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PARAMETRIC SOLUTIONS OF THE QUARTIC DIOPHANTINE EQUATION $f(x,y) = f(u,v)$

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ABSTRACT. There are very few quartic diophantine equations of the type $f(x,y) = f(u,v)$, where $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ is a binary quartic form, for which parametric solutions have been obtained. In this paper we obtain parametric solutions of such quartic equations when the coefficients a, b, c, d, e satisfy certain conditions.

1. Introduction. This paper is concerned with quartic diophantine equations of the type

$$f(x,y) = f(u,v) \tag{1.1}$$

where

$$f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \tag{1.2}$$

is a binary quartic form in the variables x and y, and the coefficients a, b, c, d and e are integers. At present no method is known of determining all integer solutions or even a single non-trivial solution of a given quartic equation of type (1.1). A necessary and sufficient condition for the solvability of equation (1.1) has been given by Choudhry [**3**]. Even when this condition is satisfied we may not get a parametric solution of the given equation. The only two equations of type (1.1) for which parametric solutions have been explicitly obtained are the classical equation

$$x^4 + y^4 = u^4 + v^4$$

for which several solutions are known [**1**], [**5**, p. 201], [**6**], and the equation

$$x^4 + 4y^4 = u^4 + 4v^4$$

for which a parametric solution has been given by Choudhry [**2**]. Further, Segre [**8**, pp. 388–390] has indicated a method of obtaining

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parametric solutions of some equations of type (1.1) without, however, giving these solutions explicitly.

In this paper we show that parametric solutions of quartic equations of the type (1.1) may be obtained when the coefficients a, b, c, d and e satisfy certain conditions such that the form $f(x, y)$ can be reduced by a nonsingular rational linear transformation $x = \alpha X + \beta Y$, $y = \gamma X + \delta Y$, either to a form of the type $pX^4 + qXY^3 + rY^4$ or to a form of the type $pX^4 + qX^2Y^2 + rY^4$ where p, q, r are arbitrary rational numbers. Thus the substitutions

$$\begin{aligned} x &= \alpha X + \beta Y, \quad y = \gamma X + \delta Y, \\ u &= \alpha U + \beta V, \quad v = \gamma U + \delta V, \end{aligned} \tag{1.3}$$

reduce the quartic equation (1.1) to the type

$$pX^4 + qXY^3 + rY^4 = pU^4 + qUV^3 + rV^4 \tag{1.4}$$

or to the type

$$pX^4 + qX^2Y^2 + rY^4 = pU^4 + qU^2V^2 + rV^4. \tag{1.5}$$

We obtain two parametric solutions of (1.4) when p, q, r are arbitrary integers such that $pqr \neq 0$ and show how more parametric solutions of this equation may be obtained. Similarly, we obtain an explicit parametric solution of (1.5) when p, q, r are integers such that $r = p$, and we show how more parametric solutions may be obtained. We also show how parametric solutions of (1.5) may be obtained when the integers p, q, r satisfy certain other conditions. The parametric solutions of equations (1.4) and (1.5) may be used to obtain parametric solutions of the original equation (1.1) by using the relations (1.3). For applying this method we obtain necessary and sufficient conditions under which the binary quartic form $f(x, y)$ may be reduced by a linear transformation to one of the two types referred above. While the reduction of the binary form $f(x, y)$ to the form $pX^4 + qXY^3 + rY^4$ has apparently not been considered earlier, it seems that Segre [**7**] has given the conditions under which a quartic form $f(x, y)$ may be reduced to the form $pX^4 + qX^2Y^2 + rY^4$ by a linear transformation, but this paper has remained inaccessible to me. Accordingly, in this paper, we have derived our conditions ab initio for both cases. We also note that our conditions give very simple algebraic tests to determine whether a quartic binary form $f(x,y)$ can be reduced by a rational linear transformation to a binary form of the type $pX^4+qXY^3+rY^4$ or $pX^4+qX^2Y^2+rY^4$, and when the conditions are satisfied, we give the requisite linear transformation explicitly.

