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QUADRIC, CUBIC AND QUARTIC CONES

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ABSTRACT. There are 2 irreducible quadric cones (real and imaginary) required for obtaining the affine classification of the 4 irreducible conic sections. According to Newton there are 5 irreducible cubic cones required for obtaining his classification of 59 irreducible cubic sections. In this historical survey paper we show how it follows from Gudkov's classification of forms of real projective quartic curves that 1037 quartic cones are required for obtaining a similar classification of irreducible quartic sections. We also present the singular-isotopy classification of the unions of irreducible affine cubic curves with their asymptotes, which consists of 99 classes. This classification sheds a new light on Newton's famous classification consisting of 78 species.

1. Conic sections. Menaechmus (ca. 350 B.C.) was the first to describe a classification of real nonempty irreducible conic sections. His approach was geometrical. He considered three kinds of cones: acute, right and obtuse cones. For each of the cones, he drew a plane perpendicular to a generator of the cone through a point of the generator other than the vertex. For the acute, right and obtuse cones, he obtained an ellipse, parabola and hyperbola, respectively. He considered cones with one nappe, which are generated by a semi-line, and thus his hyperbola only had one branch. If one considers a cone generated by a line, then the cone of two nappes is obtained and the second branch of the hyperbola appears. Usually one obtains ellipse, parabola and hyperbola by intersecting one cone with various planes.

The next advance in the study of conic sections is connected with the algebraization of the subject whereby a conic section is described by an equation. Such an algebraization can be found in *Conics* by Apollonius. For example, in the case of the parabola he proved that for any point M lying on the parabola with vertex P the equation

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$KM^2 = 2 \cdot LP \cdot KP$ holds. If we denote the segments as $KM = y$, $LP = p$, and $KP = x$ then the equation can be written in the form $y^2 = 2px$, and the meaning of the segments becomes clear via this notation.

The introduction of the Cartesian coordinate system and complex numbers allowed a systematic algebraic approach to the classification of cones and the classification of their plane sections based on an exhaustive study of their equations. Now the sections become identified with the solution sets of equations in two variables. These sections are now called conic, cubic, quartic, quintic, $\dots$, curves, or curves of degree two, three, four, five, $\dots$, respectively.

Two cones (two conic curves) are *affine equivalent* if there is an invertible affine transformation of $\mathbf{R}^3$ ($\mathbf{R}^2$) that carries one cone onto another (one conic curve onto another, respectively).

This equivalence relation leads to five standard forms of equations of the quadric cones. If the polynomial in three variables that defines such a cone is irreducible, then together with the real cone of two nappes $x^2 + y^2 - z^2 = 0$ the coordinate method suggests consideration of an imaginary cone $x^2 + y^2 + z^2 = 0$, whose real part consists only of one point. If such a polynomial is reducible, then there are three reducible cones: $x^2 + y^2 = 0$, which is the union of two imaginary planes that intersect in a real line; $x^2 - y^2 = 0$, which is the union of two intersecting real planes; and $x^2 = 0$, which is a double real plane. These five cones also represent the classification of the projective conic curves: two irreducible and three reducible curves.

In the case of conic curves, the equivalence relation leads to nine standard forms of equations. Note that we adhere to the traditional projective point of view and consider only intersections of cones with planes that do not pass through the vertex of the cones. Intersections with irreducible cones give: ellipse $x^2+y^2-1 = 0$, parabola $y-x^2 = 0$, hyperbola $x^2 - y^2 - 1 = 0$ and imaginary ellipse $x^2 + y^2 + 1 = 0$. Intersections with reducible cones give: two imaginary lines that intersect in a real point $x^2 + y^2 = 0$, two imaginary parallel lines $x^2 + 1 = 0$, two real intersecting lines $x^2 - y^2 = 0$, two real parallel lines $x^2 - 1 = 0$ and a double line $x^2 = 0$.

2. Cubic sections. The introduction of the Cartesian coordinate system in the early 17th century was one of the most dramatic events in the history of mathematics. Even the classification of conic sections was subject to a revision, as it found a new beauty in the form of equations. Isaac Newton eagerly embraced the new coordinate method and wrote his first manuscript on cubic curves in late 1667 (or early 1668) [**33**]. He returned to this subject again and again throughout his life, obtaining at least three important classifications, consisting of 5, 59 and 78 equivalence classes.

His well-known classification containing 78 species (we call it the 78-classification) was completed in 1695 and published in 1704 in his famous *Enumeratio Linearum Tertii Ordinis* [**34**] as an appendix to his *Treatise on Optics.* In the 78-classification Newton takes into account cubic curves together with their asymptotes and diameters. He shows that the general equation of an irreducible cubic curve can be written in one of the following canonical forms:

$$\begin{aligned} xy^2 + ey &= ax^3 + bx^2 + cx + d, \qquad & xy &= ax^3 + bx^2 + cx + d,\\ y^2 &= ax^3 + bx^2 + cx + d, \qquad & y &= ax^3 + bx^2 + cx + d \end{aligned}$$

and divides these four classes into 14 genera and 78 species. In [**34**] Newton described 72 of 78 species. Later six more species were added to the 78-classification: four species by James Stirling [**39**] depicted in Figure 1 and two by François Nicole [**36**] depicted in Figure 2 (we continue Newton's enumeration of species). Species 73 and 74 are the eleventh and twelve ones in the genus of *monodiametral redundant hyperbolas*[1], species 75 and 76 are the third and fourth ones in the genus of *tridiametral redundant hyperbolas*, species 77 and 78 are the fifth and sixth ones in the genus of *monodiametral parabolic hyperbolas.*

The union of curves in Figure 1 and Figure 2 with Newton's curves in Figures 1–77 from *Enumeratio* provides all pictures for the 78-classification. The 78-classification is the deepest account and contains practically all information about real cubic curves. We refer the reader to the classic survey of Ball [**2**] on this subject and to the remarkable introductions and footnotes of Whiteside, editor of *The Mathematical Papers of Isaac Newton* [**33**–**35**][3].

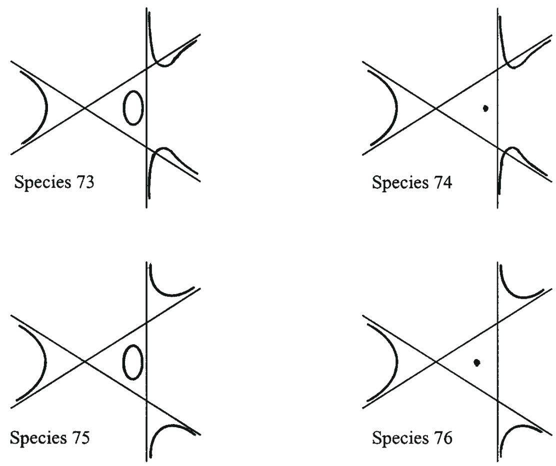

FIGURE 1.

From a modern perspective, the underlying problem with the coordinate method is that while the group of affine transformations of $\mathbf{R}^2$ (projective transformations of $\mathbf{R}P^2$) is big enough to give a finite classification of affine (projective, respectively) conics, these groups are too small to give a finite classification of curves of degree three or higher. Under these groups, the equivalence classes seem to fall into a finite number of families, each of which contains cubic curves having the same "form".

