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INDUCED REPRESENTATIONS OF LOCALLY C^*-ALGEBRAS

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ABSTRACT. In this paper, by analogy with the case of C^*-algebras, we define the notion of induced representation of a locally C^*-algebra and then we prove the imprimitivity theorem for induced representations of locally C^*-algebras.

1. Introduction. Locally C^*-algebras generalize the notion of C^*-algebra. A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C^*-seminorm p on A. The terminology "locally C^*-algebra" is due to Inoue, see [**2**]. Locally C^*-algebras were also studied by Phillips, under the name of pro -C^*-algebra, see [**7**], Fragoulopoulou, and other people.

A representation of A on a Hilbert space H is a continuous $*$-morphism φ from A to $L(H)$, the C^*-algebra of all bounded linear operators on H. Given a locally C^*-algebra A which acts non-degenerately on a Hilbert module E over a locally C^*-algebra B and a non-degenerate representation (φ, H) of B, exactly as in the case of C^*-algebras, see [**8**], we construct a representation of A, called the Rieffel-induced representation from B to A via E, and then we prove some properties of this representation. Thus, we prove that the theorem on induction in stages, Theorem 5.9 in [**8**], is also true in the context of locally C^*-algebras, Theorem 3.6. In Section 4, we prove that if A and B are two locally C^*-algebras which are strong Morita equivalent, then any non-degenerate representation of A is induced from a non-degenerate representation of B, Theorem 4.4.

2. Preliminaries. Let A be a locally C^*-algebra, and let $S(A)$ be the set of all continuous C^*-seminorms on A. If $p \in S(A)$,

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then $A_p = A/\ker p$ is a C^*-algebra in the norm induced by p and $A = \lim_{\substack{\leftarrow \\ p\in S(A)}} A_p$. The canonical map from A onto A_p is denoted by π_p and the image of a under π_p by a_p.

An isomorphism from a locally C^*-algebra A to a locally C^*-algebra B is a bijective, continuous $*$-morphism Φ from A to B such that Φ^{-1} is continuous.

If (φ, H) is a representation of A, then there is $p \in S(A)$ and a representation (φ_p, H) of A_p such that $\varphi = \varphi_p \circ \pi_p$. We say that (φ_p, H) is a representation of A_p associated to (φ, H). The representation (φ, H) is non-degenerate if $\varphi(A)H$ is dense in H. Clearly, (φ, H) is non-degenerate if and only if (φ_p, H) is non-degenerate. We say that the representations (φ_1, H_1) and (φ_2, H_2) of A are unitarily equivalent if there is a unitary operator U from H_1 onto H_2 such that $U \circ \varphi_1(a) = \varphi_2(a) \circ U$ for all $a \in A$.

Definition 2.1. A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is **C**- and A-linear in its second variable and satisfies the following relations:

(i) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;

(ii) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;

(iii) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of semi-norms $\{\|\cdot\|_p\}_{p\in S(A)}$, where $\|\xi\|_p = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$, Definition 4.1 of [**7**].

Let E be a Hilbert A-module. For $p \in S(A)$, the vector space $E_p = E/\mathcal{E}_p$, where $\mathcal{E}_p = \{\xi \in E; p(\langle \xi, \xi \rangle) = 0\}$, is a Hilbert A_p-module with the action of A_p on E_p defined by $(\xi + \mathcal{E}_p)(a + \ker p) = \xi a + \mathcal{E}_p$ and the inner product defined by $\langle \xi + \mathcal{E}_p, \eta + \mathcal{E}_p \rangle = \pi_p(\langle \xi, \eta \rangle)$, [**7**, Lemma 4.5]. The canonical map from E onto E_p is denoted by σ_p and the image of ξ under σ_p by ξ_p. Thus, for $p, q \in S(A)$, $p \geq q$, there is a canonical morphism of vector spaces σ_{pq} from E_p into E_q such that $\sigma_{pq}(\xi_p) = \xi_q$, $\xi_p \in E_p$. Then $\{E_p, A_p, \sigma_{pq} : E_p \to$

