ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

BOUNDED SOLUTIONS OF THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. We consider the nonlinear difference equation

$$\Delta\Big(a_n\Delta(b_n\Delta x_n)\Big) = q_n f(x_{n+2}), \quad n \in N,$$

where $\{a_n\}$, $\{b_n\}$, $\{q_n\}$ are positive real sequences, f is a real function with $xf(x) > 0$ for all $x \neq 0$. We obtain sufficient conditions for the boundedness of all nonoscillatory solutions of the above equation. Some examples are also given.

1. Introduction. Consider the third order difference equation

$$\text{(E)} \qquad \Delta\Big(a_n\Delta(b_n\Delta x_n)\Big) = q_n f(x_{n+2}), \quad n = 1, 2, \dots$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\{a_n\}$, $\{b_n\}$, $\{q_n\}$ are sequences of positive real numbers, $f : R \to R$ is a real function with $xf(x) > 0$ for $x \neq 0$.

The following convention is used:

$$\sum_{i=k}^{k-t} a_i := 0 \quad \text{for any} \quad k,\, t \in N.$$

By a solution of equation (E) we mean a real sequence $\{x_n\}$, which satisfies equation (E) for all sufficiently large n and is not eventually identically zero. A solution of equation (E) is called nonoscillatory, if it is eventually positive or eventually negative. Otherwise it is called oscillatory. A sequence $\{x_n\}$ is called quickly oscillatory if and only if $x_n = (-1)^n z_n$ for all $n \in N$, where $\{z_n\}$ is a sequence of positive numbers or a sequence of negative numbers.

In recent years there has been an increasing interest in the study of the qualitative behavior of solutions of difference equations. In

Received by the editors on May 1, 2003.
Key words and phrases. Bounded solutions, oscillatory solutions, nonoscillatory solutions, quickly oscillatory solutions.

comparison with second order difference equations, the study of higher order equations and in particular third order equations, has received considerably less attention. For example, the linear difference equation

$$\Delta^3 x_n = p_n x_{n+2}$$

has been investigated in [**5**]. The third order neutral difference equation of the form

$$\Delta\big(c_n \Delta(d_n \Delta(y_n + p_n y_{n-k}))\big) + q_n f(y_{n-m}) = e_n$$

has been considered in [**7**]. See also [**3, 4, 6**] and the references cited therein.

The purpose of this paper is to establish some sufficient conditions for the boundedness of all nonoscillatory solutions of equation (E). Cheng and Li in [**1**] had been concerned with boundedness of solutions of the equation

$$\text{(e)} \qquad \Delta\big(p_{n-1} \Delta(x_{n-1})\big) = q_n f(x_n).$$

The results obtained in the above paper motivated the studies on equation (E). We illustrate our results with examples.

2. Main results. There are known sufficient conditions for equation (E) to have nonoscillatory bounded solution in the case the function f is continuous, see [**2**]. It is however not known what conditions are sufficient for all nonoscillatory solutions of equation (E) to be bounded. Some of these conditions will be given below. We began with the following lemma, which will be useful in the proofs of the main results.

Lemma 1. *Any eventually positive solution* $\{x_n\}$ *of equation* (E) *belongs to one of the following four classes*:

(M_1) $x_n > 0$, $\Delta x_n > 0$, $\Delta(b_n \Delta x_n) > 0$;

(M_2) $x_n > 0$, $\Delta x_n > 0$, $\Delta(b_n \Delta x_n) < 0$;

(M_3) $x_n > 0$, $\Delta x_n < 0$, $\Delta(b_n \Delta x_n) > 0$;

(M_4) $x_n > 0$, $\Delta x_n < 0$, $\Delta(b_n \Delta x_n) < 0$

for all sufficiently large n.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (E). Then from (E) we have $\Delta\big(a_n\Delta(b_n\Delta x_n)\big) > 0$ for large n. Hence it is easy to see that $\{\Delta(b_n\Delta x_n)\}$, $\{\Delta x_n\}$ and $\{x_n\}$ are eventually of one sign. Thus, we have proved our lemma. $\square$

Note that if we assume

$$\sum_{n=1}^{\infty}\frac{1}{a_n}=\sum_{n=1}^{\infty}\frac{1}{b_n}=\infty,$$

then, by Kirguadze's lemma, any eventually positive solution of equation (E) must be of M_1-type or M_2-type.

