K$ is a symmetric domain. We assume that there are a homomorphism $\rho : G \to \mathrm{Sp}\,(n, \mathbf{R})$ of Lie groups and a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ such that

$$\tau(gz) = \rho(g)\tau(z) \tag{3.1}$$

for all $z \in \mathcal{D}$ and $g \in G$. Then G operates on the Heisenberg group $H = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ described in Section 2 by

$$[u, v, \lambda] \cdot g = [(u, v) \cdot \rho(g), \lambda]$$

for $g \in G$ and $[u, v, \lambda] \in H$. We define the *generalized Jacobi group* G^J to be the semi-direct product $G \ltimes H$ with multiplication given by

$$\begin{aligned}(g, [u, v, \lambda]) \cdot (g', [u', v', \lambda']) &= (gg', ([u, v, \lambda] \cdot g) \cdot [u', v', \lambda']) \\ &= (gg', [\hat{u} + u', \hat{v} + v', \lambda + \lambda' + \hat{u}v'^t - \hat{v}u'^t]),\end{aligned}$$

where $\hat{u}, \hat{v} \in \mathbf{R}^n$ with $(\hat{u}, \hat{v}) = (u, v)\rho(g')$.

Given positive integers ℓ and m, we denote by $J^{\rho,\tau}_{\ell,m} : G^J \times (\mathcal{D} \times \mathbf{C}^n) \to \mathbf{C}$ the map given by

$$J^{\rho,\tau}_{\ell,m}((g, [u, v, \lambda]), (z, w)) = J_{\ell,m}((\rho(g), [u, v, \lambda]), (\tau(z), w)) \tag{3.2}$$

for all $(g, [u, v, \lambda]) \in G^J$ and $(z, w) \in \mathcal{D} \times \mathbf{C}^n$. Then, using (2.4) and (3.1), it can be easily shown that the map $J^{\rho,\tau}_{\ell,m}$ is an automorphy factor of G^J, that is, it satisfies

$$J^{\rho,\tau}_{\ell,m}(\tilde{g}\tilde{g}', (z, w)) = J^{\rho,\tau}_{\ell,m}(\tilde{g}, \tilde{g}' \cdot (z, w)) \cdot J^{\rho,\tau}_{\ell,m}(\tilde{g}', (z, w)) \tag{3.3}$$

for $\tilde{g}, \tilde{g}' \in G^J$ and $(z, w) \in \mathcal{D} \times \mathbf{C}^n$. We also see that the action in (2.1) can be extended to the action of G^J on $\mathcal{D} \times \mathbf{C}^n$ given by

$$(g, [u, v, \lambda]) \cdot (z, w) = (gz, (w + u\tau(z) + v)(C_\rho z + D_\rho)^{-1})$$

for all $(z, w) \in \mathcal{D} \times \mathbf{C}^n$, $[u, v, \lambda] \in H$ and $g \in G$ with $\rho(g) = \left(\begin{smallmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{smallmatrix}\right) \in \mathrm{Sp}\,(n, \mathbf{R})$. Thus, for such $g \in G$ and $(z, w) \in \mathcal{D} \times \mathbf{C}^n$, we obtain an operation of G^J on complex-valued functions F on $\mathcal{D} \times \mathbf{C}^n$ by setting

$$\begin{aligned}&(F \mid^{\rho,\tau}_{\ell,m} (g, [u, v, \lambda]))(z, w) \\ &= J^{\rho,\tau}_{\ell,m}((g, [u, v, \lambda]), (z, w))^{-1} \cdot F((g, [u, v, \lambda]) \cdot (z, w)) \\ &= J^{\rho,\tau}_{\ell,m}((g, [u, v, \lambda]), (z, w))^{-1} \cdot F(gz, (w + u\tau(z) + v)(C_\rho z + D_\rho)^{-1}).\end{aligned} \tag{3.4}$$

We now take a discrete subgroup Γ of G such that $\rho(\Gamma) \subset \Gamma_n = \mathrm{Sp}\,(n, \mathbf{Z})$, and set

$$\Gamma^J = \{(\gamma, [u, v, \lambda]) \in G^J \mid \gamma \in \Gamma;\ u, v \in \mathbf{Z}^n;\ \lambda \in \mathbf{Z}\},$$

so that Γ^J becomes a discrete subgroup of G^J.