$E_q, p \geq q; p, q \in S(A)\}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}(\xi_p a_p) = \sigma_{pq}(\xi_p)\pi_{pq}(a_p), \xi_p \in E_p, a_p \in A_p$; $\langle \sigma_{pq}(\xi_p), \sigma_{pq}(\eta_p)\rangle = \pi_{pq}(\langle \xi_p, \eta_p\rangle), \xi_p, \eta_p \in E_p$; $\sigma_{pp}(\xi_p) = \xi_p,\ \xi_p \in E_p$ and $\sigma_{qr} \circ \sigma_{pq} = \sigma_{pr}$ if $p \geq q \geq r$ and $\lim_{\overleftarrow{p}} E_p$ is a Hilbert A-module which may be identified with the Hilbert A-module E, [**7**, Proposition 4.4].

A Hilbert A-module E is full if the ideal of A generated by $\{\langle \xi, \eta\rangle,\ \xi, \eta \in E\}$ is dense in A.

Let E and F be two Hilbert A-modules. The set of all adjointable linear operators from E to F is denoted by $L_A(E, F)$, and we write $L_A(E)$ for $L_A(E, E)$. We consider on $L_A(E, F)$ the topology determined by the family of semi-norms $\{\widetilde{p}\}_{p\in S(A)}$, where $\widetilde{p}(T) = \sup\left\{\|T\xi\|_p ; \|\xi\|_p \leq 1\right\}$. Then $L_A(E, F)$ is isomorphic to $\lim_{\overleftarrow{p}} L_{A_p}(E_p, F_p)$, [**7**, Proposition 4.7], and $L_A(E)$ becomes a locally C^*-algebra. The canonical maps from $L_A(E, F)$ to $L_{A_p}(E_p, F_p)$, $p \in S(A)$ are denoted by $(\pi_p)_*$ and $(\pi_p)_* (T) (\sigma_p(\xi)) = \sigma_p (T\xi)$.

We say that the Hilbert A-modules E and F are unitarily equivalent if there is a unitary operator in $L_A(E, F)$.

A locally C^*-algebra A acts nondegenerately on a Hilbert B-module E if there is a continuous $*$-morphism Φ from A to $L_B(E)$ such that $\Phi(A)E$ is dense in E.

The closed vector subspace of $L_A(E, F)$ spanned by $\{\theta_{\eta,\xi};\ \xi \in E, \eta \in F\}$, where $\theta_{\eta,\xi}(\zeta) = \eta \langle \xi, \zeta\rangle$, is denoted by $K_A(E, F)$, and we write $K_A(E)$ for $K_A(E, E)$. Moreover, the locally C^*-algebras $K_A(E, F)$ and $\lim_{\overleftarrow{p}} K_{A_p}(E_p, F_p)$ are isomorphic as well as the C^*-algebras $(K_A(E, F))_p$ and $K_{A_p}(E_p, F_p)$ for all $p \in S(A)$. Since $K_A(E)E$ is dense in E, $K_A(E)$ acts non-degenerately on E.

3. Induced representations. Let A and B be two locally C^*-algebras, let E be a Hilbert B-module, let $\Phi : A \to L_B(E)$ be a non-degenerate continuous $*$-morphism and let (φ, H) be a non-degenerate representation of B. We will construct a non-degenerate representation $\left({}_E^A\varphi, {}_E H\right)$ of A from (φ, H) via E.

Construction 3.1 (for C^*-algebras, see [**8**]). *Define a sesquilinear form* $\langle \cdot, \cdot \rangle_0^{\varphi}$ *on the vector space* $E \otimes_{\text{alg}} H$ *by*

$$\langle \xi \otimes h_1, \eta \otimes h_2 \rangle_0^{\varphi} = \langle h_1, \varphi(\langle \xi, \eta \rangle_E) h_2 \rangle_{\varphi}$$

where $\langle \cdot, \cdot \rangle_{\varphi}$ *denotes the inner product on the Hilbert space* H*. It is easy to see that* $(E \otimes_{\text{alg}} H)/N_{\varphi}$*, where* N_{φ} *is the vector subspace of* $E \otimes_{\text{alg}} H$ *generated by* $\{\xi \otimes h \in E \otimes_{\text{alg}} H; \langle \xi \otimes h, \xi \otimes h \rangle_0^{\varphi} = 0\}$*, is a pre-Hilbert space with the inner product defined by*

$$\langle \xi \otimes h_1 + N_{\varphi}, \eta \otimes h_2 + N_{\varphi} \rangle^{\varphi} = \langle \xi \otimes h_1, \eta \otimes h_2 \rangle_0^{\varphi}.$$