Theorem 1. *Assume f is a nondecreasing function, $f(x)/x$ is nonincreasing for $x>0$ and*

$$\sum_{k=1}^{\infty}\frac{1}{b_k}\sum_{j=1}^{k-1}\frac{1}{a_j}\sum_{i=1}^{j-1}q_i<\infty. \tag{1}$$

Then every M_1-type solution of equation (E) *is bounded.*

Proof. Let $\{x_n\}$ be an unbounded solution of equation (E) of M_1-type. By Lemma 1, we have $x_n>0$, $\Delta x_n>0$ and $\Delta(b_n\Delta x_n)>0$ for $n\geq N$. From (E) we get

$$\begin{aligned} q_n &= \frac{\Delta\big(a_n\Delta(b_n\Delta x_n)\big)}{f(x_{n+2})} = \frac{a_{n+1}\Delta(b_{n+1}\Delta x_{n+1})}{f(x_{n+2})} - \frac{a_n\Delta(b_n\Delta x_n)}{f(x_{n+2})} \\ &\geq \frac{a_{n+1}\Delta(b_{n+1}\Delta x_{n+1})}{f(x_{n+2})} - \frac{a_n\Delta(b_n\Delta x_n)}{f(x_{n+1})} \\ &= \Delta\left[\frac{a_n\Delta(b_n\Delta x_n)}{f(x_{n+1})}\right] \quad \text{for} \quad n\geq N. \end{aligned} \tag{2}$$

Summing both sides of (2) from $i=N$ to $i=j-1$ we obtain

$$\sum_{i=N}^{j-1}q_i+\frac{a_N\Delta(b_N\Delta x_N)}{f(x_{N+1})}\geq\frac{a_j\Delta(b_j\Delta x_j)}{f(x_{j+1})}$$

and therefore

$$\frac{1}{a_j}\sum_{i=N}^{j-1} q_i + \frac{a_N\Delta(b_N\Delta x_N)}{a_j f(x_{N+1})} \geq \frac{\Delta(b_j\Delta x_j)}{f(x_{j+1})} \geq \frac{b_{j+1}\Delta x_{j+1}}{f(x_{j+1})} - \frac{b_j\Delta x_j}{f(x_j)}$$
$$= \Delta\left[\frac{b_j\Delta x_j}{f(x_j)}\right].$$

Summing once again, from $j = N$ to $j = k-1$, we obtain

$$\sum_{j=N}^{k-1}\frac{1}{a_j}\sum_{i=N}^{j-1} q_i + \sum_{j=N}^{k-1}\frac{a_N\Delta(b_N\Delta x_N)}{a_j f(x_{N+1})} \geq \frac{b_k\Delta x_k}{f(x_k)} - \frac{b_N\Delta x_N}{f(x_N)}.$$

Hence

(3)
$$\frac{\Delta x_k}{f(x_k)} \leq \frac{1}{b_k}\sum_{j=N}^{k-1}\frac{1}{a_j}\sum_{i=N}^{j-1} q_i + \frac{1}{b_k}\frac{a_N\Delta(b_N\Delta x_N)}{f(x_{N+1})}\sum_{j=N}^{k-1}\frac{1}{a_j} + \frac{b_N\Delta x_N}{b_k f(x_N)}.$$

Since $f(x)/x$ is nonincreasing for $x > 0$, from (3) we have

(4)
$$\frac{\Delta x_k}{x_k} \leq \frac{f(x_N)}{x_N}\frac{\Delta x_k}{f(x_k)} \leq \frac{f(x_N)}{x_N}\frac{1}{b_k}\sum_{j=N}^{k-1}\frac{1}{a_j}\sum_{i=N}^{j-1} q_i$$
$$+ \frac{f(x_N)}{x_N}\frac{a_N\Delta(b_N\Delta x_N)}{f(x_{N+1})}\frac{1}{b_k}\sum_{j=N}^{k-1}\frac{1}{a_j} + \frac{b_N\Delta x_N}{x_N b_k}.$$

Let $g(t) = x_k + (t-k)\Delta x_k$ for $k \leq t \leq k+1$. Then $g'(t) = \Delta x_k$ and $g(t) \geq x_k$ for $k \leq t \leq k+1$. Hence

(5)
$$\frac{\Delta x_k}{x_k} = \int_k^{k+1}\frac{g'(t)}{x_k}\,dt \geq \int_k^{k+1}\frac{g'(t)}{g(t)}\,dt = \ln[x_k + \Delta x_k] - \ln[x_k]$$
$$= \ln[x_{k+1}] - \ln[x_k].$$