2. Solution of the diophantine equation $f(x,y) = f(u,v)$ by reduction to the type $pX^4+qXY^3+rY^4 = pU^4+qUV^3+rV^4$. We first obtain two parametric solutions of equation (1.4). A solution of equation (1.4) is easily obtained by writing $X = U$. This solution may be written in parametric terms as follows:

$$\begin{aligned} X &= -r(m+n)(m^2+n^2), \quad Y = qm(m^2+mn+n^2), \\ U &= -r(m+n)(m^2+n^2), \quad V = qn(m^2+mn+n^2). \end{aligned} \tag{2.1}$$

Next we show that, by using a known solution of the quartic equation (1.1), another solution can, in general, be obtained. If $(x_0,\, y_0,\, u_0,\, v_0)$ is a known solution of (1.1) so that $f(x_0,\, y_0) = f(u_0,\, v_0)$, we substitute

$$\begin{aligned} x &= \xi\theta + x_0, \quad u = \xi\theta + u_0, \\ y &= \eta\theta + y_0, \quad v = \eta\theta + v_0, \end{aligned} \tag{2.2}$$

(where $\xi,\ \eta$ are arbitrary) in equation (1.1) so that it reduces to

$$f(\xi\theta + x_0,\, \eta\theta + y_0) - f(\xi\theta + u_0,\, \eta\theta + v_0) = 0, \tag{2.3}$$

or

$$\begin{aligned} &\frac{\theta^3}{6}\left[\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^3 f\right\}_{x=x_0,y=y_0} \right.\\ &\left. -\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^3 f\right\}_{x=u_0,\,y=v_0}\right] + \frac{\theta^2}{2}\left[\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^2 f\right\}_{x=x_0,y=y_0}\right.\\ &\left. -\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^2 f\right\}_{x=u_0,\,y=v_0}\right] + \theta\left[\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right) f\right\}_{x=x_0,y=y_0}\right.\\ &\qquad\qquad\left. -\left\{\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right) f\right\}_{x=u_0,\,y=v_0}\right] = 0. \end{aligned} \tag{2.4}$$

We choose ξ and η such that the linear condition obtained by equating the coefficient of θ to zero is satisfied after which (2.4) can, in general, be readily solved to get a nonzero solution for θ. This leads to a rational solution of (1.1) and, on clearing denominators, we obtain an integral solution of (1.1). Thus, when one parametric solution of equation (1.1) is known, we may apply the above method to obtain a new parametric solution of (1.1) and the process may be applied repeatedly to obtain more parametric solutions of this equation.

Applying the above method by taking the parametric solution (2.1) as the initial known solution of equation (1.4), we obtain the following parametric solution of (1.4):

$$\begin{aligned} X &= -r(m+n)(m-n)^2(m^4+4m^3n+8m^2n^2+4mn^3+n^4),\\ Y &= qm(m+2n)(m-n)(m^2+mn+n^2)^2,\\ U &= -r(m+n)(m-n)^2(m^4+4m^3n+8m^2n^2+4mn^3+n^4),\\ V &= -qn(m-n)(2m+n)(m^2+mn+n^2)^2. \end{aligned} \tag{2.5}$$

Both the solutions (2.1) and (2.5) necessarily have $X = U$. With (X, Y, U, V) defined by (2.1), we may take $(X, Y, -U, -V)$ as our initial known solution of equation (1.4), and then the above method gives a parametric solution of degree 27 such that $X \neq \pm U$.

We will now give a necessary and sufficient condition that the binary form $f(x, y)$ may be reduced by a linear transformation to the form $pX^4 + qXY^3 + rY^4$ so that equation (1.1) may be reduced by suitable substitutions to an equation of type (1.4). For this purpose we first define three functions $\phi_i(\theta)$, $i = 1, 2, 3$ as follows:

$$\begin{aligned} \phi_1(\theta) = {} & (8ac-3b^2)\theta^4 + (24ad-4bc)\theta^3 + (48ae+6bd-4c^2)\theta^2\\ & + (24be-4cd)\theta + 8ce - 3d^2, \end{aligned}$$

$$\begin{aligned} \phi_2(\theta) = {} & (8a^2d-4abc+b^3)\theta^6 + (32a^2e+4abd-8ac^2+2b^2c)\theta^5\\ & + (40abe-20acd+5b^2d)\theta^4 + (-20ad^2+20b^2e)\theta^3\\ & + (-40ade+20bce-5bd^2)\theta^2\\ & + (-32ae^2-4bde+8c^2e-2cd^2)\theta - 8be^2 + 4cde - d^3, \end{aligned}$$