The completion of $(E \otimes_{\text{alg}} H)/N_{\varphi}$ *with respect to the inner product* $\langle \cdot, \cdot \rangle^{\varphi}$ *is denoted by* ${}_EH$*. Let* $T \in L_B(E)$*. Define a linear map* ${}_E\varphi(T)$ *from* $E \otimes_{\text{alg}} H$ *into* $E \otimes_{\text{alg}} H$ *by*

$${}_E\varphi(T)(\xi \otimes h) = T\xi \otimes h.$$

If (φ_q, H) *is a representation of* B_q *associated to* (φ, H)*, then we have*

$$\begin{aligned}
\langle {}_E\varphi(T)(\xi \otimes h), {}_E\varphi(T)(\xi \otimes h) \rangle_0^{\varphi} &= \langle h, \varphi(\langle T\xi, T\xi \rangle_E) h \rangle_{\varphi} \\
&= \left\langle h, \varphi_q \left(\left\langle (\pi_q)_*(T)\sigma_q(\xi), (\pi_q)_*(T)\sigma_q(\xi) \right\rangle_{E_q} \right) h \right\rangle_{\varphi} \\
&\leq \widetilde{q}(T) \left\langle h, \varphi_q \left(\langle \sigma_q(\xi), \sigma_q(\xi) \rangle_{E_q} \right) h \right\rangle_{\varphi} \\
&= \widetilde{q}(T) \langle h, (\varphi_q \circ \pi_q)(\langle \xi, \xi \rangle_E) h \rangle_{\varphi} \\
&= \widetilde{q}(T) \langle \xi \otimes h, \xi \otimes h \rangle_0^{\varphi}
\end{aligned}$$

for all $\xi \in E$ *and* $h \in H$*. From this we conclude that* ${}_E\varphi(T)$ *may be extended to a bounded linear operator* ${}_E\varphi(T)$ *on* ${}_EH$*. In this way we have obtained a map* ${}_E\varphi$ *from* $L_B(E)$ *to* $L({}_EH)$*. It is easy to see that* $({}_E\varphi, {}_EH)$ *is a representation of* $L_B(E)$ *on* ${}_EH$*. Moreover,* ${}_E\varphi$ *is non-degenerate. Then* ${}_E\varphi \circ \Phi$ *is a non-degenerate representation of* A *on* ${}_EH$ *and it is denoted by* ${}_E^A\varphi$*.*

Definition 3.2. The representation $\left({}_E^A\varphi, {}_EH\right)$ constructed above is called the Rieffel-induced representation from B to A via E.

Remark 3.3. 1. Let (φ_1, H_1) and (φ_2, H_2) be two non-degenerate representations of B. If (φ_1, H_1) and (φ_2, H_2) are unitarily equivalent, then $\left({}_E^A\varphi_1, {}_E H_1\right)$ and $\left({}_E^A\varphi_2, {}_E H_2\right)$ are unitarily equivalent.

2. Let F be a Hilbert B-module which is unitarily equivalent to E. If U is a unitary element in $L_B(E, F)$ and A acts on F by $a \to U \circ \Phi(a) \circ U^*$, then the representations $\left({}_E^A\varphi, {}_E H\right)$ and $\left({}_F^A\varphi, {}_F H\right)$ of A are unitarily equivalent.

Proof. (1) If U is a unitary operator from H_1 onto H_2, then it is not hard to check that the linear operator V from $E \otimes_{\text{alg}} H_1$ onto $E \otimes_{\text{alg}} H_2$ defined by $V(\xi \otimes h) = \xi \otimes Uh$ may be extended to a unitary operator V from ${}_E H_1$ onto ${}_E H_2$ and, moreover, $V \circ_E^A \varphi_1(a) = {}_E^A\varphi_2(a) \circ V$ for all a in A.