Now, summing both sides of (5) from $k = N$ to $k = n-1$ we obtain

$$\sum_{k=N}^{n-1}\frac{\Delta x_k}{x_k} \geq \sum_{k=N}^{n-1}(\ln x_{k+1} - \ln x_k) = \ln x_n - \ln x_N.$$

Hence, by (4) we get

$$\begin{aligned}\ln(x_n) - \ln(x_N) \le{}& \frac{f(x_N)}{x_N} \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \sum_{i=N}^{j-1} q_i \\ &+ \frac{f(x_N)}{x_N} \frac{a_N \Delta(b_N \Delta x_N)}{f(x_{N+1)}} \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \\ &+ \frac{b_N \Delta x_N}{x_N} \sum_{k=N}^{n-1} \frac{1}{b_k}.\end{aligned}$$

From (1) there follows the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{b_k},$$

therefore $\{\ln(x_n)\}$ is bounded. This contradiction completes our proof. $\square$

Theorem 2. *Assume the condition* (1) *holds. Then every M_2-type solution of equation* (E) *is bounded.*

Proof. If $\{x_n\}$ is a solution of the equation (E) of M_2-type then there exists such an N that $x_n > 0$, $\Delta x_n > 0$ and $\Delta(b_n \Delta x_n) < 0$ for all $n \ge N$. Summing both sides of the inequality $\Delta(b_n \Delta x_n) < 0$ from $n = N$ to $n = i - 1$ we obtain

$$\Delta x_i < \frac{b_N \Delta x_N}{b_i} \quad \text{for} \quad i \ge N.$$

Summing once again the last inequality from $i = N$ to $i = n-1$ we get

$$x_n < b_N \Delta x_N \sum_{i=N}^{n-1} \frac{1}{b_i} + x_N.$$

Because the condition (1) implies

$$\sum_{k=N}^{\infty} \frac{1}{b_k} < \infty \quad \text{and} \quad b_N \Delta x_N > 0$$

so the solution $\{x_n\}$ must be bounded. This completes our proof. $\square$

Consequently by Lemma 1, Theorem 1 and Theorem 2 the following theorem is obtained.

Theorem 3. *Assume f is a nondecreasing function, $f(x)/x$ is nonincreasing for $x > 0$ and*

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} \sum_{i=1}^{j-1} q_i < \infty.$$

Then every eventually positive solution of equation (E) *is bounded.*

If the assumption, that $f(x)$ is nondecreasing and $f(x)/x$ is nonincreasing for $x > 0$, is replaced by the assumption that $f(x)$ is nonincreasing and $f(x)/x$ is nondecreasing for $x < 0$, then we may conclude that every eventually negative solution of equation (E) is bounded.

Example 1. Consider the equation

$$\text{(E1)} \qquad \Delta\big((n+1)^2 \Delta(n^2 \Delta x_n)\big) = \frac{1}{(n+1)^{1/3}(n+2)^{2/3}(n+3)}\, x_{n+2}^{1/3}, \quad n \geq 2.$$

All conditions of Theorem 3 are satisfied. Hence every eventually positive solution of (E1) is bounded. One such solution is $\{x_n\} = \{1 - (1/n)\}$.

Theorem 4. *Suppose f is nonincreasing for $x > 0$ and* (1) *holds. Then every M_1-type solution of equation* (E) *is bounded.*

Proof. Let $\{x_n\}$ be a solution of M_1-type of equation (E). Then there exists such an N that $x_n > 0$, $\Delta x_n > 0$ and $\Delta(b_n \Delta x_n) > 0$ for $n \geq N$. Since $f(x_{n+1}) \leq f(x_n)$ for $n \geq N$, we have

$$\Delta\big(a_n \Delta(b_n \Delta x_n)\big) = q_n f(x_{n+2}) \leq q_n f(x_N) \quad \text{for} \quad n \geq N.$$

Summing both sides of the above inequality from $i = N$ to $i = j-1$ we get

$$a_j \Delta(b_j \Delta x_j) - a_N \Delta(b_N \Delta x_N) \leq f(x_N) \sum_{i=N}^{j-1} q_i.$$

Hence

$$\Delta(b_j \Delta x_j) \leq \frac{1}{a_j}\, a_N \Delta(b_N \Delta x_N) + \frac{1}{a_j}\, f(x_N) \sum_{i=N}^{j-1} q_i. \tag{6}$$

Summing (6) from $j = N$ to $j = k-1$ we obtain

$$\Delta x_k \leq \frac{b_N}{b_k}\, \Delta x_N + \frac{1}{b_k}\, a_N \Delta(b_N \Delta x_N) \sum_{j=N}^{k-1} \frac{1}{a_j} + \frac{f(x_N)}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \sum_{i=N}^{j-1} q_i.$$

A final summation gets

$$\begin{aligned} x_k \leq x_N + b_N \Delta x_N \sum_{k=N}^{n-1} \frac{1}{b_k} + a_N \Delta(b_N \Delta x_N) \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \\ + f(x_N) \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \sum_{i=N}^{j-1} q_i. \end{aligned}$$

Since (1) implies

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{b_k} < \infty,$$

thus $\{x_n\}$ is bounded.

