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PERIODIC SOLUTIONS AND ASYMPTOTIC BEHAVIOR OF A PDE WITH HYSTERESIS IN THE SOURCE TERM

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ABSTRACT. A parabolic PDE with hysteresis in the source term is considered. The existence of periodic solutions for a general hysteresis operator is proven and an asymptotic result for solutions of this equation, using ideas due to Krejčí, is obtained.

1. Introduction.

Let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be an open bounded set of Lipschitz class, denote by $\partial\Omega$ the boundary of Ω and set $Q := \Omega \times (0, \infty)$, $\Sigma := \partial\Omega \times (0, \infty)$.

In this paper we consider the following model equation

$$\frac{\partial u}{\partial t} - \triangle u + \mathcal{F}(u) = f \quad \text{in} \quad Q, \tag{1}$$

coupled with initial and boundary conditions, where

$$\mathcal{F} : M\left(\Omega; C^0([0,\infty))\right) \longrightarrow M\left(\Omega; C^0([0,\infty))\right)$$

is a continuous operator with memory, $M\left(\Omega; C^0([0,\infty))\right)$ denotes the Fréchet space of (strongly) measurable functions $\Omega \to C^0([0,\infty))$ and f is a given function.

Sufficient conditions for the existence and uniqueness of solutions of (1) are well known and we present them in the next section.

We study the question of existence of periodic solutions of (1) as well as asymptotic behavior of solutions as $t \to \infty$. To our knowledge there are so far only two papers dealing with such problems, [**1**] and [**5**]. In [**1**] they investigated the asymptotic behavior, as $t \to \infty$, of both the solution of (1) and the corresponding memory term $\mathcal{F}(u)$, where $\mathcal{F}(u)$ is a hysteresis operator. They showed that under some

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assumptions on the hysteresis boundary curves there exists $u_\infty \in H^1_0(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$, for all $p \in [1,\infty)$, such that $u(.,t) \to u_\infty$ weakly in $H^1_0(\Omega)$, $w(x,t) = \mathcal{F}(u(x,t)) \to -\triangle u_\infty$ strongly in $L^p(\Omega)$, for all $p \in [1,\infty)$, and almost everywhere in Ω as $t \to \infty$. They assumed $\mathcal{F}$ is a generalized play operator and their proof of asymptotic stability relied on the specific properties of this operator. The question of existence of periodic solutions of (1) was considered by Longfeng in [**5**], but also only in a very special case, where $\mathcal{F}$ is assumed to be a specific type of hysteresis operator. We prove the existence of a periodic solution of (1) with a more general hysteresis operator. The proof is based on a homotopy version of the Leray-Schauder fixed point theorem. The fourth section contains an asymptotic result for (1), where we assume $\mathcal{F}$ to be any Lipschitz continuous hysteresis operator. This is a much more general assumption than the one in [**1**].

2. A parabolic problem. We denote by $M\big(\Omega; C^0([0,T])\big)$ the Fréchet space of (strongly) measurable functions $\Omega \to C^0([0,T])$, see e.g., the Appendix in [**7**]. Let

$$\mathcal{F} : M\big(\Omega; C^0([0,T])\big) \quad \longrightarrow \quad M\big(\Omega; C^0([0,T])\big) \tag{2}$$

be a causal and strongly continuous operator. We fix a relatively open subset Γ_1 of $\partial\Omega$, and set

$$V := H^1_{\Gamma_1}(\Omega) := \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_1\} \tag{3}$$

where γ_0 denotes the trace operator. Thus if $\Gamma_1 = \varnothing$, then $V = H^1(\Omega)$; if $\Gamma_1 = \partial\Omega$, then $V = H^1_0(\Omega)$. We identify the space $L^2(\Omega)$ with its dual $L^2(\Omega)'$. As V is a dense subspace of $L^2(\Omega)$, $L^2(\Omega)'$ can be identified with a subspace of V'. So we get

$$V \subset L^2(\Omega) = L^2(\Omega)' \subset V', \tag{4}$$

with continuous, dense and compact injections. We define the operator $A : V \to V'$, $u \mapsto Au$ as follows :

$$_{V'}\langle Au, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall\, v \in V; \tag{5}$$

hence $Au = -\triangle u$ in $\mathcal{D}'(\Omega)$, where $\mathcal{D}(\Omega) = \{\phi; \phi$ infinitely differentiable on Ω and with compact support in $\Omega\}$ and $\mathcal{D}'(\Omega) =$ dual of $\mathcal{D}(\Omega) =$ space of distributions on Ω. We assume that