(2) Consider the linear operator W from $E \otimes_{\text{alg}} H$ onto $F \otimes_{\text{alg}} H$ defined by $W(\xi \otimes h) = U\xi \otimes h$. Then we have

$$\begin{aligned}\left({}_F^A\varphi(a) \circ W\right)(\xi \otimes h) &= (U \circ \Phi(a) \circ U^*)(U\xi) \otimes h = U(\Phi(a)\xi) \otimes h \\ &= W(\Phi(a)\xi \otimes h) = \left(W \circ_E^A \varphi(a)\right)(\xi \otimes h)\end{aligned}$$

for all a in A, ξ in E and h in H. It is not difficult to see that W may be extended to a unitary operator from ${}_E H$ onto ${}_F H$ and ${}_F^A\varphi(a) \circ W = W \circ_E^A \varphi(a)$ for all a in A. □

Proposition 3.4. *Let (φ, H) be a non-degenerate representation of B. If (φ_q, H) is a non-degenerate representation of B_q associated to (φ, H), then there is $p \in S(A)$ such that A_p acts non-degenerately on E_q and the representations $\left({}_E^A\varphi, {}_E H\right)$ and $\left({}_{E_q}^{A_p}\varphi_q \circ \pi_p, {}_{E_q} H\right)$ of A are unitarily equivalent.*

Proof. Define a linear map U from $E \otimes_{\text{alg}} H$ into $E_q \otimes_{\text{alg}} H$ by

$$U(\xi \otimes h) = \sigma_q(\xi) \otimes h.$$

Since

$$\begin{aligned}\langle U(\xi \otimes h), U(\xi \otimes h)\rangle_0^{\varphi_q} &= \left\langle h, \varphi_q\left(\langle \sigma_q(\xi), \sigma_q(\xi)\rangle_{E_q}\right) h\right\rangle_{\varphi} \\ &= \langle h, (\varphi_q \circ \pi_q)(\langle \xi, \xi\rangle_E) h\rangle_{\varphi} \\ &= \langle \xi \otimes h, \xi \otimes h\rangle_0^{\varphi}\end{aligned}$$

for all $\xi \in E$ and $h \in H$, U may be extended to a bounded linear operator U from ${}_E H$ onto ${}_{E_q} H$. It is easy to verify that U is unitary and $U \circ_E \varphi(T) = \left({}_{E_q}\varphi \circ (\pi_q)_*\right)(T) \circ U$ for all $T \in L_B(E)$. Hence the representations $({}_E\varphi, {}_E H)$ and $\left({}_{E_q}\varphi_q \circ (\pi_q)_*, {}_{E_q}H\right)$ of $L_B(E)$ are unitarily equivalent.

The continuity of Φ implies that there is $p \in S(A)$ such that $\widetilde{q}(\Phi(a)) \leq p(a)$ for all a in A and so there is a $*$-morphism Φ_p from A_p to $L_{B_q}(E_q)$ such that $\Phi_p \circ \pi_p = (\pi_q)_* \circ \Phi$. Moreover, Φ_p is non-degenerate. From

$$\begin{aligned} U \circ^A_E \varphi(a) &= U \circ_E \varphi(\Phi(a)) = \left({}_{E_q}\varphi_q \circ (\pi_q)_*\right)(\Phi(a)) \circ U \\ &= \left({}_{E_q}\varphi_q\left(\Phi_p\left(\pi_p(a)\right)\right)\right) \circ U = \left({}^{A_p}_{E_q}\varphi_q \circ \pi_p\right)(a) \circ U \end{aligned}$$

for all $a \in A$, we conclude that the representations $\left({}^A_E\varphi, {}_E H\right)$ and $\left({}^{A_p}_{E_q}\varphi_q \circ \pi_p, {}_{E_q}H\right)$ of A are unitarily equivalent and the proposition is proved. □

Corollary 3.5. *If* $(\varphi, H) = (\oplus_{i\in I}\varphi_i, \oplus_{i\in I}H_i)$, *then* $\left({}^A_E\varphi, {}_E H\right)$ *is unitarily equivalent to* $\left(\oplus_{i\in I}\, {}^A_E\varphi_i, \oplus_{i\in I}\, {}_E H_i\right)$.