$$\begin{aligned}\phi_3(\theta) = {} & (256a^3e - 64a^2bd + 16ab^2c - 3b^4)\theta^8 \\ & + (512a^2be - 128a^2cd - 80ab^2d + 64abc^2 - 8b^3c)\theta^7 \\ & + (256a^2ce - 192a^2d^2 + 352ab^2e - 96abcd + 64ac^3 \\ & \quad - 52b^3d + 8b^2c^2)\theta^6 \\ & + (-256a^2de + 512abce - 224abd^2 + 128ac^2d + 48b^3e \\ & \quad - 120b^2cd + 32bc^3)\theta^5 \\ & + (-256a^2e^2 - 192abde + 384ac^2e + 80acd^2 + 80b^2ce \\ & \quad - 146b^2d^2 - 16bc^2d + 16c^4)\theta^4 \\ & + (-256abe^2 + 512acde + 48ad^3 - 224b^2de + 128bc^2e \\ & \quad - 120bcd^2 + 32c^3d)\theta^3 \\ & + (256ace^2 + 352ad^2e - 192b^2e^2 - 96bcde - 52bd^3 \\ & \quad + 64c^3e + 8c^2d^2)\theta^2 \\ & + (512ade^2 - 128bce^2 - 80bd^2e + 64c^2de - 8cd^3)\theta \\ & + 256ae^3 - 64bde^2 + 16cd^2e - 3d^4.\end{aligned}$$

Theorem 1. *A necessary and sufficient condition that the binary form*

$$f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

may be reduced by a rational linear transformation to the form $pX^4 + qXY^3 + rY^4$ *is that the quartic equation* $\phi_1(\theta) = 0$ *has a rational solution* θ_0 *such that* $f(\theta_0, 1) \neq 0$, $\phi_2(\theta_0) \neq 0$ *and* $\phi_3(\theta_0) \neq 0$. *When this condition is satisfied, the requisite linear transformation is as follows:*

$$\begin{aligned} x &= \theta_0 X + (b\theta_0^3 + 2c\theta_0^2 + 3d\theta_0 + 4e)Y, \\ y &= X - (4a\theta_0^3 + 3b\theta_0^2 + 2c\theta_0 + d)Y. \end{aligned} \tag{2.6}$$

Proof. If there exists a rational θ_0 satisfying the conditions of the theorem, it is seen by direct computation that the binary form reduces under the linear transformation (2.6) to the form $f(\theta_0, 1) \times \{X^4 - 8\phi_2(\theta_0)XY^3 + \phi_3(\theta_0)Y^4\}$ which shows that the condition stated in the theorem is sufficient. To show that the condition is also necessary, we apply to the form $f(x,y)$ an arbitrary nonsingular rational linear transformation

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y \tag{2.7}$$

where, without loss of generality, we may take $\gamma \neq 0$, and we equate to zero the coefficient of X^3Y in the resulting form which gives us the following condition:

$$(4a\alpha^3 + 3b\alpha^2\gamma + 2c\alpha\gamma^2 + d\gamma^3)\beta + (b\alpha^3 + 2c\alpha^2\gamma + 3d\alpha\gamma^2 + 4e\gamma^3)\delta = 0. \tag{2.8}$$

It follows from (2.8) that

$$\begin{aligned} \beta &= (b\alpha^3 + 2c\alpha^2\gamma + 3d\alpha\gamma^2 + 4e\gamma^3)t, \\ \delta &= -(4a\alpha^3 + 3b\alpha^2\gamma + 2c\alpha\gamma^2 + d\gamma^3)t \end{aligned} \tag{2.9}$$

where t is arbitrary and, with these values of β and δ, the form $f(x, y)$ is reduced by the linear transformation (2.7) to the form

$$\gamma^4 f(\alpha/\gamma, 1)\{X^4 + 2t^2\gamma^4\phi_1(\alpha/\gamma)X^2Y^2 - 8t^3\gamma^6\phi_2(\alpha/\gamma)XY^3 + t^4\gamma^8\phi_3(\alpha/\gamma)Y^4\}. \tag{2.10}$$