Definition 3.1. A holomorphic map $F : \mathcal{D} \times \mathbf{C}^n \to \mathbf{C}$ is a *Jacobi form of weight ℓ and index m for Γ^J associated to ρ and τ* if

$$F \mid^{\rho,\tau}_{\ell,m} \tilde{g} = F$$

for all $\tilde{g} \in \Gamma^J$. We denote by $\mathcal{J}_{\ell,m}(\Gamma^J, \rho, \tau)$ the set of all Jacobi forms of weight ℓ and index m for Γ^J associated to ρ and τ.

Given $a, b \in \mathbf{Q}^n$ and a positive integer m, we consider the associated theta function $\vartheta^m_{a,b}(z, w)$ on $\mathcal{D} \times \mathbf{C}^n$ defined by

$$\vartheta^m_{a,b}(z, w) = \theta_{a,b}(2m\tau(z), 2mw) \tag{3.5}$$

for all $(z, w) \in \mathcal{D} \times \mathbf{C}^n$, where $\theta_{a,b}$ is the theta function in (2.6).

Proposition 3.1. *If $\Phi : \mathcal{D} \times \mathbf{C}^n \to \mathbf{C}$ is an element of $\mathcal{J}_{\ell,m}(\Gamma^J, \rho, \tau)$, then there are holomorphic functions $f_r : \mathcal{D} \to \mathbf{C}$ with $r \in \Xi_m$ such that*

$$\Phi(z, w) = \sum_{r \in \Xi_m} f_r(z) \vartheta^m_{r,0}(z, w) \tag{3.6}$$

for all $z \in \mathcal{D}$ and $w \in \mathbf{C}^n$, where Ξ_m is as in (2.7).

Proof. Given $z \in \mathcal{D}$, using (3.5) and Proposition 2.1, we see that the set $\{\vartheta^m_{a,b}(z, \cdot) \mid r \in \Xi_m\}$ of functions $w \mapsto \vartheta^m_{a,b}(z, w)$ on $\mathbf{C}^n$ is a basis for $R_m(\tau(z))$. If $\Phi(z, w) \in \mathcal{J}_{\ell,m}(\Gamma^J, \rho, \tau)$, the function $w \mapsto \Phi(z, w)$ is an element of $R_m(\tau(z))$; hence we obtain

$$\Phi(z, w) = \sum_{r \in \Xi_m} f_r(z) \vartheta^m_{r,0}(z, w)$$

for some functions $f_r : \mathcal{D} \to \mathbf{C}$ with $r \in \Xi_m$. It can be shown that each f_r is holomorphic by adopting the argument used in [**13**, Lemma 3.4] and the fact that the symmetric domain $\mathcal{D}$ can be realized as a bounded domain in $\mathbf{C}^p$ for some positive integer p; hence the proposition follows. $\square$

Proposition 3.2. *Let r be an element of Ξ_m, and let $\gamma \in \Gamma$ with $\rho(\gamma) = \left(\begin{smallmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{smallmatrix}\right) \in \mathrm{Sp}\,(n, \mathbf{Z})$. Then we have*

$$\begin{aligned}(3.7)\quad \vartheta^m_{r,0}(\gamma z, w(C_\rho \tau(z) + D_\rho)^{-1}) \\ &= \det(C_\rho \tau(z) + D_\rho)^{1/2} \mathbf{e}^m(w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t) \\ &\quad \times \sum_{s \in \Xi_m} u^\rho_{rs}(\gamma) \vartheta^m_{r,0}(z, w)\end{aligned}$$

for all $z \in \mathcal{D}$ and $w \in \mathbf{C}^n$, where the matrix $(u^\rho_{rs}(\gamma))$ is a constant unitary $(2m)^n \times (2m)^n$ matrix depending on the choice of $\det(C_\rho \tau(z) + D_\rho)^{1/2}$. $\square$

Proof. Using (2.8), (3.1) and (3.5), we have

$$\begin{aligned}\vartheta^m_{r,0}(\gamma z, w(C_\rho \tau(z) + D_\rho)^{-1}) \\ &= \theta_{r,0}(2m\rho(\gamma)\tau(z), 2mw(C_\rho \tau(z) + D_\rho)^{-1}) \\ &= \det(C_\rho \tau(z) + D_\rho)^{1/2} \mathbf{e}^m(w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t) \\ &\quad \times \sum_{s \in \Xi_m} u_{rs}(\rho(\gamma)) \vartheta^m_{s,0}(z, w)\end{aligned}$$

for all $z \in \mathcal{D}$. Thus we obtain (3.7) by setting $u^\rho_{rs}(\gamma) = u_{rs}(\rho(\gamma))$. $\square$