Proof. Let (φ_q, H) be a representation of B_q associated to (φ, H). It is easy to see that there is a representation (φ_{iq}, H_i) of B_q such that $\varphi_{iq} \circ \pi_q = \varphi_i$ for each $i \in I$. Moreover, $\varphi_q = \oplus_{i\in I}\varphi_{iq}$. By Proposition 3.4, there is a $p \in S(A)$ such that the representations $\left({}^A_E\varphi, {}_E H\right)$ and $\left({}^{A_p}_{E_q}\varphi_q \circ \pi_p, {}_{E_q}H\right)$ of A are unitarily equivalent as well as the representations $\left({}^A_E\varphi_i, {}_E H\right)$ and $\left({}^{A_p}_{E_q}\varphi_{iq} \circ \pi_p, {}_{E_q}H_i\right)$ for all $i \in I$.

On the other hand, we know that the representations $\left({}^{A_p}_{E_q}\varphi_q, {}_{E_q}H\right)$ and $\left(\oplus_{i\in I}\, {}^{A_p}_{E_q}\varphi_{iq}, \oplus_{i\in I}\, {}_{E_q}H_i\right)$ of A_p are unitarily equivalent, [**8**, Corollary 5.4]. This implies that the representations $\left({}^{A_p}_{E_q}\varphi_q \circ \pi_p, {}_{E_q}H\right)$ and $\left(\oplus_{i\in I}\, {}^{A_p}_{E_q}\varphi_{iq} \circ \pi_p, \oplus_{i\in I}\, {}_{E_q}H_i\right)$ of A are unitarily equivalent and the corollary is proved. □

Let A, B and C be three locally C^*-algebras, let E be a Hilbert B-module and F a Hilbert C-module and let $\Phi_1 : A \to L_B(E)$ and $\Phi_2 : B \to L_C(F)$ be non-degenerate continuous $*$-morphisms. If $E \otimes_{\Phi_2} F$ is the inner tensor product of E and F using Φ_2, then $E \otimes_{\Phi_2} F = \lim\limits_{\substack{\leftarrow \\ r\in S(C)}} E \otimes_{\Phi_{2r}} F_r$ and the locally C^*-algebras $L_C(E \otimes_{\Phi_2} F)$ and $\lim\limits_{\substack{\leftarrow \\ r\in S(C)}} L_{C_r}(E \otimes_{\Phi_{2r}} F_r)$ are isomorphic as well as $K_C(E \otimes_{\Phi_2} F)$ and $\lim\limits_{\substack{\leftarrow \\ r\in S(C)}} K_{C_r}(E \otimes_{\Phi_{2r}} F_r)$, where $\Phi_{2r} = (\pi_r)_* \circ \Phi_2$, see [**3**]. Moreover, there is a non-degenerate continuous $*$-morphism $(\Phi_2)_*$ from $L_B(E)$ to $L_C(E \otimes_{\Phi_2} F)$ defined by $(\Phi_2)_* (T)(\xi \otimes_{\Phi_2} \eta) = T\xi \otimes_{\Phi_2} \eta$. Let $\Phi = (\Phi_2)_* \circ \Phi_1$. Then Φ is a non-degenerate continuous $*$-morphism from A to $L_C(E \otimes_{\Phi_2} F)$.