For the equivalence relation determined by the group of invertible affine transformations, it is a very elegant result that there are a finite

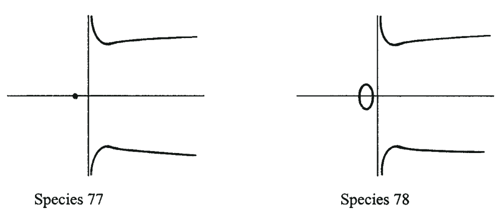

FIGURE 2.

number of affine classes of conic curves. To obtain a finite classification of curves of degree three or higher, it is necessary to consider a new equivalence relation. Such an equivalence relation was intuitively clear to Newton and his contemporaries. We would like to provide a number of precise definitions in modern terms that lead to finite classifications of cubic and quartic curves.

Let X denote the real affine plane $\mathbf{R}^2$, or the real projective plane $\mathbf{R}P^2$. Two curves C_0 and C_1 in X are called *topologically equivalent* if there exists a homeomorphism of X that carries the pair (X, C_0) onto the pair (X, C_1).

Two curves C_0 and C_1 in X are called *isotopy equivalent* if there exists a homeomorphism of X isotopic to the identity that carries the pair (X, C_0) onto the pair (X, C_1)[4].

These two definitions have been the foundation of classifications of nonsingular algebraic curves. But their application to singular curves leads to only a coarse treatment of singular points; for example, in the case of cubic curves, the curves $y = x^3$ and $y^2 = x^3$ are isotopy equivalent in $\mathbf{R}^2$ and moreover are isotopy equivalent to a triple line, say $x^3 = 0$. To obtain a more detailed classification, we shall require that the isotopy preserve the singularities of a curve.

Two curves C_0 and C_1 without multiple components in X are called *singular-isotopy equivalent* if they are 1) isotopy equivalent and 2) the isotopy φ_t connecting C_0 and C_1 preserves the real singularities for each curve C_t, where $t \in [0, 1]$ or, in other words, if, for each singular point, the restriction of $\varphi_t|_U$ to a small neighborhood U of the point is a diffeomorphism.

If one wishes to classify a set of curves having only zero-modal singularities, then this definition leads to a finite number of singular-isotopy classes. Note that cubic curves without multiple components, irreducible quartic curves, and unions of irreducible affine cubic curves with their real asymptotes are examples of such sets of curves.

It is clear that isotopy equivalence follows from singular-isotopy equivalence, and that topological equivalence follows from isotopy equivalence, and that all three classifications of nonsingular curves coincide. In this sense, the singular-isotopy classification is situated between the affine and isotopy classifications. For irreducible affine conic curves, the affine, topological, isotopy and singular-isotopy classifications coincide. For all affine conic curves, the topological and isotopy classifications coincide and contain 6 classes. It is easy to show that 1) the isotopy classification of all affine cubic curves contains 21 classes [**41**] and 2) the isotopy classification of projective cubic curves contains 8 classes [**30**].

The notion of singular-isotopy equivalence lends a precise meaning to Newton's classification of irreducible cubic cones, containing five singular-isotopy classes, and to his particular classification of irreducible cubic curves that contains 59 singular-isotopy classes [**35**].

Although the 78-classification is very impressive, the elegance afforded to the affine classification of conics by the coordinate method is unfortunately lost when applied to the cubic curves. It is cumbersome in the sense of requiring a large number of calculations with regard to the equations. These calculations are not exhibited in the *Enumeratio*, but anyone wishing to understand and check this classification must do all these calculations. We think that at the end of his investigations, Newton realized that elegance could be restored by returning to the geometric idea of ancient times.

The coordinate method gives rise to curves as geometrical interpretations of equations. On the other hand, homogenization of the equations gives rise to cones. The latter phenomenon allowed Newton to classify cones of degree three and to use them as the basis of a geometrical classification of cubic sections.

It was Newton's remarkable observation that all irreducible cubic curves can be obtained from plane sections of five cubic cones. He expressed this on at least two occasions: "And just as the circle by projecting its shadow generates all conics, so the five divergent parabolas by their shadows generate and exhibit all other curves of second kind ... ," ([**34**, p. 635]), and " ... In this way the ancients derived from the circle all figures of the second order and thence named them conic sections .... So, too, all figures of higher orders can be derived from certain simpler figures of the same order by means of successive projections .... On this principle there is but a single class of lines of second order, in that all may be derived from the circle; but of the third order there are five kinds," ([**35**, pp. 411, 413]).

Felix Klein made the following remark about this in 1893: "His [Newton's] *Enumeratio Linearum Tertii Ordinis* shows that he had a very clear conception of projective geometry; for he says that all curves of the third order can be derived by central projection from five fundamental types," [**29**, p. 25].

There are five singular-isotopy classes of irreducible projective cubics, which essentially coincide with Newton's five cubic cones. Starting from this point, we will not distinguish between cones with vertex at the origin in $\mathbf{R}^3$ and projective curves in $\mathbf{R}P^2$. We would like to stress that Newton considered only irreducible cubics. There are perhaps two reasons for this. First, a reducible cubic cone is simply a union of cones of lower degrees. Second, reducible cones can lead to an undesirable confusion of types; for example, the irreducible cone $y^2z = x^3$ is isotopy equivalent to the reducible cone $x^3 = 0$ in $\mathbf{R}P^2$.

The first rigorous proofs that there are only five singular-isotopy classes of irreducible cubic cones were attributed to Nicole and Clairaut, December of 1731. However, there is evidence in the final *Geometriæ Libri Duo* that Newton had such a proof because all of the important ingredients are contained in his general equation $y^2=(1/m)x^3+n(x+a)$, written in modern terms by Whiteside [**35**, p. 413]. The absence of the y^3-, xy^2-, and x^2y-terms in the equation means that Newton chooses an inflection point of the curve at infinity such that the line at infinity is tangent to the curve at this point and the y-axis passes through this inflection point. He eliminates the xy- and y-terms for the purpose of symmetry. From this equation Newton describes the five families of cubic cones by selecting appropriate values of the parameters m and n. We exhibit representatives of the five classes in Figures 3–7.

The Newton five cubic cone classification is the foundation for the singular-isotopy classification of cubic curves. In the final *Geometriæ Libri Duo* [**35**] Newton describes all 59 singular-isotopy classes (59-classification) of cubic curves obtained from the five types of cubic cones. In the 1690's he arrived at the idea of the superiority of the methods of the ancient Greeks. With the renewed inspiration of these methods and possessing a thorough familiarity of the cubic curves, he created this new classification in 1693 in unpublished work [**33**]. This new classification, born of ancient conceptions, had a simple elegance that turned out to be much closer in spirit to ideas that emerged in the 20th century. For each of the five types shown in Figures 3–7 Newton

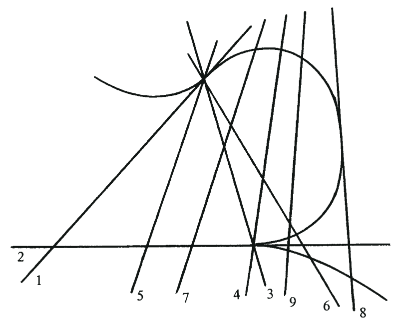

FIGURE 3.

enumerates all positions of lines that can be regarded as the line at infinity and giving rise to different singular-isotopy classes of cubic curves. The positions of these lines are shown in Figures 3–7. Each set of cubic curves generated from one of the five cones Newton calls a *Genere* (we save the Latin term). Thus, there are five *Genera* in the 59-classification which contain 9, 14, 12, 9 and 15 singular-isotopy classes, respectively [**35**]. Following Newton's enumeration, we exhibit representatives of the 59 classes in Figures 8–12, where we show in parentheses the species numbers for the 78-classification. We also show both linear and parabolic asymptotes[5].