$$u_0, w_0 \in L^2(\Omega), \quad f \in L^2(0,T;V'). \tag{6}$$

Problem 1. *To find* $u \in M\big(\Omega; C^0([0,T])\big) \cap L^2(0,T;V)$ *such that* $\mathcal{F}(u) \in L^2(Q)$ *and*

$$\iint_Q \left(-u\,\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v + \mathcal{F}(u)v\right) dx\,dt = \int_0^T {}_{V'}\langle f, v\rangle_V\,dt + \int_\Omega u_0(x)v(x,0)\,dx \tag{7}$$

for all $v \in L^2(0,T;V) \cap H^1\left(0,T;L^2(\Omega)\right)$, $v(\cdot,T) = 0$, *a.e. in* Ω.

Interpretation. Equation (7) yields

$$\frac{\partial u}{\partial t} + Au + \mathcal{F} = f \quad \text{in} \quad \mathcal{D}'(0,T;V'). \tag{8}$$

By comparing the terms of this equation, we see that $\partial u/\partial t \in L^2(0,T;V')$, thus $u \in L^2(0,T;V) \cap H^1(0,T;V')$ and (8) holds in V' almost everywhere in $(0,T)$. The functions of this space admit time traces in $L^2(\Omega)$. Hence, integrating by parts in (7) and using (8), we get

$$u(x,0) = u_0(x) \quad \text{in} \quad L^2(\Omega) \quad \text{(in the sense of traces).} \tag{9}$$

Let us now interpret (8) for $V = H^1_{\Gamma_1}(\Omega)$. Let $\Gamma_2 := \Gamma/\Gamma_1$, fix any

$$f_1 \in L^2(Q), \qquad f_2 \in L^2(\Gamma_2 \times (0,T)), \tag{10}$$

and define $f \in L^2\big(0,T;L^2(\Omega)\big) \oplus L^2(0,T;V')$ by

$${}_{V'}\langle f(t), v\rangle_V := \int_\Omega f_1(x,t)v(x)\,dx + \int_{\Gamma_2} f_2(\sigma,t)\gamma_0 v(\sigma)\,d\sigma \qquad \forall\, v \in V, \quad \text{a.e. in} \quad (0,T).$$

Then (7) corresponds to the differential equation

$$\frac{\partial}{\partial t} u - \triangle u + \mathcal{F}(u) = f_1 \quad \text{in} \quad \mathcal{D}'(\Omega), \quad \text{a.e. in} \quad (0,T), \tag{11}$$

coupled with the boundary conditions

$$\gamma_0 u = 0 \qquad \text{on} \quad \Gamma_1 \times (0,T), \tag{12}$$

$$\frac{\partial u}{\partial \nu} = f_2 \quad \text{in} \quad \mathcal{D}'(\Gamma_2 \times (0,T)), \tag{13}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative.

The following theorem is proved in [**7**]:

Theorem 1. *Assume that* (2)–(4) *hold. Let* $\mathcal{F}$ *be affinely bounded, in the sense that*

$$\begin{gathered} \exists L \in \mathbf{R}^+, \quad \exists g \in L^2(\Omega); \quad \forall\, v \in M(\Omega, C^0([0,T])); \\ \|[\mathcal{F}(v)](x,\cdot)\|_{C^0([0,T])} \le L\|v(x,\cdot)\|_{C^0([0,T])} + g(x) \quad \textit{a.e. in} \quad \Omega. \end{gathered} \tag{14}$$

Moreover, let

$$\begin{gathered} f = f_1 + f_2,\ f_1 \in L^2(\Omega), \quad f_2 \in W^{1,1}(0,T,V'), \\ u_0 \in V, \quad w_0 \in L^2(\Omega). \end{gathered} \tag{15}$$

Then Problem 1 *has at least one solution such that*

$$u \in H^1\big(0,T; L^2(\Omega)\big) \cap L^\infty(0,T;V), \tag{16}$$

$$\mathcal{F}(u) \in L^2\big(\Omega; C^0([0,T])\big). \tag{17}$$

If $\mathcal{F}$ *also has the global Lipschitz continuity property*

$$\exists K > 0; \quad \forall\, t \in (0,T], \quad \forall\, v_1, v_2 \in L^2\big(\Omega; C^0([0,t])\big), \tag{18}$$

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^2(\Omega;C^0([0,t]))} \le K\|v_1 - v_2\|_{L^2(\Omega;C^0([0,t]))}, \tag{19}$$

then Problem 1 *has only one solution satisfying* (16).