Theorem 3.6. *Let A, B, C, E, F, Φ_1 and Φ_2 be as above. If (φ, H) is a non-degenerate representation of C, then the representations $\left({}^A_G\varphi, {}_GH\right)$, where $G = E \otimes_{\Phi_2} F$, and $\left({}^A_E\left({}^B_F\varphi\right), {}_E\left({}_FH\right)\right)$ of A are unitarily equivalent.*

Proof. Let (φ_r, H) be a non-degenerate representation of C_r associated to (φ, H). Then there is $q \in S(B)$ and a non-degenerate continuous $*$-morphism $\Psi_{2q} : B_q \to L_{C_r}(F_r)$ such that $\Psi_{2q} \circ \pi_q = (\pi_r)_* \circ \Phi_2$ and there is $p \in S(A)$ and a non-degenerate continuous $*$-morphism $\Psi_{1p} : A_p \to L_{B_q}(E_q)$ such that $\Psi_{1p} \circ \pi_p = (\pi_q)_* \circ \Phi_1$ and a non-degenerate continuous $*$-morphism $\Phi_p : A_p \to L_{C_r}(G_r)$ such that $\Phi_p \circ \pi_p = (\pi_r)_* \circ \Phi$.

According to Proposition 3.4, the representations $\left({}^A_G\varphi, {}_GH\right)$ and $\left({}^{A_p}_{G_r}\varphi_r \circ \pi_p, {}_{G_r}H\right)$ of A are unitarily equivalent as well as the representations $\left({}^B_F\varphi, {}_FH\right)$ and $\left({}^{B_q}_{F_r}\varphi_r \circ \pi_q, {}_{F_r}H\right)$ of B. Since the representations $\left({}^B_F\varphi, {}_FH\right)$ and $\left({}^{B_q}_{F_r}\varphi_r \circ \pi_q, {}_{F_r}H\right)$ of B are unitarily equivalent, by Proposition 3.4 and Remark 3.3 (1) we deduce that the representations $\left({}^A_E\left({}^B_F\varphi\right), {}_E\left({}_FH\right)\right)$ and $\left({}^{A_p}_{E_q}\left({}^{B_q}_{F_r}\varphi_r\right) \circ \pi_p, {}_{E_q}\left({}_{F_r}H\right)\right)$ of A are unitarily equivalent.

To show that the representations $\left({}^A_G\varphi, {}_GH\right)$ and $\left({}^A_E\left({}^B_F\varphi\right), {}_E\left({}_FH\right)\right)$ of A are unitarily equivalent it is sufficient to prove that the representations $\left({}^{A_p}_{G_r}\varphi_r, {}_{G_r}H\right)$ and $\left({}^{A_p}_{E_q}\left({}^{B_q}_{F_r}\varphi_r\right), {}_{E_q}\left({}_{F_r}H\right)\right)$ of A_p are unitarily equivalent. But we know that the representations $\left({}^{A_p}_{X_r}\varphi_r, {}_{X_r}H\right)$, where $X_r = E_q \otimes_{\Psi_{2q}} F_r$, and $\left({}^{A_p}_{E_q}\left({}^{B_q}_{F_r}\varphi_r\right), {}_{E_q}\left({}_{F_r}H\right)\right)$ of A_p are unitarily equivalent, [**8**, Theorem 5.9], and so it is sufficient to prove that the representations $\left({}^{A_p}_{X_r}\varphi_r, {}_{X_r}H\right)$ and $\left({}^{A_p}_{G_r}\varphi_r, {}_{G_r}H\right)$ of A_p are unitarily equivalent.

It is not hard to check that the linear map $U : G_r \to X_r$ defined by $U\left(\xi \otimes_{\Phi_{2r}} \eta\right) = \sigma_q\left(\xi\right) \otimes_{\Psi_{2q}} \eta$ is a unitary operator in $L_{C_r}(G_r, X_r)$ and moreover, $\left(\Phi_{2r}\right)_*(T) = U^* \circ \left(\Psi_{2q}\right)_*\left(\left(\pi_q\right)_*(T)\right) \circ U$ for all T in $L_B(E)$, see the proof of Proposition 4.4 in [**3**]. Since

$$\begin{aligned}\Phi_p(\pi_p(a)) &= \left(\pi_r\right)_*\left(\left(\Phi_2\right)_*\left(\Phi_1\left(a\right)\right)\right) = \left(\Phi_{2r}\right)_*\left(\Phi_1(a)\right)\\ &= U^* \circ \left(\Psi_{2q}\right)_*\left(\left(\pi_q\right)_*\left(\Phi_1(a)\right)\right) \circ U\\ &= U^* \circ \left(\left(\Psi_{2q}\right)_* \circ \Psi_{1p}\right)\left(\pi_p(a)\right) \circ U\end{aligned}$$

for all a in A and by Remark 3.3 (2), the representations $\left({}^{A_p}_{G_r}\varphi_r, {}_{G_r}H\right)$ and $\left({}^{A_p}_{X_r}\varphi_r, {}_{X_r}H\right)$ of A_p are unitarily equivalent and the theorem is proved. □