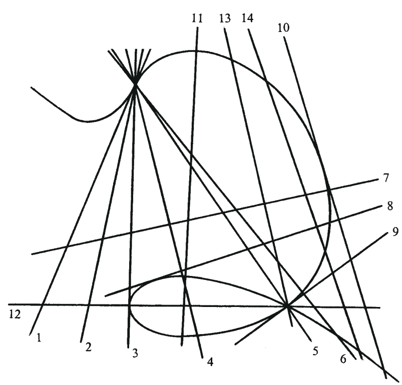

FIGURE 4.

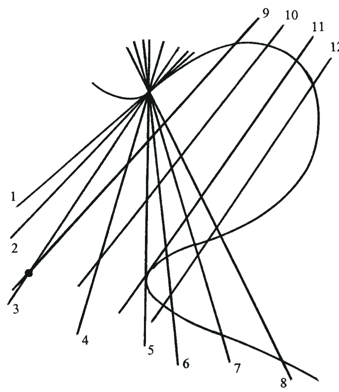

FIGURE 5.

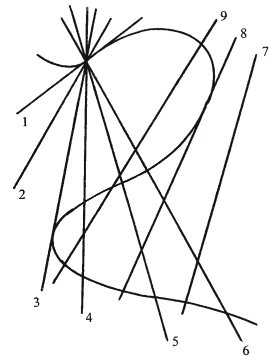

FIGURE 6.

3. Quartic sections. We have seen so far that in the case of conic sections two irreducible cones generate the ellipse, parabola, hyperbola and imaginary ellipse. In the case of degree three Newton created a new scheme of classification, which we have identified as being based on the singular-isotopy equivalence relation. In this scheme five cubic cones generate all 59 singular-isotopy classes of irreducible affine cubic curves. In this section we would like to address the question of how many quartic cones are required for obtaining an analogous classification of irreducible quartic curves.

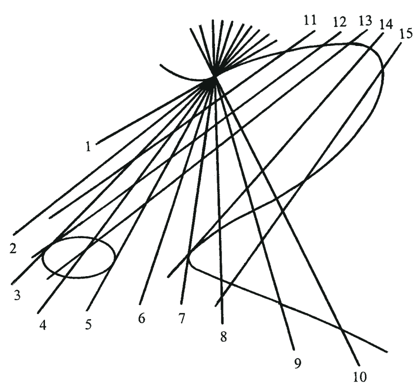

FIGURE 7.

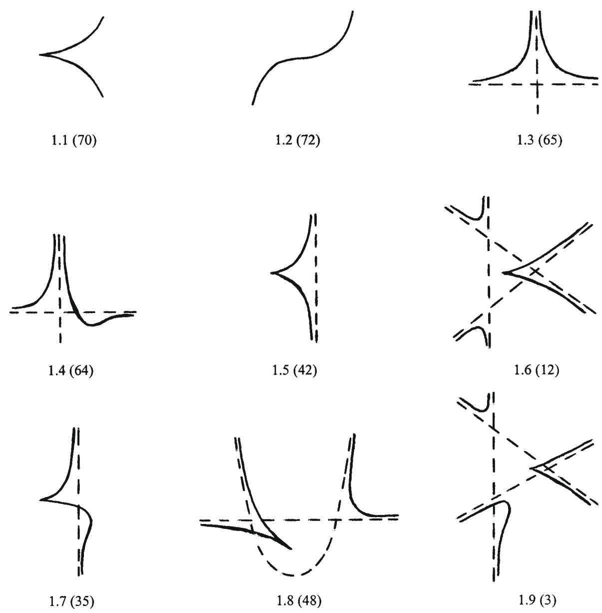

FIGURE 8. The *Primo Genere* of Newton's 59-classification.

In [**31**] we proved that the isotopy classification 1) of irreducible projective quartic curves contains 42 classes, 2) of all projective quartic curves contains 66 classes, 3) of affine quartic curves contains 647 classes, and that 4) the topological classification of pairs ($\mathbf{R}^2$, *quartic curve*) contains 516 classes.

Irreducible quartic curves can have only the following singular points [**11, 12, 25**]:

$$A_1,\ A_1^*,\ A_2,\ A_3,\ A_3^*,\ A_4,\ A_5,\ A_5^*,\ A_6,\ D_4,\ D_4^*,\ D_5,\ E_6,\ 2A_1^{im},\ 2A_2^{im}.$$

The names of these singular points follow Arnold's notation for singularities [**1**] and Gudkov's special convention [**4**]: 1) if there is no asterisk in the notation of a point, then the point is real and all branches centered in it are real, 2) if there is an asterisk, then the point is real and two branches centered in this point are complex conjugate, 3) if there is an upper index *im*, then the point is complex (imaginary), 4) an integer factor before a letter denotes the number of such points.

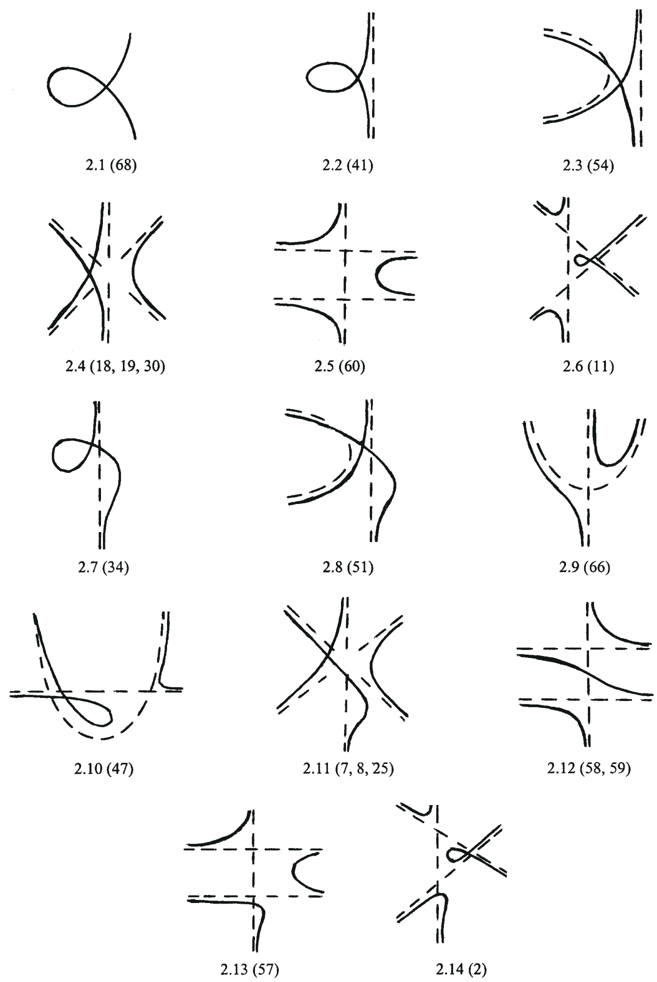

FIGURE 9. The *Secundo Genere* of Newton's 59-classification.