3. Periodic solutions. We consider the question of existence of periodic solutions for (1) coupled with suitable boundary conditions. Here f will be a given function $\omega-$periodic in t.

We will make use of various subsets of the following assumptions:

(A1) Global Lipschitz continuity:

$$\exists K > 0; \quad \forall\, t \in (0,\infty), \quad \forall\, v_1, v_2 \in L^2\big(\Omega; C^0([0,t])\big),$$
$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^2(\Omega;C^0([0,t]))} \le K\|v_1 - v_2\|_{L^2(\Omega;C^0([0,t]))}.$$

(A2) Monocyclicity: If $u(x,t)$ is $\omega-$periodic in t, then $[\mathcal{F}(u)](x, t+\omega) = [\mathcal{F}(u)](x,t)$ for all $t \ge \omega$, $x \in \Omega$.

(A3) Affine boundedness:

$$\exists K_1 \in \mathbf{R}^+, \exists g \in L^2(\Omega); \quad \forall\, v \in M(\Omega, C^0([0,\infty)));$$
$$\|[\mathcal{F}(v)](x,\cdot)\|_{C^0([0,\infty))} \le K_1\|v(x,\cdot)\|_{C^0([0,\infty))} + g(x) \text{ a.e. in } \Omega.$$

(A4) Saturation:

$$|\mathcal{F}(u)(x,t)| \le C, \quad \text{for all} \quad x \in \Omega,\ t \in [0,\infty),$$

where C is some positive constant.

Remark 1. The term monocyclicity was introduced in [**2**] by M.A.Krasnosel'skii and A.V.Pokrovskii. For a periodic input $u(.)$, the least $\delta > 0$ such that the identity

$$[\mathcal{F}(u)](x,t+\omega) = [\mathcal{F}(u)](x,t), \quad t \ge \delta$$

holds is called a periodicity stabilization time of the output. If, for any periodic input, this time does not exceed the value of one period, then the operator is monocyclic. More details as well as the proof of the fact that the generalized play operator, and therefore also the generalized Prandtl-Ishlinskii operator of play type, is monocyclic can be found in [**2**]. The property (A4) is physically sensible for many problems.

Let $\omega > 0$, and let B be a Banach space. A measurable function $u : \mathbf{R}^+ \to B$ is called $\omega-$periodic if $u(t+\omega) = u(t)$ for almost all

$t \in \mathbf{R}^+$. By $L^2_\omega(0,\infty;B)$ we denote the Banach space of all (classes of) $\omega-$periodic functions $u:(0,\infty)\to B$ for which $u|_{(0,\omega)} \in L^2(0,\omega;B)$. The norm is given by

$$\|u|_{(0,\omega)}\|_{L^2((0,\omega);B)} = \left(\int_0^\omega \|u(x,t)\|_B^2\,dt\right)^{1/2}.$$

We can similarly define other spaces of functions, ω-periodic in t, for more details see, e.g., [**6**].

Define $D = H^{1,2}_\omega(Q)\cap G$, where

$$G = \overline{\{u \in C^\infty_\omega(\overline{Q}),\ u = 0 \text{ on } \partial\Omega,\ t \in \mathbf{R}^+\}} \quad \text{in} \quad H^1_\omega(Q).$$

We will prove the following theorem:

Theorem 2. *If* $f \in L^2_\omega\big(0,\infty,L^2(\Omega)\big)$ *is given and* $\mathcal{F}$ *satisfies the assumptions* (A1), (A2), *and at least one of* (A4) *or* (A3) *with* $K_1 < K = (\omega_n/|\Omega|)^{2n}$, *where* $|\Omega|$ *denotes the volume of* Ω *and* ω_n *is the volume of the* n*-dimensional unit ball, then there exists* $u \in D$, *which is periodic and satisfies the equation* (1) *almost everywhere in* Q, *for* $t \geq \omega$.

Remark 2. The operator $-\triangle$ in the equation (1) can be replaced by any symmetric uniformly elliptic operator.

Proof. To prove the theorem we will need the following lemma, for the proof see [**6**, Theorem III. 1.3.1].