Definition 3.2. Let $W : \Gamma \to \mathbf{C}^{k,k}$ be a function on Γ with values in the space $\mathbf{C}^{k,k}$ of complex $k \times k$ matrices, and let $j \in (1/2)\mathbf{Z}$. A $\mathbf{C}^n$-valued holomorphic function $\mathbf{h} = (h_1, \ldots, h_k) : \mathcal{D} \to \mathbf{C}^n$ on $\mathcal{D}$ is a *vector-valued modular form for Γ of weight j with respect to W, ρ and τ* if

$$\mathbf{h}(\gamma z) W(\gamma) = \det(C_\rho \tau(z) + D_\rho)^j \mathbf{h}(\mathbf{z})$$

for all $z \in \mathcal{D}$ and $\gamma \in \Gamma$ with $\rho(\gamma) = \left(\begin{smallmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{smallmatrix}\right) \in \mathrm{Sp}\,(n, \mathbf{Z})$. We denote by $\mathcal{M}^W_j(\Gamma, \rho, \tau)$ the complex vector space consisting of all vector-valued modular forms for Γ of weight j with respect to W, ρ and τ.

Theorem 3.1. *Let $U : \Gamma \to \mathbf{C}^{2m,2m}$ be the map given by $U(\gamma) = (u^\rho_{rs}(\gamma))$ for all $\gamma \in \Gamma$, where the functions $u^\rho_{rs} : \Gamma \to \mathbf{C}$ are as in*

Proposition 3.2. *Then the space $\mathcal{J}_{\ell,m}(\Gamma^J,\rho,\tau)$ of Jacobi forms of weight ℓ and index m for Γ^J associated to ρ and τ is canonically isomorphic to the space $\mathcal{M}^U_{\ell-1/2}(\Gamma,\rho,\tau)$ of vector-valued modular forms for Γ of weight $\ell-1/2$ with respect to U, ρ and τ.*

Proof. Let $\Phi(z,w)\in\mathcal{J}_{\ell,m}(\Gamma^J,\rho,\tau)$, and let $\gamma\in\Gamma$ with $\rho(\gamma)=\left(\begin{smallmatrix}A_\rho B_\rho\\ C_\rho D_\rho\end{smallmatrix}\right)\in\mathrm{Sp}\,(n,\mathbf{Z})$. Using (2.3), (3.2) and (3.4), we have

$$\begin{aligned}\Phi(z,w)&=\Phi\mid^{\rho,\tau}_{\ell,m}(\gamma,[0,0,0])(z,w)\\ &=J^{\rho,\tau}_{\ell,m}((\gamma,[0,0,0]),(z,w))^{-1}\Phi(\gamma z,w(C_\rho\tau(z)+D_\rho)^{-1})\\ &=\det(C_\rho\tau(z)+D_\rho)^{-\ell}\mathbf{e}^m(-w(C_\rho\tau(z)+D_\rho)^{-1}C_\rho w^t)\\ &\quad\times\Phi(\gamma z,w(C_\rho\tau(z)+D_\rho)^{-1})\end{aligned}$$

for all $(z,w)\in\mathcal{D}\times\mathbf{C}^n$. Thus we see that

$$\begin{aligned}&\Phi(\gamma z,w(C_\rho\tau(z)+D_\rho)^{-1})\\ &\qquad=\det(C_\rho\tau(z)+D_\rho)^{\ell}\mathbf{e}^m(w(C_\rho\tau(z)+D_\rho)^{-1}C_\rho w^t)\Phi(z,w).\end{aligned}\tag{3.8}$$

Let $(f_r(z))_{r\in\Xi_m}$ be the system of holomorphic functions on $\mathcal{D}$ associated to $\Phi(z,w)$ satisfying (3.6). Then the relation (3.8) can be written in the form

$$\begin{aligned}&\sum_{r\in\Xi_m}f_r(\gamma z)\vartheta^m_{r,0}(\gamma z,w(C_\rho\tau(z)+D_\rho)^{-1})\\ &\qquad=\det(C_\rho\tau(z)+D_\rho)^{\ell}\mathbf{e}^m(w(C_\rho\tau(z)+D_\rho)^{-1}C_\rho w^t)\\ &\qquad\quad\times\sum_{r\in\Xi_m}f_r(z)\vartheta^m_{r,0}(z,w).\end{aligned}$$