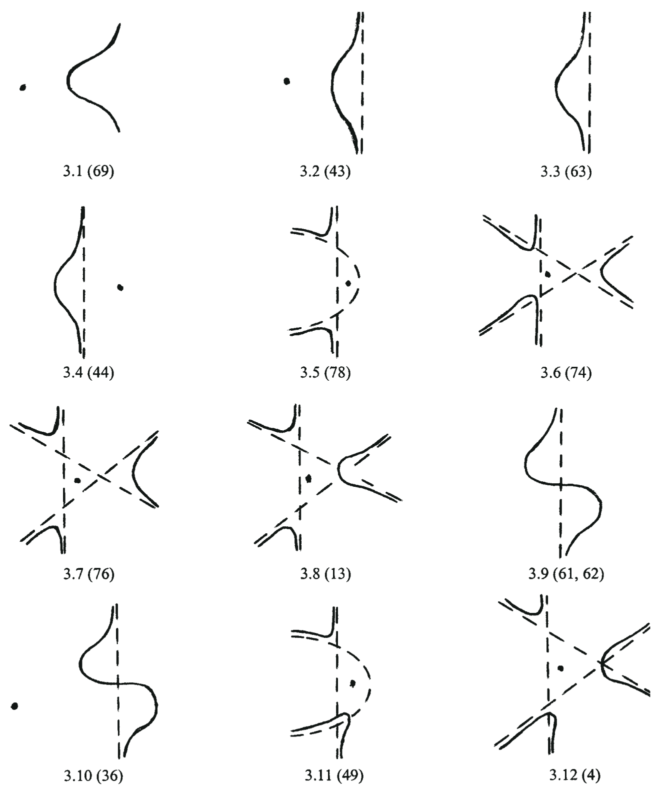

FIGURE 10. The *Tertio Genere* of Newton's 59-classification.

Notice that a new phenomenon has appeared for quartic curves: complex singular points. Now we should distinguish between curves with and without complex singular points. Singular-isotopy equivalence does not take them into account. If one would like to take imaginary singular points into account together with the real part of the curve, then one can use the following definition.

Two curves C_0 and C_1 are called *algebraic-isotopy equivalent* if they are 1) singular-isotopy equivalent and 2) the isotopy φ_t connecting C_0 and C_1 preserves the imaginary singularities for each curve C_t, where $t \in [0, 1]$.

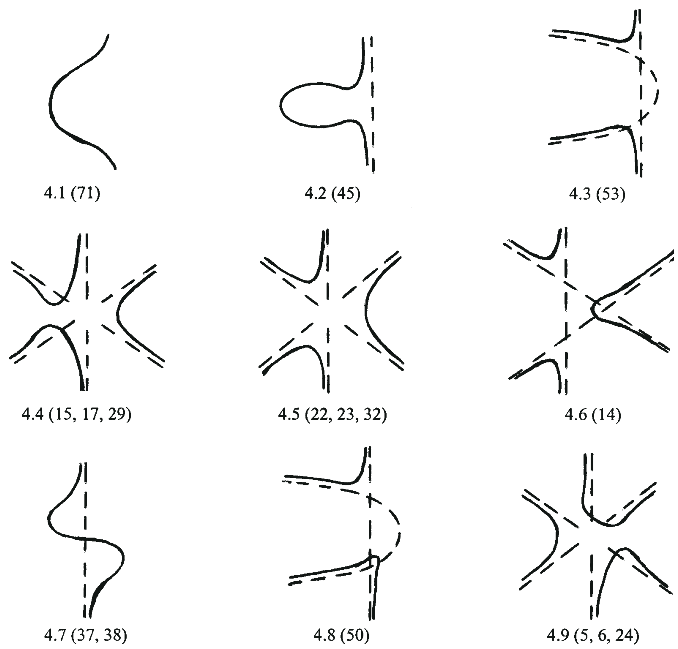

FIGURE 11. The *Quarto Genere* of Newton's 59-classification.

If a curve doesn't have imaginary singular points, then its singular-isotopy and algebraic-isotopy classes are the same. In particular, the singular-isotopy and algebraic-isotopy classifications of cubic curves coincide because such curves don't have imaginary singular points. This definition provides a finite number of algebraic-isotopy classes of cubic curves without multiple components and irreducible quartic curves with both real and imaginary singular points because each singular-isotopy class of these curves contains no more than three algebraic-isotopy classes (there are three possible cases: the set of imaginary singular points is either empty, or consists of $\{2A_1^{im}\}$, or $\{2A_2^{im}\}$). It is clear that singular-isotopy equivalence follows from algebraic-isotopy equivalence. In this sense the algebraic-isotopy classification is situated between the affine and singular-isotopy classifications.

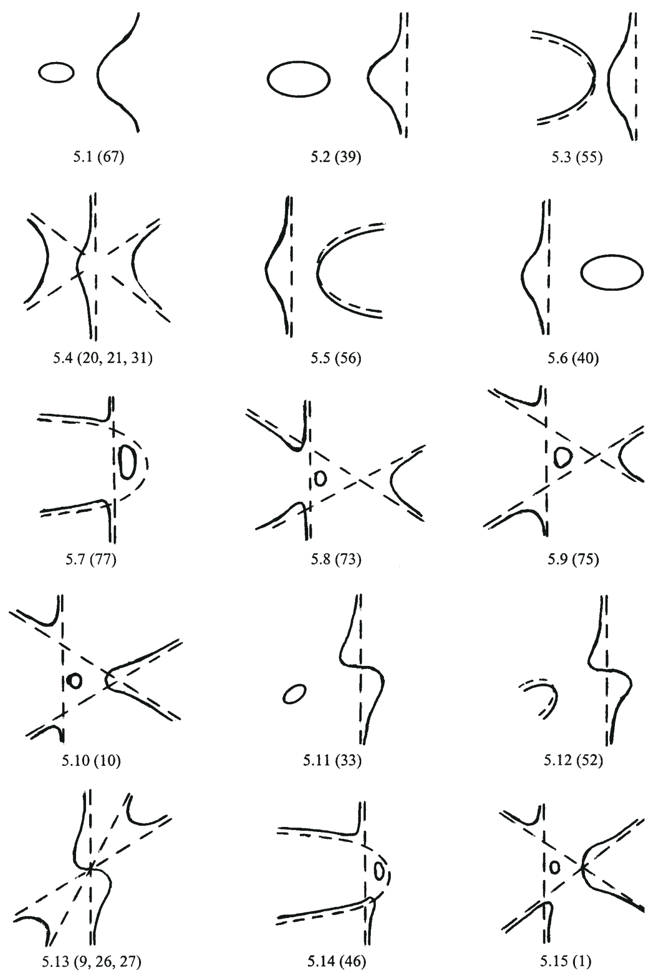

FIGURE 12. The *Quinto Genere* of Newton's 59-classification.

The singular-isotopy classification of irreducible projective quartic curves without complex singular points contains 99 classes and was proved in [**25**]. The additional 18 algebraic-isotopy classes of irreducible quartic curves with complex conjugate singular points were studied in [**19, 20**]. Thus, the algebraic-isotopy classification of irreducible projective quartic curves (117-classification) contains 117 classes. We will continue this discussion in Section 3.1.

The spirit of Newton's approach to the classification of affine cubic curves is to obtain a finite classification that closely follows the affine classification. The famous 78-classification of Newton [**34**] is close to the singular-isotopy classification of unions of cubic curves with their real asymptotes, see the Appendix. His other 59-classification [**35**] follows the ancient idea of obtaining affine cubics as sections of cubic cones. Both classifications take into account singular and real inflection points and the behavior of the curve at infinity.