It is now an immediate consequence that, for the form $f(x, y)$ to be reduced by the linear transformation (2.7) to the form $pX^4 + qXY^3 + rY^4$, we must have $\phi_1(\alpha/\gamma) = 0$, $f(\alpha/\gamma, 1) \neq 0$, $\phi_2(\alpha/\gamma) \neq 0$ and $\phi_3(\alpha/\gamma) \neq 0$. This shows that the condition of the theorem is necessary with $\theta_0 = \alpha/\gamma$ being the requisite rational solution of the equation $\phi_1(\theta) = 0$. This completes the proof. □

As an example, the diophantine equation

$$x^4 - 4x^2y^2 + 6y^4 = u^4 - 4u^2v^2 + 6v^4 \tag{2.11}$$

may be solved by using the above theorem and the method of solving equation (1.4). Here, the value $\theta_0 = 1$ satisfies the condition of the theorem, and the linear transformation $x = X + 16Y, y = X + 4Y$ reduces the form $x^4 - 4x^2y^2 + 6y^4$ to the form $3(X^4 + 2560XY^3 + 16896Y^4)$ so that equation (2.11) may be reduced to the equation

$$X^4 + 2560XY^3 + 16896Y^4 = U^4 + 2560UV^3 + 16896V^4, \tag{2.12}$$

and we readily obtain the following solution of (2.11):

$$\begin{aligned} x &= 47m^3 + 47m^2n + 47mn^2 - 33n^3, \\ y &= -13m^3 - 13m^2n - 13mn^2 - 33n^3, \\ u &= -33m^3 + 47m^2n + 47mn^2 + 47n^3, \\ v &= -33m^3 - 13m^2n - 13mn^2 - 13n^3. \end{aligned} \tag{2.13}$$

Using (2.13) as the initial known solution of (2.11), we may obtain more parametric solutions of (2.11) as already indicated.

As another example, the linear transformation $x = X + 112Y, y = X - 84Y$ reduces the binary form $3x^4 + 36x^2y^2 + 10y^4$ to the form $49(X^4 - 526848XY^3 + 84822528Y^4)$ and accordingly we may obtain parametric solutions of the equation

$$3x^4 + 36x^2y^2 + 10y^4 = 3u^4 + 36u^2v^2 + 10v^4. \tag{2.14}$$

3. Solution of the diophantine equation $f(x, y) = f(u, v)$ by reduction to the type $pX^4+qX^2Y^2+rY^4 = pU^4+qU^2V^2+rV^4$. We will obtain parametric solutions of equation (1.5) when p, q, r satisfy certain conditions. We first consider equation (1.5) with $r = p$, that is, the equation

$$pX^4 + qX^2Y^2 + pY^4 = pU^4 + qU^2V^2 + pV^4. \tag{3.1}$$

We take $(X, Y, U, V) = (m, n, n, -m)$ as the initial known solution of (3.1) and, applying the method described in Section 2 of finding a solution of (1.1) starting from a known solution, we obtain the following solution of equation (3.1):

$$\begin{aligned}
X &= 8p^3m^7 + 2pq^2m^6n + (8p^3 + 2pq^2 + q^3)m^5n^2 \\
&\quad + (16p^2q - q^3)m^4n^3 + (-16p^3 + 16p^2q + 2pq^2)m^3n^4 \\
&\quad + 24p^3m^2n^5 + (8p^3 - 4p^2q + 2pq^2)mn^6 + 4p^2qn^7, \\
Y &= -4p^2qm^7 + (8p^3 - 4p^2q + 2pq^2)m^6n - 24p^3m^5n^2 \\
&\quad + (-16p^3 + 16p^2q + 2pq^2)m^4n^3 + (q^3 - 16p^2q)m^3n^4 \\
&\quad + (8p^3 + 2pq^2 + q^3)m^2n^5 - 2pq^2mn^6 + 8p^3n^7, \\
U &= 4p^2qm^7 + (8p^3 - 4p^2q + 2pq^2)m^6n + 24p^3m^5n^2 \\
&\quad + (-16p^3 + 16p^2q + 2pq^2)m^4n^3 + (16p^2q - q^3)m^3n^4 \\
&\quad + (8p^3 + 2pq^2 + q^3)m^2n^5 + 2pq^2mn^6 + 8p^3n^7, \\
V &= -8p^3m^7 + 2pq^2m^6n + (-2pq^2 - q^3 - 8p^3)m^5n^2 \\
&\quad + (16p^2q - q^3)m^4n^3 + (16p^3 - 16p^2q - 2pq^2)m^3n^4 \\
&\quad + 24p^3m^2n^5 + (4p^2q - 8p^3 - 2pq^2)mn^6 + 4p^2qn^7.
\end{aligned} \tag{3.2}$$