Consequently by Lemma 1, Theorem 2 and Theorem 4 we get the following theorem.

Theorem 5. *Suppose f is nonincreasing for $x > 0$ and*

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} \sum_{i=1}^{j-1} q_i < \infty.$$

Then every eventually positive solution of equation (E) *is bounded.*

Example 2. Consider the equation

$$\text{(E2)} \qquad \Delta\big((n+2)^2 \Delta(n^2 \Delta x_n)\big) = \frac{1}{(n+1)(n+2)^2} \frac{1}{x_{n+2}}, \quad n \geq 1.$$

All conditions of Theorem 5 are satisfied. Hence every eventually positive solution of (E2) is bounded. One such solution is $\{x_n\} = \{1/n\}$.

We remark that, if the assumption that $f(x)$ is nonincreasing for $x > 0$ is replaced by the assumption that $f(x)$ is nondecreasing for $x < 0$, then we may conclude that every eventually negative solution of (E) is bounded.

Now, we turn our attention to unbounded solutions.

Theorem 6. *Suppose f is nondecreasing and*

$$\text{(7)} \qquad \sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} \sum_{i=1}^{j-1} q_i = \infty.$$

Then every M_1-type solution of equation (E) *is unbounded.*

Proof. Let $\{x_n\}$ be a M_1-type solution of equation (E). Then there exists an integer $N \geq 1$ such that $x_n > 0$, $\Delta x_n > 0$ and $\Delta(b_n \Delta x_n) > 0$ for all $n \geq N$. Then, from (E), we have

$$\text{(8)} \qquad \Delta\big(a_n \Delta(b_n \Delta x_n)\big) = q_n f(x_{n+2}) \geq q_n f(x_N), \quad n \geq N.$$

Similarly as in the proof of Theorem 4 summing three times both sides of (8) we obtain

$$\begin{aligned} x_n \geq x_N + b_N \Delta x_N \sum_{k=N}^{n-1} \frac{1}{b_k} + a_N \Delta(b_N \Delta x_N) \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \\ + f(x_N) \sum_{k=N}^{n-1} \frac{1}{b_k} \sum_{j=N}^{k-1} \frac{1}{a_j} \sum_{i=N}^{j-1} q_i, \end{aligned}$$

and it, in view of (7), implies $x_n \to \infty$ as $n \to \infty$. □

Example 3. The difference equation

$$\Delta\big(2^n \Delta(2^n \Delta x_n)\big) = \frac{10}{3}\, 4^n x_{n+2}, \quad n \geq 1. \tag{E3}$$

satisfies conditions of Theorem 6 and hence any M_1-type solution of equation (E3) is unbounded. One such solution is $\{x_n\} = \{(3/2)^n\}$.

For M_2-type solution we need a stronger condition than (7).

Theorem 7. *Suppose f is nondecreasing,*

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} \sum_{i=1}^{j-1} q_i = \infty \tag{7}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} < \infty. \tag{9}$$

Then every M_2-type solution of equation (E) *is unbounded.*

The proof is similar to the proof of Theorem 6, thus we omit it.

If the condition (9) is not satisfied, then the above result may fail. It is shown in Example 4.

Example 4. Consider the difference equation

$$\Delta\big((n+1)\Delta((n+1)\Delta x_n)\big) = \frac{n+2}{n(n+1)}\, x_{n+2}, \quad n \geq 1. \tag{E4}$$

All conditions of Theorem 7 are satisfied except condition (9), namely

$$\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{j=1}^{k-1} \frac{1}{a_j} = \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{j=1}^{k-1} \frac{1}{j+1} = \infty.$$

So we cannot say that every M_2-type solution is unbounded. In fact, equation (E4) got the bounded solution

$$\{x_n\} = \left\{1 - \frac{1}{n}\right\}.$$

From Theorem 1 in [**2**] we have the following result.