In 1981–90, Gudkov and his students obtained the so-called *classification of forms* of irreducible projective quartic curves, which takes into account the location of singular, real inflection and *planar* (standard local equation $y = x^4$) points. Gudkov improved the definition of the form of a quartic curve several times [**5, 12, 19, 20**]. Based on his ideas, we will give one more improvement of the definition of the form of a quartic curve.

Let A be the set of real points of an irreducible quartic curve. Let $\{P_1, \dots, P_k\}$ be the set which is the union of the set of real nonisolated singular points and the set of inflection and planar points of the curve A. Let $\{Q_1, \dots, Q_l\}$ be the set of isolated singular points of the curve. Let $A_1, \dots, A_m$ be the connected components of the set $A \setminus [(\cup_{i=1}^k P_i) \cup (\cup_{i=1}^l Q_i)]$. It is clear that each component A_i is homeomorphic to either an open interval or a circle. Let $B_1, \dots, B_n$ be the connected components of the set $P^2 \setminus [(\cup_{i=1}^k P_i) \cup (\cup_{i=1}^m A_i)]$. It is clear that each component B_j is homeomorphic to an open disk, perhaps with holes (the number of holes can be zero). Let $A_i \subseteq \partial B_j$. The component A_i is called *convex with respect to* B_j if for any point $R \in A_i$, there exists a circular neighborhood U of R in $\mathbf{R}P^2$ such that, for any two points $R_1, R_2 \in U \cup A_i$, the open straight line interval $R_1 R_2 \subset U$ belongs to B_j; otherwise, the component A_i is called *concave with respect to* B_j.

Two real projective curves C_0 and C_1 are said to *represent the same form* if they are 1) algebraic-isotopy equivalent, 2) the isotopy φ_t connecting C_0 and C_1 preserves the set of inflection and planar points for each curve C_t where $t \in [0, 1]$ and 3) the isotopy φ_t connecting C_0 and C_1 preserves the convexity or concavity of arcs of the curves with respect to corresponding components. The equivalence class in this case is called a *form* of the curve.

It is clear that the number of forms of irreducible quartic curves is finite because, as it follows from Klein's formula [**27**], the number τ of inflection points and the number π of planar points of a quartic curve satisfy the inequality $\tau + 2\pi \leq 8$. It is clear that singular-isotopy equivalence follows from equivalence of forms, and that the classification of forms is situated between the affine and algebraic-isotopy classifications.

Note that a) according to Klein's formula [**27**], a cubic curve has at most three real inflection points, and b) these inflection points are collinear. These two facts provide requirement 2) in the previous definition of forms. Thus, for affine and projective cubic curves the singular-isotopy, algebraic-isotopy (no imaginary points) and classification of forms coincide and, in particular, the 59-classification becomes the classification of forms of affine cubic curves.

Note that a) according to Harnack's theorem [**26**], a projective quartic curve can have at most four connected components, b) according to the Klein-Viro formula [**27, 40**] the maximum number of real inflection points a quartic curve can have is eight and c) a planar point can dissipate into two inflection points (real or imaginary).

The idea and spirit of the research on real quartic curves is due to Gudkov. This research naturally falls into three big parts which we describe in the following Sections 3.1–3.3.

3.1 Classification of coarse forms. In [**11, 12**] and [**13–19**], they considered the arrangement of the real inflection points of an irreducible quartic curve when the inflection points are in general position, i.e., these points do not coincide with each other (no planar points) and do not coincide with singular points (no inflection points at a center of a real branch at a singular point of the curve). Later in papers [**6–10**], Gudkov called the forms of such curves *coarse forms* (or *rough forms*). They proved that the 117 algebraic-isotopy classes of irreducible quartic curves with inflection points in general position have 384 coarse forms (= 349 forms with only real singular points + 35 forms with complex ones [**21**])[6].

The classification of coarse forms of projective quartic curves was given in [**11, 12, 13–15**].

3.2 Classification of special forms. In papers [**6–10**] and [**21**], Gudkov completed the classification of the so-called *special forms* of irreducible quartic curves. This involves the case when the inflection points of a quartic curve are not in general position, i.e., either 1) two inflection points coincide to form a planar point (in Gudkov's notation Π), or 2) one real branch of an ordinary double point has an inflection point at its center (in Gudkov's notation A_1^1) or 3) each of two real branches at an ordinary double point has an inflection point at its center (in Gudkov's notation A_1^2).

The scheme of the proof of the classification of forms is the following.

1) It follows from the 117-classification that there are 47 sets of singular points that an irreducible projective quartic curve can have, 39 sets of which have only real singular points [**25**] and 8 sets have the imaginary ones [**21**] (see the Table below). Each of the 47 sets has representative curves in the 117-classification.

2) Denote the sum of the Milnor numbers of the singular points of a curve as μ (the *Milnor number of a curve*). One can check that there are seven values $0, 1, \dots, 6$ for μ; and Gudkov divided the curves from the 117-classification into seven divisions with respect to their Milnor numbers.

3) For each curve from the 117-classification, he calculates the class m^* and the number w of inflection points of the curve using Plücker's formulas

$$m^* = m(m-1) - \sum_{j=1}^{p} \kappa(z_j) \quad \text{and} \quad w = 3m(m-2) - \sum_{j=1}^{p} h(z_j),$$

where m is the degree of the curve, m^* is the class of the curve, $z_1, \dots, z_p$ is the collection of all its singular points, $\kappa(z_j)$ is the class of the singular point z_j and $h(z_j)$ is its Hess (the multiplicity of the intersection of the curve with its Hessian at the point z_j [**11**]).

4) Using the Klein-Viro formula [**40**]

$$m - \sum_{P_1} [\mathrm{Ord}\,(P_1) - 1] - \sum_{P_2} \mathrm{Ord}\,(P_2) = m^* - \sum_{P_1} [\mathrm{Ord}^*(P_1) - 1] - \sum_{P_3} \mathrm{Ord}^*(P_3),$$

where the first and third sums run over all branches P_1 with real centers and real tangent lines, the second sum runs over all branches P_2 with real centers and imaginary tangent lines, the fourth sum runs over all branches P_3 with imaginary centers and real tangent lines, $\mathrm{Ord}\,(P)$ is the order of the branch P and $\mathrm{Ord}^*(P)$ is the order of the dual branch, he calculates for each curve the invariant

$$I = \tau + 2t = m^* - 4 + \sum_{P_1} [\mathrm{Ord}\,(P_1) - 1] + \sum_{P_2} \mathrm{Ord}\,(P_2),$$

where τ is the number of real inflection points and t is the number of isolated double tangent lines of the curve. For quartic curves this invariant has a maximum value of 8. Thus, the number of nonnegative integer solutions of the equation

$$\tau + 2t = I,$$

for τ and t is no more than 5.

5) For each curve of the 117-classification and for each solution of the equation $\tau + 2t = I$, he enumerates the admissible forms with all possible arrangements of inflection and planar points.

6) Finally, Gudkov either constructs a projective quartic curve having the same form or proves that such a form does not contain a quartic curve.

7) Gudkov's construction of projective quartic forms is based on Shustin's inequality

$$\sum_{j=1}^{p} b(z_j) \leq 11,$$

where now $z_1, \dots, z_p$ is the collection of planar and singular points of the curve and $b(z_j)$ is Brusotti number of z_j [**38**]. In particular,

$b(\Pi) = 3$, $b(A_k) = b(A_k^*) = k-1$, $b(A_1^1) = 4$, $b(A_1^2) = 6$, $b(D_k) = b(D_k^*) = k-1$, $b(E_6) = 5$.