4. The imprimitivity theorem. Let A and B be locally C^*-algebras. We recall that A and B are strongly Morita equivalent, written $A \sim_M B$, if there is a full Hilbert A-module E such that the locally C^*-algebras B and $K_A(E)$ are isomorphic. The strong Morita equivalence is an equivalence relation in the set of all locally C^*-algebras, see [**4**]. Also the vector space $K_A(E, A)$, denoted by $\widetilde{E}$, is a full Hilbert $K_A(E)$-module with the action of $K_A(E)$ on $K_A(E, A)$ defined by $(T, S) \to T \circ S$, $S \in K_A(E)$ and $T \in K_A(E, A)$, and the inner product defined by $\langle T, S\rangle = T^* \circ S$, $T, S \in K_A(E, A)$. Moreover, the linear map α from A to $K_{K_A(E)}\left(\widetilde{E}\right)$ defined by $\alpha(a)\left(\theta_{b,\xi}\right) = \theta_{ab,\xi}$ is an isomorphism of locally C^*-algebras, see [**4**]. Since the locally C^*-algebras B and $K_A(E)$ are isomorphic, $\widetilde{E}$ may be regarded as a Hilbert B-module.

It is not hard to check that the linear operator U_p from $\left(\widetilde{E}\right)_p$ to $\widetilde{E_p}$ defined by $U_p\left(T + \ker\left(\widetilde{p}\right)\right) = \left(\pi_p\right)_*(T)$ is unitary. Thus the Hilbert $K_{A_p}(E_p)$-modules $\left(\widetilde{E}\right)_p$ and $\widetilde{E_p}$ may be identified.

Lemma 4.1. *If $A \sim_M B$, then for each $p \in S(A)$ there is a $q_p \in S(B)$ such that $A_p \sim_M B_{q_p}$. Moreover, the set $\{q_p \in S(B); p \in S(A)$ and $A_p \sim_M B_{q_p}\}$ is a cofinal subset of $S(B)$.*

Proof. If Φ is an isomorphism of locally C^*-algebras from B onto $K_A(E)$, then the map $\tilde{p}\circ\Phi$, denoted by q_p, is a continuous C^*-seminorm on B. Since $\ker \pi_{q_p} = \ker (\pi_p)_* \circ \Phi$, there is a unique continuous $*$-morphism Φ_{q_p} from B_{q_p} onto $K_{A_p}(E_p)$ such that $\Phi_{q_p} \circ \pi_{q_p} = (\pi_p)_* \circ \Phi$. Moreover, Φ_{q_p} is an isomorphism of C^*-algebras, and since E_p is a full Hilbert A_p-module, we conclude that $A_p \sim_M B_{q_p}$.

To show that $\{q_p \in S(B); p \in S(A)$ and $A_p \sim_M B_{q_p}\}$ is a cofinal subset of $S(B)$, let $q \in S(B)$. Then there is $p_0 \in S(A)$ such that

$$q\left(\Phi^{-1}\left(\Phi\left(b\right)\right)\right) \leq \widetilde{p_0}\left(\Phi\left(b\right)\right)$$

for all $b \in B$, whence, since $q\left(\Phi^{-1}\left(\Phi\left(b\right)\right)\right) = q(b)$ and $\widetilde{p_0}\left(\Phi\left(b\right)\right) = q_{p_0}(b)$, we deduce that $q \leq q_{p_0}$. □

Remark 4.2. If E is a Hilbert B-module which gives the strong Morita equivalence between the locally C^*-algebras A and B, then E_p gives the strong Morita equivalence between the C^*-algebras A_p and B_{q_p}.