Theorem 8. *Suppose f is nondecreasing and*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n} = \infty,$$
$$\sum_{k=1}^{\infty} q_k \sum_{j=1}^{k} \frac{1}{a_j} \sum_{i=1}^{j} \frac{1}{b_i} = \infty.$$

Then every nonoscillatory solution of equation (E) *is unbounded.*

Example 5. Consider the equation

$$\Delta\left(\frac{1}{6^n}\, \Delta\left(\frac{1}{3^n}\, \Delta x_n\right)\right) = \frac{17 \times 2^{-n} \times 3^{-2n-3}}{n+5}\, x_{n+2}, \quad n \geq 1. \tag{E5}$$

All conditions of Theorem 8 are satisfied. Hence every nonoscillatory solution of (E5) is unbounded. One such solution is $\{x_n\} = \{n+3\}$.

It is known that all solutions of equation (e) are nonoscillatory, see Theorem 1 in [**1**]. For the equation (E) this property does not hold (for some sequences $\{q_n\}$). It is shown in the following example.

Example 6. Consider the equation

$$\Delta^3 x_n = \frac{1}{4}\, x_{n+2}, \quad n \geq 1. \tag{E6}$$

The general solution of equation (E6) is in the form

$$\begin{aligned} x_n = c_1 2^n &+ c_2 \Big\{ \Big(\frac{\sqrt{2}}{2}\Big)^n \sin\Big(n \operatorname{arc} tg\, \frac{\sqrt{7}}{5}\Big)\Big\} \\ &+ c_3 \Big\{ \Big(\frac{\sqrt{2}}{2}\Big)^n \cos\Big(n \operatorname{arc} tg\, \frac{\sqrt{7}}{5}\Big)\Big\} \end{aligned}$$

for $c_i \in C$, $i = 1, 2, 3$. Hence the equation (E6) has oscillatory solutions.

Now we show

Theorem 9. *Equation* (E) *cannot have a quickly oscillatory solution.*

Proof. Let $z_n > 0$ for all $n \in N$ and suppose that $x_n = (-1)^n z_n$ is a solution of equation (E). Then

$$\begin{aligned} &\Delta\big(a_n \Delta(b_n \Delta x_n)\big) \\ &\quad = (-1)^{n+1}\big[a_{n+1} b_{n+2} z_{n+3} + (a_{n+1} b_{n+2} + a_{n+1} b_{n+1}) z_{n+2} \\ &\qquad\qquad + (a_{n+1} b_{n+1} + a_n b_{n+1} + a_n b_n) z_{n+1} + a_n b_n z_n\big]. \end{aligned}$$

Therefore equation (E) can be written in the form

$$\begin{aligned} (-1)^{n+1}\big[&a_{n+1} b_{n+2} z_{n+3} + (a_{n+1} b_{n+2} + a_{n+1} b_{n+1}) z_{n+2} \\ &+ (a_{n+1} b_{n+1} + a_n b_{n+1} + a_n b_n) z_{n+1} + a_n b_n z_n\big] \\ &\qquad = q_n f\big((-1)^{n+2} z_{n+2}\big). \end{aligned}$$

Hence for n even we have

$$\begin{aligned} -\ [&a_{n+1} b_{n+2} z_{n+3} + (a_{n+1} b_{n+2} + a_{n+1} b_{n+1}) z_{n+2} \\ &+ (a_{n+1} b_{n+1} + a_n b_{n+1} + a_n b_n) z_{n+1} + a_n b_n z_n] \\ &\qquad = q_n f\big((-1)^{n+2} z_{n+2}\big). \end{aligned}$$

where

$$\begin{aligned}-\ \big[&a_{n+1}b_{n+2}z_{n+3} + (a_{n+1}b_{n+2} + a_{n+1}b_{n+1})z_{n+2}\\ &+ (a_{n+1}b_{n+1} + a_nb_{n+1} + a_nb_n)z_{n+1} + a_nb_nz_n\big] < 0\end{aligned}$$

and, by the assumption $xf(x) > 0$, $q_nf(z_{n+2}) > 0$. On the other hand, for n odd

$$\begin{aligned}\big[&a_{n+1}b_{n+2}z_{n+3} + (a_{n+1}b_{n+2} + a_{n+1}b_{n+1})z_{n+2}\\ &+ (a_{n+1}b_{n+1} + a_nb_{n+1} + a_nb_n)z_{n+1} + a_nb_nz_n\big] = q_nf(-z_{n+2}).\end{aligned}$$

The left side of the above equation is always positive, and the right side is always negative. This contradiction proves our theorem. □

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