Gudkov's classification of forms begins with the enumeration of the sets of singular points that a projective quartic curve can have. He enumerates all such sets and then applies the Klein-Viro formula [**27, 40**], Shustin's inequality [**38**] and Plücker's formulas to calculate the possible numbers of inflection points and double tangent lines of a quartic curve with a fixed set of singular points. Shustin's inequality describes the condition that allows one to perturb independently the singular points of an irreducible algebraic curve. Gudkov used Shustin's inequality for the construction of quartic curves. Without this inequality, the construction would be much more complicated.

This classification of special forms of irreducible quartic curves contains 653 forms (= 629 forms with only real singular points + 24 forms with complex ones [**21**]). As an auxiliary result of his method, Gudkov in [**6**–**10**] rederived the classification of coarse forms and corrected mistakes that were made in [**11**–**19**]. Thus the total number of all forms of irreducible projective quartic curves is 1037 forms (= 653 special + 384 coarse forms).

This is the fourth number in the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 1037, \dots$, giving the number of irreducible cones that are required for obtaining all cone sections for degree $d = 1, 2, 3, 4, \dots$, respectively. Now we have the answer to the question posed at the beginning of this section. In retrospect, there is a thread of history that connects Menaechmus and Apollonius of ancient Greece to Newton and to Gudkov. On the most superficial level, there is an intriguing sequence of numbers—the number of cones required for each degree. The deep part is what is behind the numbers; the ultimate view of structure and beauty is contained in the proofs.

As a supplementary result of his method, Gudkov supplied each form of quartic curve with the so-called singular lines, if any. A real projective line is called a *singular line with respect to a quartic curve* or simply a *singular line* if either 1) the line is tangent to the quartic curve at two distinct real points lying on the same complete real branch of the quartic curve (double tangent line), or 2) the line is tangent to the quartic curve at two complex-conjugate points (double isolated tangent line), or 3) the line is tangent to the quartic curve at a planar point, or 4) the line is tangent to a real quadratic branch at a singular point, or 5) the line is tangent to a real cubic branch at a singular point, or 6) the line passes through two singular points of the quartic curve, or 7) the line is a double tangent at a singular point of the quartic curve. These lines in Gudkov's classification of forms play a role which is methodologically analogous to the asymptotes in Newton's 78-classification.

To summarize the exposition on Gudkov's classification, we present a table where μ is the Milnor number of a curve f, $\mathrm{Sing}\,(f)$ is the set of singular points of the curve f, T is the number of types from the 117-classification with the set of singular points $\mathrm{Sing}\,(f)$, S is the number of special forms, C is the number of coarse forms.

Note that in [**6**] and [**7**], Gudkov calculated and included the numbers of types and the numbers of special and coarse forms in statements of his theorems. In [**8**–**10**], there are no numbers of special and coarse forms; in some theorems, he lists pictures of forms, in some of them, he presents pictures of the main special forms and restricts his explanation to the phrase: " ... *and there exist the special and coarse forms that can be obtained by means of formal deformations of the special forms.*" We counted that there are 653 special and 384 coarse forms in [**6**–**10**] and [**21**][7].

3.3 Stratification of the space of quartic curves. In [**22**–**24**] Gudkov and Polotovskii proved that the set of projective quartic curves of the same algebraic-topological type represents one stratum (connected component) in the space $\mathbf{R}P^{14}$ of all quartic curves. There are two definitions of the algebraic-topological type given in [**23**] and [**24**]: one for irreducible and another for reducible quartic curves. We cite both of them. They use the following notation. If $\mathbf{R}P^{14}$ denotes the space of all quartic curves and $A \in \mathbf{R}P^{14}$, then they denote as $\mathbf{R}A \subset \mathbf{R}P^2$ the set of real points of the curve A.

"We say that two irreducible curves $A, B \in \mathbf{R}P^{14}$ have the same (*algebraic-topological*) *type* if 1) the curves A and B have the same collections of singular points, 2) there exists a diffeomorphism $d : \mathbf{R}P^2 \to \mathbf{R}P^2$ such that $d(\mathbf{R}A) = \mathbf{R}B$, and the diffeomorphism d carries the (real) singular points of the curve A onto the singular points of the same type of the curve B" [**23**].

TABLE. Gudkov's classification of forms of projective quartic curves.

No.	μ	Sing (f)	T	S	C	Theorem
1	6	$3A_2$	1	0	1	10 [**6**]
2	6	A_6	1	0	1	11 [**6**]
3	6	$A_2 \sqcup A_4$	1	0	1	12 [**6**]
4	6	E_6	1	1	2	13 [**6**]
5	6	$A_2 \sqcup 2A_2^{\mathrm{im}}$	1	0	1	1 [**21**]
6	5	$A_1 \sqcup 2A_2$	2	1	3	14 [**6**]
7	5	$A_1^* \sqcup 2A_2$	1	0	1	15 [**6**]
8	5	$A_2 \sqcup A_3^*$	1	0	1	16 [**6**]
9	5	$A_2 \sqcup A_3$	2	0	2	17 [**6**]
10	5	$A_1 \sqcup A_4$	1	1	2	18 [**6**]
11	5	$A_1^* \sqcup A_4$	1	1	2	19 [**6**]
12	5	A_5	2	1	3	20 [**6**]
13	5	A_5^*	2	1	3	21 [**6**]
14	5	D_5	2	1	3	22 [**6**]
15	5	$A_1 \sqcup 2A_2^{\mathrm{im}}$	1	0	1	2 [**21**]
16	5	$A_1^* \sqcup 2A_2^{\mathrm{im}}$	2	1	3	3 [**21**]
17	4	D_4	2	1	3	23 [**6**]
18	4	D_4^*	1	3	3	24 [**6**]
19	4	$A_1 \sqcup A_3^*$	1	2	2	25 [**6**]
20	4	$A_1^* \sqcup A_3^*$	2	3	4	26 [**6**]
21	4	$A_1^* \sqcup A_3$	2	1	3	27 [**6**]
22	4	$A_1 \sqcup A_3$	4	4	7	28 [**6**]
23	4	$2A_1^* \sqcup A_2$	1	1	2	29 [**6**]
24	4	$A_1 \sqcup A_1^* \sqcup A_2$	1	5	4	30 [**6**]
25	4	$2A_1 \sqcup A_2$	3	5	7	31 [**6**]
26	4	$2A_2$	3	2	5	32 [**6**]
27	4	A_4	2	3	5	33 [**6**]
28	4	$2A_2^{\mathrm{im}}$	3	2	5	4 [**21**]
29	4	$2A_1^{\mathrm{im}} \sqcup A_2$	2	1	3	5 [**21**]
30	3	$3A_1^*$	2	6	5	1 [**7**]
31	3	$A_1 \sqcup 2A_1^*$	1	12	6	2 [**7**]

TABLE. (Continued).