Thus, by using (3.7) we see that

$$\begin{aligned}&\sum_{s\in\Xi_m}\sum_{r\in\Xi_m}f_r(\gamma z)u^\rho_{rs}(\gamma)\vartheta^m_{r,0}(z,w)\\ &\qquad=\det(C_\rho\tau(z)+D_\rho)^{\ell-1/2}\sum_{s\in\Xi_m}f_s(z)\vartheta^m_{r,0}(z,w).\end{aligned}$$

Hence we obtain

$$\sum_{r\in\Xi_m}f_r(\gamma z)u^\rho_{rs}(\gamma)=\det(C_\rho\tau(z)+D_\rho)^{\ell-1/2}f_s(z)$$

for each $s \in \Xi_m$, and therefore it follows that $\mathbf{C}^{2n}$-valued function $\mathbf{f}(z) = (f_r(z))_{r\in\Xi_m}$ is an element of $\mathcal{M}^U_{\ell-1/2}(\Gamma,\rho,\tau)$. On the other hand, if $\mathbf{f}(z) = (f_r(z))_{r\in\Xi_m}$ is an element of $\mathcal{M}^U_{\ell-1/2}(\Gamma,\rho,\tau)$, the relations used above show that the function $\Phi(z,w)$ given by (3.6) is an element of $\mathcal{J}_{\ell,m}(\Gamma^J,\rho,\tau)$; hence the theorem follows. $\square$

4. Eisenstein series. In this section we construct Eisenstein series for a discrete subgroup of the Jacobi group associated to a semi-simple Lie group of Hermitian type, which provide examples of Jacobi forms on symmetric domains. We also express those in terms of theta functions on symmetric domains.

Let G^J_n be the Jacobi group associated to $\mathrm{Sp}\,(n,\mathbf{R})$ described in Section 2, and set

$$\Gamma_{n,0} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}\,(n,\mathbf{Z}) \;\middle|\; C = 0 \right\},$$
$$\Gamma^J_{n,0} = \big\{(\gamma,[u,v,\lambda]) \in \Gamma^J_n \mid \gamma \in \Gamma_{n,0},\ u = 0\big\},$$

where $\Gamma^J_n \subset G^J_n$ is as in (2.2). By (2.5) the constant function 1 on $\mathcal{H}_n \times \mathbf{C}^n$ satisfies $1 \mid_{\ell,m} \tilde{M} = 1$ for all $\tilde{M} \in \Gamma^J_{n,0}$; hence we can consider the Eisenstein series for Jacobi forms for the Siegel upper half space $\mathcal{H}_n$ given by

$$E_{\ell,m}(\zeta,w) = \sum_{\tilde{M}\in\Gamma^J_{n,0}\backslash\Gamma^J_n} (1 \mid_{\ell,m} \tilde{M})(\zeta,w) \tag{4.1}$$

for all $(\zeta,w) \in \mathcal{H}_n \times \mathbf{C}^n$, see [**16**].

Let $\mathcal{D}$, G, ρ, τ, G^J, $\Gamma \subset G$ and $\Gamma^J \subset G^J$ be as in Section 3, and define the subgroup Γ^J_0 of Γ^J by

$$\Gamma^J_0 = \big\{(\gamma,[u,v,\lambda]) \in \Gamma^J \mid \rho(\gamma)g \in \Gamma_{n,0},\ u = 0\big\}. \tag{4.2}$$

Then by using (2.3) and (3.2) we see that the constant function 1 on $\mathcal{D} \times \mathbf{C}^n$ satisfies $1 \mid^{\rho,\tau}_{\ell,m} \tilde{\delta} = 1$ for all $\tilde{\delta} \in \Gamma^J_0$. Thus we can consider the Eisenstein series defined by

$$E^{\rho,\tau}_{\ell,m}(z,w) = \sum_{\tilde{\delta}\in\Gamma^J_0\backslash\Gamma^J} (1 \mid^{\rho,\tau}_{\ell,m} \tilde{\delta})(z,w) \tag{4.3}$$

for $(z,w) \in \mathcal{D} \times \mathbf{C}^n$.