Theorem 4.3. *Let A and B be two locally C^*-algebras such that $A \sim_M B$ and let (φ, H) be a non-degenerate representation of A. Then (φ, H) is unitarily equivalent to $\left({}^{A}_{\widetilde{E}}\left({}^{B}_{E}\varphi\right), {}_{\widetilde{E}}\left({}_{E}H\right)\right)$, where E is a Hilbert A-module which gives the strong Morita equivalence between A and B.*

Proof. Let (φ_p, H) be a non-degenerate representation of A_p associated to (φ, H). By Lemma 4.1 there is a $q \in S(B)$ such that $A_p \sim_M B_q$. Moreover, the Hilbert A_p-module E_p gives the strong Morita equivalence between A_p and B_q, Remark 4.2. Then the representations (φ_p, H) and $\left({}^{A_p}_{\widetilde{E_p}}\left({}^{B_q}_{E_p}\varphi_p\right), {}_{\widetilde{E_p}}\left({}_{E_p}H\right)\right)$ of A_p are unitarily equivalent, [**8**, Theorem 6.23] and by Remark 3.3 (2), the representations $\left({}^{A_p}_{\widetilde{E_p}}\left({}^{B_q}_{E_p}\varphi_p\right), {}_{\widetilde{E_p}}\left({}_{E_p}H\right)\right)$ and $\left({}^{A_p}_{\widetilde{E}_p}\left({}^{B_q}_{E_p}\varphi_p\right), {}_{\widetilde{E}_p}\left({}_{E_p}H\right)\right)$ of A_p

are unitarily equivalent. From these facts we conclude that the representations (φ, H) and $\left({}^{A_p}_{\widetilde{E}_p}\left({}^{B_q}_{E_p}\varphi_p\right)\circ\pi_p, {}_{\widetilde{E}_p}\left({}_{E_p}H\right)\right)$ of A are unitarily equivalent.

On the other hand, according to Proposition 3.4, the representations $\left({}^{B}_{E}\varphi, {}_{E}H\right)$ and $\left({}^{B_q}_{E_p}\varphi_p\circ\pi_q, {}_{E_p}H\right)$ of B are unitarily equivalent. From this, using Remark 3.3 (1) and Proposition 3.4, we deduce that the representations $\left({}^{A}_{\widetilde{E}}\left({}^{B}_{E}\varphi\right), {}_{\widetilde{E}}\left({}_{E}H\right)\right)$ and $\left({}^{A_p}_{\widetilde{E}_p}\left({}^{B_q}_{E_p}\varphi_p\right)\circ\pi_p, {}_{\widetilde{E}_p}\left({}_{E_p}H\right)\right)$ of A are unitarily equivalent and the theorem is proved. □

Theorem 4.4. *Let A and B be locally C^*-algebras. If $A \sim_M B$, then there is a bijective correspondence between equivalence classes of non-degenerate representations of A and B which preserves direct sums and irreducibility.*

Proof. Let E be a Hilbert A-module which gives the strong Morita equivalence between A and B. By Theorem 4.3 and Remark 3.3 (1) the map from the set of all non-degenerate representations of A to the set of all non-degenerate representations of B which maps (φ, H) onto $\left({}^{B}_{E}\varphi, {}_{E}H\right)$ induces a bijective correspondence between equivalence classes of non-degenerate representations of A and B. Moreover, this correspondence preserves direct sums, Corollary 3.5.

Let (φ, H) be an irreducible, non-degenerate representation of A. Suppose that $\left({}^{B}_{E}\varphi, {}_{E}H\right)$ is not irreducible. Then $\left({}^{B}_{E}\varphi, {}_{E}H\right) = (\psi_1 \oplus \psi_2, H_1 \oplus H_2)$ and by Corollary 3.5 and Theorem 4.3 the representations $\left({}^{A}_{\widetilde{E}}\psi_1 \oplus {}^{A}_{\widetilde{E}}\psi_2, {}_{\widetilde{E}}H_1 \oplus {}_{\widetilde{E}}H_2\right)$ and (φ, H) of A are unitarily equivalent, a contradiction. So the bijective correspondence defined above preserves irreducibility.

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