Next we show how a parametric solution of equation (1.5) may be obtained in certain other cases. The substitution $V = Y$ reduces (1.5) to the quadratic equation

$$p(X^2 + U^2) + qY^2 = 0, \tag{3.3}$$

while the substitution $U = X$ reduces equation (1.5) to the quadratic equation

$$qX^2 + r(Y^2 + V^2) = 0. \tag{3.4}$$

Both (3.3) and (3.4) are quadratic diophantine equations whose solvability is readily determined by a theorem of Legendre [**4**, p. 117]. An integer solution of one of these equations may be used to obtain its parametric solution, and this will, in turn, lead to a parametric solution of equation (1.5).

As an example, we note that equation (2.11) can be solved by substituting $y = v$ and proceeding as above, we obtain the following parametric solution of (2.11):

$$\begin{aligned} x &= 2(m^2 - n^2), \quad & y &= m^2 + n^2, \\ u &= 4mn, & v &= m^2 + n^2. \end{aligned} \tag{3.5}$$

It is easy to see that the two parametric solutions (2.13) and (3.5) generate different integer solutions of equation (2.11). We also note that neither the substitution $x = u$ nor the substitution $y = v$ yields a solution of equation (2.14) which is solvable by the method described in the preceding section.

We will now prove a theorem giving a necessary and sufficient condition under which a binary form $f(x, y)$ is reduced by a linear transformation to the form $pX^4+qX^2Y^2+rY^4$. The functions $\phi_i(\theta)$, $i = 1$, 2, 3 are as defined in Section 2.

Theorem 2. *A necessary and sufficient condition that the binary form*

$$f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

may be reduced by a rational linear transformation to the form $pX^4 + qX^2Y^2 + rY^4$ *is that the sextic equation* $\phi_2(\theta) = 0$ *has a rational* *solution* θ_0 *such that* $f(\theta_0, 1) \neq 0$, $\phi_1(\theta_0) \neq 0$ *and* $\phi_3(\theta_0) \neq 0$. *When this condition is satisfied, the requisite linear transformation is as follows:*

$$x = \theta_0 X + (b\theta_0^3 + 2c\theta_0^2 + 3d\theta_0 + 4e)Y,$$
$$y = X - (4a\theta_0^3 + 3b\theta_0^2 + 2c\theta_0 + d)Y. \tag{3.6}$$

Proof. As before, if there exists a rational θ_0 satisfying the conditions of the theorem, direct computation shows that the binary form $f(x, y)$ reduces under the linear transformation (3.6) to the form $f(\theta_0, 1)\{X^4 + 2\phi_1(\theta_0)X^2Y^2 + \phi_3(\theta_0)Y^4\}$ which shows that the condition stated in the theorem is sufficient. To show that the condition is necessary, as before we apply to the binary form $f(x, y)$ the arbitrary nonsingular rational linear transformation (2.7), and again we must take β and δ as defined by (2.9) so that $f(x, y)$ is transformed to the form (2.10), and it immediately follows that the condition of the theorem is necessary.

When the condition stated in the theorem is satisfied, the quartic equation (1.1) may be reduced by suitable substitutions to an equation of type (1.5), and we have already seen above that parametric solutions of such an equation can be obtained under certain conditions. It now follows that we can obtain parametric solutions of equation (1.1) if there exists a rational θ_0 such that $\phi_2(\theta_0) = 0$ and one of the following conditions is satisfied:

(i) $\phi_3(\theta_0)$ is a perfect fourth power. In this case, the quartic equation (1.1) can be reduced by a further substitution to an equation of type (3.1).