Using (3.3) and (3.4), for each $\tilde{\gamma} \in \Gamma^J$ we have

$$\begin{aligned}(E^{\rho,\tau}_{\ell,m} \mid^{\rho,\tau}_{\ell,m} \tilde{\gamma})(z,w) &= \sum_{\tilde{\delta} \in \Gamma^J_0 \backslash \Gamma^J} (1 \mid^{\rho,\tau}_{\ell,m} \tilde{\delta} \mid^{\rho,\tau}_{\ell,m} \tilde{\gamma})(z,w) \\ &= \sum_{\tilde{\delta} \in \Gamma^J_0 \backslash \Gamma^J} (1 \mid^{\rho,\tau}_{\ell,m} \tilde{\delta}\tilde{\gamma})(z,w) = E^{\rho,\tau}_{\ell,m}(z,w).\end{aligned}$$

Thus we see that $E^{\rho,\tau}_{\ell,m}(z,w)$ satisfies the transformation formula for Jacobi forms of weight ℓ and index m for Γ^J associated to ρ and τ. The fact that $E^{\rho,\tau}_{\ell,m}(z,w)$ is indeed an element of $\mathcal{J}_{\ell,m}(\Gamma^J, \rho, \tau)$ follows from the next theorem.

Theorem 4.1. *Let ℓ be an even integer with $\ell > n+2$. Then the Eisenstein series in* (4.2) *converges absolutely and uniformly on any compact subset of $\mathcal{D} \times \mathbf{C}^n$. Furthermore, it can be written in the form*

$$E^{\rho,\tau}_{\ell,m}(z,w) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} (\vartheta^m_{0,0} \mid^{\rho,\tau}_{\ell,m} (\gamma, [0,0,0]))(z,w) \tag{4.4}$$

for all $(z,w) \in \mathcal{D} \times \mathbf{C}^n$, where $\Gamma_0 = \{\gamma \in \Gamma \mid \rho(\gamma) \in \Gamma_{n,0}\}$.

Proof. If $\tilde{\delta} = (\delta, [u,v,\lambda]) \in \Gamma^J$, then by (2.5), (3.2) and (3.3) we see that

$$\begin{aligned}(1 \mid^{\rho,\tau}_{\ell,m} \tilde{\delta})(z,w) &= J^{\rho,\tau}_{\ell,m}((\delta,[u,v,\lambda]),(z,w))^{-1} \\ &= J_{\ell,m}((\rho(\delta),[u,v,\lambda])),(\tau(z),w))^{-1} \\ &= (1 \mid_{\ell,m} (\rho(\delta),[u,v,\lambda])))(\tau(z),w)\end{aligned}$$

for all $(z,w) \in \mathcal{D} \times \mathbf{C}^n$. Thus by using this and (4.3) we obtain

$$\begin{aligned}E^{\rho,\tau}_{\ell,m}(z,w) &= \sum_{\tilde{\delta} \in \Gamma^J_0 \backslash \Gamma^J} (1 \mid^{\rho,\tau}_{\ell,m} (\delta,[u,v,\lambda]))(z,w) \\ &= \sum_{\tilde{\delta} \in \Gamma^J_0 \backslash \Gamma^J} (1 \mid_{\ell,m} (\rho(\delta),[u,v,\lambda]))(\tau(z),w).\end{aligned}$$

However, $(\rho(\delta),[u,v,\lambda])$ belongs to Γ^J_n, respectively $\Gamma^J_{n,0}$, if $(\delta,[u,v,\lambda])$ belongs to Γ^J, respectively Γ^J_0; hence, we see that $E^{\rho,\tau}_{\ell,m}(z,w)$ is a subseries of $E_{\ell,m}(\tau(z),w)$. Thus the convergence of $E^{\rho;\tau}_{\ell,m}(z,w)$ follows from the corresponding property of the Eisenstein series $E_{\ell,m}(\zeta,w)$ on the Siegel upper half space $\mathcal{H}_n$ in (4.1), see [**16**, Theorem 3.1]. On the other hand, if Δ is a complete set of representatives of $\Gamma_0\backslash\Gamma$, then the set

$$\{(\gamma,[(u,0)\rho(\gamma),0])\in\Gamma^J \mid \gamma\in\Delta,\ u\in\mathbf{Z}^n\}$$

is a complete set of representatives of $\Gamma_0^J\backslash\Gamma^J$. If $\gamma\in\Delta$ with $\rho(\gamma)=\left(\begin{smallmatrix}A_\rho & B_\rho\\ C_\rho & D_\rho\end{smallmatrix}\right)\in\mathrm{Sp}\,(n,\mathbf{R})$, then we have $(u,0)\rho(\gamma)=(uA_\rho,uB_\rho)$ and