32	3	$2A_1 \sqcup A_1^*$	2	17	9	3 [**7**]
33	3	$3A_1$	5	10	12	4 [**7**]
34	3	$A_1^* \sqcup A_2$	2	10	8	5 [**7**]
35	3	$A_1 \sqcup A_2$	4	23	15	6 [**7**]
36	3	A_3^*	3	8	8	7 [**7**]
37	3	A_3	6	11	14	8 [**7**]
38	3	$A_1 \sqcup 2A_1^{\mathrm{im}}$	2	3	4	6 [**21**]
39	3	$A_1^* \sqcup 2A_1^{\mathrm{im}}$	3	6	7	7 [**21**]
40	2	$2A_1^*$	3	31	14	8, 9 [**8**]
41	2	$A_1 \sqcup A_1^*$	3	62	22	11,13,14 [**8**]
42	2	$2A_1$	7	65	32	16–22 [**8**]
43	2	A_2	4	29	19	24,26–28 [**8**]
No.	μ	Sing (f)	T	S	C	Theorem
44	2	$2A_1^{\mathrm{im}}$	4	11	11	8 [**21**]
45	1	A_1^*	5	74	30	2–6 [**9**]
46	1	A_1	6	124	43	8–13 [**9**]
47	0	$\varnothing$	6	110	42	1–6 [**10**]
		Total	117	653	384	

"We say that two reducible (real projective plane) quartic curves A and B have the same (*algebraic-topological*) *type* if 1) the curves A and B have irreducible (over $\mathbf{C}$) components of the same degree and multiplicity, 2) the number of pairs of imaginary complex-conjugate irreducible components of the curves A and B is the same, 3) the curves A and B have the same collections of isolated singular points

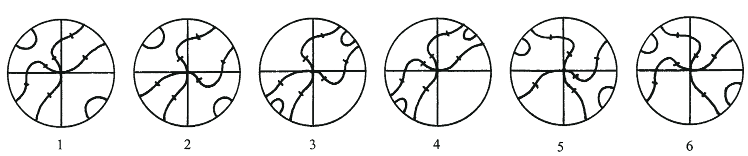

FIGURE 13.

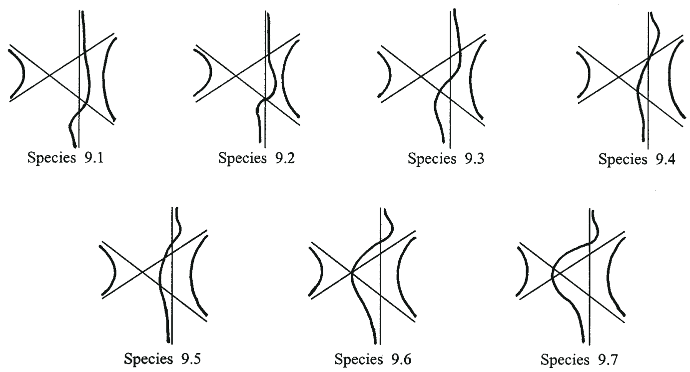

FIGURE 14.

(including imaginary singular points), 4) there exists a homeomorphism $\gamma : \mathbf{R}P^2 \to \mathbf{R}P^2$ such that $\gamma(\mathbf{R}A) = \mathbf{R}B$, and the irreducible components of the curves A and B, which correspond to each other under the homeomorphism γ, have the same multiplicity" [**24**].

One can check that the definition of the algebraic-topological type and our definition of algebraic-isotopy equivalence, both applied to irreducible quartic curves, produce the same equvalence classes. We already counted that there are 117 such algebraic-topological classes. In [**24**] Gudkov and Polotovskii proved that there are 95 algebraic-topological types[8] of reducible quartic curves. Thus, there are 212 strata in the space $\mathbf{R}P^{14}$ of quartic curves. By the way, they proved in [**24**] that the set of projective conic (cubic) curves of the same algebraic-topological type represents one stratum in the space $\mathbf{R}P^5$ ($\mathbf{R}P^9$) of all conic (cubic, respectively) curves. There are 5 and 15 strata of them, respectively.

Note that Bruce and Giblin [**3**] proved that the stratification of the space $\mathbf{C}P^{14}$ of all complex projective quartic curves consists of 54 strata, of which 20 correspond to irreducible curves, 26 to reducible curves with no multiple components and 8 to curves with multiple components. Also note that if we rewrite *mutatis mutandis* the definition of singular-isotopy equivalence for quartic curves, then it follows from their work, there are 20 singular-isotopy classes of irreducible complex projective quartic curves.

According to V.A. Rokhlin's remark (item 4.1 in [**37**]), the fact that each isotopy class of projective nonsingular quartic curves represents one stratum was known to Klein [**28**, p. 112]. Rokhlin found an example of two projective nonsingular quintic curves that represent the same isotopy class but belong to distinct strata in the space $\mathbf{R}P^{20}$ of quintic curves, see item 3 in [**37**].

We hope the reader shares our sense of wonder over how a point of light shining in Ancient Greece has illuminated the development of ideas about equations, curves and cones in the 17th and 20th centuries.

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Appendix

In this section we present the singular-isotopy classification of unions of irreducible affine cubic curves with their asymptotes. This classification is the natural extension of Newton's 78-classification. With respect to this singular-isotopy classification, there are three sorts of species to consider.

1. Species 37 and 38 represent the same singular-isotopy class.

2. Each of the following 67 species represents a single singular isotopy class, all of which are distinct: 2, 3, 6, 8, 11, 12, 15–19, 21–36, 39–78.

3. Each of species 1, 4, 5, 7, 9, 10, 13, 14, and 20 represents more than one singular-isotopy class.

We now consider the third case in detail. Let us recall Newton's definitions of various components of cubic curves. He writes: "I call a branch *hyperbolic* which infinitely approaches some asymptote, … " ([**35**, p. 593]). And " … we shall call a hyperbola … *circumscribed* when it cuts the asymptotes and embraces the parts cut off in its fold, *double-kinded* when it is inscribed in one of its infinite branches and circumscribed in the other, … , *conchoidal* when it is applied along the asymptote with a concave vertex and divergent branches, *snaky* when it cuts its asymptote with an inflection and is extended either way in opposing branches, ..." ([**35**, pp. 597 and 599]).

It remains to answer the following questions. First, does the circumscribed hyperbola in species 1, 4, 5, 10, 13, and 14 (see Figures 12.5.15, 10.3.12, 11.4.9, 12.5.10, 10.3.8, 11.4.6 in this paper or Figures 1 (or 2), 7, 8, 17, 20, and 20^9 in [**34**], respectively), or the double-kinded hyperbola in species 7 (see Figure 9.2.11 in this paper or Figure 9 (or 10) in [**34**]), or the conchoidal hyperbola in species 20 (see Figure 12.5.4 in this paper or Figure 27 in [**34**]) intersect two sides, pass through the vertex or fail to intersect the sides of the triangle whose sides are segments of the asymptotes? We enumerate the singular-isotopy classes in a manner that corresponds to Newton's enumeration of species in the 78-classification as follows: $1.i$, $4.i$, $5.i$, $7.i$, $10.i$, $13.i$, $14.i$, and $20.i$ where $i = 1, 2, 3$ refers to 'intersects two sides', 'passes through the vertex', and 'does not intersect the sides', respectively. And second, can the snaky hyperbola in species 9 be placed as shown in Figure 14?

Note that Newton considers a similar question: he distinguishes species 26, when the snaky hyperbola does not pass through the triple point of intersection of asymptotes, from species 27, when it does.