(ii) the quartic equation

$$X^4 + 2\phi_1(\theta_0)X^2Y^2 + \phi_3(\theta_0)Y^4 = U^4 + 2\phi_1(\theta_0)U^2V^2 + \phi_3(\theta_0)V^4 \tag{3.7}$$

reduces to a solvable quadratic diophantine equation by either substituting $V = Y$ or $U = X$ in (3.7).

As an example, we consider the diophantine equation

$$\psi(x, y) = \psi(u, v) \tag{3.8}$$

where

$$\psi(x, y) = 7x^4 - 42x^3y + 99x^2y^2 - 108xy^3 + 46y^4. \tag{3.9}$$

For the form $\psi(x,y)$ to be reduced by a linear transformation to the form $pX^4+qX^2Y^2+pY^4$, we must find a rational solution of $\phi_2(\theta)=0$, which in this case reduces to the following condition:

$$(\theta-1)(\theta-2)(2\theta-3)(2\theta^2-6\theta+5)=0. \tag{3.10}$$

Taking $\theta=1$ as the rational solution θ_0 of the equation (3.10), we get from (3.6) the following linear transformation

$$x=X+16Y, \quad y=X+8Y \tag{3.11}$$

which reduces the form $\psi(x,y)$ to $2(X^4+96X^2Y^2+4096Y^4)$ and since $4096=8^4$ is a perfect fourth power, by writing $X=X_1$, $Y_1=8Y$, this form is further reduced to $2X_1^4+3X_1^2Y_1^2+2Y_1^4$. Thus the substitutions

$$x=X_1+2Y_1, \quad y=X_1+Y_1, \quad u=U_1+2V_1, \quad v=U_1+V_1 \tag{3.12}$$

reduce equation (3.8) to the equation

$$2X_1^4+3X_1^2Y_1^2+2Y_1^4=2U_1^4+3U_1^2V_1^2+2V_1^4. \tag{3.13}$$

This is a special case of (3.1) with $p=2$, $q=3$ and the solution (3.2) of equation (3.1) gives a parametric solution of (3.13) which on using (3.12) yields the following solution of (3.8):

$$\begin{aligned}
x &= -32m^7+140m^6n-257m^5n^2+365m^4n^3-230m^3n^4\\
&\quad +446m^2n^5-20mn^6+176n^7,\\
y &= 16m^7+88m^6n-65m^5n^2+265m^4n^3-65m^3n^4\\
&\quad +319m^2n^5+16mn^6+112n^7,\\
u &= -80m^7+124m^6n-62m^5n^2+430m^4n^3-35m^3n^4\\
&\quad +511m^2n^5-68mn^6+160n^7,\\
v &= -16m^7+88m^6n+65m^5n^2+265m^4n^3+65m^3n^4\\
&\quad +319m^2n^5-16mn^6+112n^7.
\end{aligned} \tag{3.14}$$

When (X_1, Y_1, U_1, V_1) is a solution of (3.13), then $(\pm X_1, \pm Y_1, \pm U_1, \pm V_1)$, $(\pm X_1, \pm Y_1, \pm V_1, \pm U_1)$, $(\pm Y_1, \pm X_1, \pm U_1, \pm V_1)$ and $(\pm Y_1, \pm X_1, \pm V_1, \pm U_1)$ are all solutions of (3.13), and all of these may be used to obtain more parametric solutions of equation (3.8). It is easily seen from (3.12) that four of these solutions, namely, (X_1, Y_1, U_1, V_1), $(X_1, Y_1, U_1, -V_1)$, (X_1, Y_1, V_1, U_1) and $(X_1, Y_1, V_1, -U_1)$ lead to the same values of x, y, and hence our parametric solution of (3.13) leads to a parametric solution of the diophantine chain

$$\psi(x_1, y_1) = \psi(x_2, y_2) = \psi(x_3, y_3) = \psi(x_4, y_4) = \psi(x_5, y_5).$$

Finally we note that the rational solutions $\theta = 2$ and $\theta = 3/2$ of equation (3.10) may also be used to obtain parametric solutions of (3.8) but these are not essentially different from the parametric solutions obtained by using the rational solution $\theta = 1$ of equation (3.10).

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