$$(1\mid^{\rho;\tau}_{\ell,m}(\gamma,[(u,0)\rho(\gamma),0]))(z,w)=\det(C_\rho\tau(z)+D_\rho)^{-\ell}\mathbf{e}^m(S)$$

for all $u\in\mathbf{Z}^n$, where

$$\begin{aligned}S=&-(w+uA_\rho\tau(z)+uB_\rho)(C_\rho\tau(z)+D_\rho)^{-1}C_\rho(w+uA_\rho\tau(z)+uB_\rho)^t\\&+uA_\rho\tau(z)(uA_\rho)^t+2uA_\rho w^t+uA_\rho(uB_\rho)^t.\end{aligned}$$

Using the fact that the matrix $(C_\rho\tau(z)+D_\rho)^{-1}C_\rho$ is symmetric, we see that

$$\begin{aligned}S=&-w(C_\rho\tau(z)+D_\rho)^{-1}C_\rho w^t+uA_\rho B_\rho^t u^t\\&+u(A_\rho\tau(z)A_\rho^t-(A_\rho\tau(z)+B_\rho)(C_\rho\tau(z)+D_\rho)^{-1}\\&\times C_\rho(A_\rho\tau(z)+B_\rho)^t)u^t\\&+2u(A_\rho-(A_\rho\tau(z)+B_\rho)(C_\rho\tau(z)+D_\rho)^{-1}C_\rho)w^t.\end{aligned}\tag{4.5}$$

Since the matrix $\tau(z)\in\mathcal{H}_n$ is symmetric and $A_\rho D_\rho^t-B_\rho C_\rho^t=I$, we have

$$\begin{aligned}&A_\rho-(A_\rho\tau(z)+B_\rho)(C_\rho\tau(z)+D_\rho)^{-1}C_\rho\\&=A_\rho-(A_\rho\tau(z)+B_\rho)C_\rho^t((C_\rho\tau(z)+D_\rho)^{-1})^t\\&=(A_\rho(C_\rho\tau(z)+D_\rho)^t-(A_\rho\tau(z)+B_\rho)C_\rho^t)((C_\rho\tau(z)+D_\rho)^{-1})^t\\&=((C_\rho\tau(z)+D_\rho)^{-1})^t.\end{aligned}\tag{4.6}$$

Hence, we obtain

$$\begin{aligned}&A_\rho\tau(z)A_\rho^t-(A_\rho\tau(z)+B_\rho)(C_\rho\tau(z)+D_\rho)^{-1}C_\rho(A_\rho\tau(z)+B_\rho)^t\\&=A_\rho\tau(z)A_\rho^t-(A_\rho-((C_\rho\tau(z)+D_\rho)^{-1})^t)(A_\rho\tau(z)+B_\rho)^t\\&=-A_\rho B_\rho^t+(A_\rho\tau(z)+B_\rho)(C_\rho\tau(z)+D_\rho)^{-1}.\end{aligned}\tag{4.7}$$

Thus, using (4.6), (4.7) and the relation

$$(A_\rho \tau(z) + B_\rho)(C_\rho \tau(z) + D_\rho)^{-1} = \rho(\gamma)\tau(z) = \tau(\gamma z),$$

we see that (4.5) reduces to

$$S = u\tau(\gamma z)u^t + 2u(C_\rho \tau(z) + D_\rho)^{-1} w^t - w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t.$$

Hence we have

$$\begin{aligned} E_{\ell,m}^{\rho,\tau}(z,w) &= \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \sum_{u \in \mathbf{Z}^n} \det(C_\rho \tau(z) + D_\rho)^{-1} \mathbf{e}^m(S) \\ &= \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \det(C_\rho \tau(z) + D_\rho)^{-1} \\ &\quad \times \mathbf{e}^m(-w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t) \\ &\quad \times \sum_{u \in \mathbf{Z}^n} \mathbf{e}^m(u\tau(\gamma z)u^t + 2u(C_\rho \tau(z) + D_\rho)^{-1} w^t) \\ &= \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \det(C_\rho \tau(z) + D_\rho)^{-1} \\ &\quad \times \mathbf{e}^m(-w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t) \\ &\quad \times \vartheta_{0,0}^m(\gamma z, w(C_\rho \tau(z) + D_\rho)^{-1}), \end{aligned}$$

where we used (2.6) and (3.5). Thus we obtain (4.4) by using this and (3.4). □

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