Theorem. *The singular-isotopy classes of species* $1.i$, $4.i$, $5.i$, $7.i$, $10.i$, $13.i$, $14.i$, *and* $20.i$ *where* $i = 1, 2, 3$ *and species* $9.i$ *where* $i = 1, \dots, 7$ *are realizable by cubic curves.*

Proof. 1) Choose a point A on the nonoval component of the projective cubic X depicted in Figure 7 such that the point A is different from an inflection point of X. Draw two tangent lines through the point A that are tangent to the nonoval component, say at points B and C. Draw the line BC and denote the third point of intersection of the line BC and curve X as D. If D happens to be an inflection point of X, then choose, in a small enough neighborhood (in X) of point A, another point and repeat the construction one more time to obtain D, which is not an inflection point of X. Draw the tangent line to the curve X passing through the point D. Let the line BCD be the line at infinity for the affine plane $\mathbf{R}^2 = \mathbf{R}P^2 \setminus BCD$. The tangent lines to the curve X at points B, C, and D become the asymptotes of the cubic curve

X in $\mathbf{R}^2$. It is easy to see that the curve together with its asymptotes realize the species 1.2.

If U is a small enough circular neighborhood (in $\mathbf{R}^2$) of A, then the set $U \setminus X$ consists of two connected components one of which is convex and the other nonconvex. If we choose a new point A in the convex (nonconvex) component and repeat the previous construction, then we obtain species 1.1 (and 1.3, respectively).

2) The species 4.1-3, 5.1-3, 7.1-3, 9.1-7, and 20.1-3 are constructed in a similar way from the projective cubic curves depicted in Figures 5, 6, 4, 7, and 7, respectively.

3) The projective curve X with the equation $xy^2 = -(x+z)(x+2z)(x+3z)$ looks in the affine chart $\{z=1\}$ like the curve 5.2(39) depicted in Figure 12 and has an inflection point $D = (0:1:0)$. The curve X is invariant with respect to the involution $\alpha : \mathbf{R}P^2 \to \mathbf{R}P^2$, $\alpha(x:y:z) = (x:(-y):z)$. Draw two tangent lines from the point $A = (-1:0:1) \in X$ to the nonoval component of X and denote the points of tangency as B and C. One can see that $\alpha(A) = A$ and $\alpha(B) = C$, and then the line BC (parallel to the axis $\{x=0\}$ in the affine chart $\{z=1\}$) passes through the point D (in $\mathbf{R}P^2$). Let the line BCD be the line at infinity for the affine plane $\mathbf{R}^2 = \mathbf{R}P^2 \setminus BCD$. The tangent lines at points B, C, and D become the asymptotes of the cubic curve X in $\mathbf{R}^2$. It is easy to see that the curve together with its asymptotes realize the species 10.2.

If we choose a new point A in the interval $(-1, -1+\varepsilon)$ (or in $(-1-\varepsilon, -1)$) of the axis $\{y=0\}$ in the affine chart $\{z=1\}$, where ε is a small enough positive number, and repeat the previous construction, then we obtain species 10.1 (and 10.3, respectively).

4) The species 13.1-3 and 14.1-3 are constructed in a similar way from the projective cubic curves $xy^2 = -(x+z)(x+2z)^2$ and $xy^2 = -(x+z)(x^2+z^2)$ depicted in Figures 10.3.2(43) and 11.4.2(45), respectively. $\square$

Corollary. *The singular-isotopy classification of the unions of affine irreducible cubic curves with their asymptotes consists of* 99 *classes.*

ENDNOTES

1. The species 74 is described in manuscript [**33**] on page 55 in the section entitled *Its Seven Forms*. In this section Newton enumerates seven items which he calls *Forms*. In the *Form 5* Newton describes three curves and refers to three figures 5, 9 and 13 which are not present in the manuscript. With respect to Newton's description these three curves represent species 74, 15 and 29 of the 78-classification [**34**]. This is a remarkable place in the manuscript [**33**]; Newton obviously describes (*sic!*) species 74, omitted in [**34**].

2. Note that Newton in [**33**] in the section entitled *De Formis Septimæ Speciei* on page 64 writes: "Septima Species habet sex$^{(70)}$ formas: ... " (the translation on page 65: "The seventh species has six$^{(70)}$ forms: ... "), but there are only four forms in the draft manuscript. The $^{(70)}$ refers to the editor's footnote remark (in [**33**]): "Read 'quatuor' (four). Newton is perhaps thinking of the six forms of species 6." We think that the number six is correct because there should be here exactly two more curves in omitted *Form 5* and *Form 6*, added by Nicole later. Namely, in this section Newton considers the equation $bxy^2 = gx^2 + kx + l$. His description of *Form 1* is correct. To obtain the right description of *Form 2* and *Form 3*, it is necessary to add the omitted condition $bg > 0$. The description of *Form 4* is also correct. The omitted *Form 5* and *Form 6* should be (in our edition) as follows.

Form 5. If the roots of the quadratic polynomial $gx^2 + kx + l$ are of the same sign and $bg < 0$, then we have the oval and two hyperbola-parabolic branches lying on opposite sides of the asymptote, see our Figure 2, Species 78.

Form 6. If the roots are equal and $bg < 0$, then we have the isolated double point and two hyperbola-parabolic branches lying on opposite sides of the asymptote, see our Figure 2, Species 77.

3. There is the same misdrawing in [Fig] 15 on page 74 of [**33**] and in Figure 7 on page 604 of [**34**]. The left hyperbola must be double-kinded: the top branch is inscribed and the bottom branch is circumscribed (see also Appendix).

4. In the literature on real algebraic geometry, another definition of isotopy equivalence in $\mathbf{R}P^2$ is used. Two curves C_0 and C_1 in $\mathbf{R}P^2$ are called *isotopy equivalent* if the pairs $(\mathbf{R}P^2, C_0)$ and $(\mathbf{R}P^2, C_1)$ are topologically equivalent. This definition is equivalent to our definition of topological equivalence in $\mathbf{R}P^2$ above because every homeomorphism from $\mathbf{R}P^2$ to $\mathbf{R}P^2$ is isotopic to the identity map.

5. A *parabolic* asymptote of a curve C is the osculating conic to the curve C at its point of tangency with the line at infinity. The multiplicity of intersection of a curve with its parabolic asymptote at the infinite point is greater than or equal to 5. Parabolic asymptotes were presented by Newton perhaps for the first time in his figures, see [**33**, pp. 80–81].

6. According to Gudkov's remark [**4**], Zeuthen [**42**] constructed all 42 existing coarse forms of nonsingular quartic curves, but didn't prove that one of the admissible curves cannot be realized by a quartic curve.

7. The number of coarse forms we counted differs from the number 396 of coarse forms shown in [**5**] and from the number 382 of coarse forms shown in [**32**]. According to Polotovskii and Nebukina (private communication), Gudkov made some special distinctions in certain specific cases. He considered the unions of curves with some of their special lines and counted such a union as a form of the curve. For example, Gudkov distinguished between the six unions shown in

Figure 13, where each projective quartic curve consists of an oval and two nonoval branches intersecting at a point of type A_1, inflection points are marked and two singular lines—tangent lines to the branches at the singular point—are present. We obtain these curves by means of deformations of three special forms of Figure 25 from [**9**].

In her Ph.D. thesis [**32**], Nebukina (Gudkov was her advisor) applied the definition of the form given in [**19**] and proved, in particular, that these six quartic curves (without singular lines) represent the same coarse form. According to our definition of form of a quartic curve, the curves in Figures 13.1, 3 and 5 represent the same coarse form, but the curves in Figures 13.2, 4 and 6 represent another coarse form. We apply our definition of form in the Table.

8. The numbers 92, 93, and 96 of the algebraic-topological types of reducible projective quartic curves shown in [**22–24**], respectively, are not correct (private communication with Polotovskii).

9. Newton applies the same Figure 20 both to his species 13 and 14. He does not specify the double isolated point of species 13 in Figure 20.

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