Lemma 3. *Suppose that* $f \in L^2_\omega\big(0,\infty;L^2(\Omega)\big)$. *Then there exists a unique periodic solution of the equation*

$$\frac{\partial u}{\partial t} - \triangle u = f \quad \text{in} \quad Q, \tag{20}$$

which satisfies the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \Sigma,$$

such that $u \in D$*. Moreover, there exists a positive constant* K_2 *such that*

$$|u|_D \le K_2 |f|_{L^2_\omega(0,\infty;L^2(\Omega))}.$$

The main tool in our proof will be the homotopy version of the Leray-Schauder fixed point theorem:

Theorem 4. *Let* B *be a Banach space,* $T : B \times [0,1] \to B$ *a compact mapping such that*

(i) $T(x,0) = 0$ *for all* $x \in B$,

(ii) *there exists a constant* M *such that* $|x|_B \le M$ *for all* $(x,\sigma) \in B \times [0,1]$ *satisfying* $x = T(x,\sigma)$.

Then the mapping T_1 *of* B *into itself given by* $T_1 x = T(x,1)$ *has a fixed point.*

We introduce the Banach space $B = L^2\big(\Omega, C^0[0,\infty)\big)$. It can be easily seen from property (A3) or (A4) and property (A2) that, for all $v \in B$,

$$\mathcal{F}(v) \in L^2\left(\Omega, C^0([0,\infty)\right) \cap L^2\left(\Omega, C^0_\omega([\omega,\infty))\right).$$

For any $\sigma \in [0,1]$ and $t \ge \omega$ we consider the equation

$$\frac{\partial u}{\partial t} - \triangle u = -\sigma \mathcal{F}(v) + \sigma f, \quad v \in B.$$

By Lemma 3, the above equation has for any $\sigma \in [0,1]$ and any $v \in B$ a unique solution $\widetilde{u} \in D$, defined and periodic for $t \ge 0$. Let $u \in D$ be the periodic extension of $\widetilde{u}$ to $[0,\infty)$.

By interpolation, see e.g. [**4**], we have

$$D \subset H^1_\omega(Q) \subset L^2\left(\Omega; C^0_\omega[0,\infty)\right) \tag{21}$$

with continuous injections and the last one is also compact. If we denote by $T : T(v,\sigma) = u$, it follows from above that $T : B \times [0,1] \to B$. We shall show that all the assumptions of the Leray-Schauder theorem are satisfied for the mapping T. Obviously, for all $v \in B$, $T(v,0) = 0$, so assumption (i) is satisfied.

To show that T is a compact mapping: For all $\sigma \in [0,1]$, v_1, $v_2 \in B$, letting $T(v_1,\sigma) = u_1, T(v_2,\sigma) = u_2$, we get

$$\frac{\partial}{\partial t}(u_1 - u_2) - \triangle(u_1 - u_2) = -\sigma[\mathcal{F}(v_1) - \mathcal{F}(v_2)], \quad \forall\, t \geq \omega. \tag{22}$$

Multiplying the last equation by $u_1 - u_2$ and integrating over Ω, we get after integration by parts

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_1 - u_2\|^2_{L^2(\Omega)} + \int_\Omega [\nabla(u_1 - u_2)]^2\,dx = -\sigma\int_\Omega (u_1 - u_2)[\mathcal{F}(v_1) - \mathcal{F}(v_2)]\,dx$$

After integrating this in t over $[\omega, 2\omega]$, we have

$$\begin{aligned} \frac{1}{2}\Big\{\|u_1(2\omega) - u_2(2\omega)\|^2_{L^2(\Omega)} - \|u_1(\omega) - u_2(\omega)\|^2_{L^2(\Omega)}\Big\} \\ + \int_\omega^{2\omega}\int_\Omega [\nabla(u_1 - u_2)]^2\,dx\,dt \\ \leq |\sigma|\int_\omega^{2\omega}\int_\Omega |u_1 - u_2|\,|\mathcal{F}(v_1) - \mathcal{F}(v_2)|\,dx\,dt \\ \leq \|u_1 - u_2\|_{L^2(\Omega;L^2(\omega,2\omega))}\omega^{1/2} \\ \times\left\{\int_\Omega \sup_t[\mathcal{F}(v_1) - \mathcal{F}(v_2)]^2\,dx\right\}^{1/2} \\ \leq L\omega^{1/2}\,\|v_1 - v_2\|_B\|u_1 - u_2\|_{L^2(\Omega,L^2(\omega,2\omega))}. \end{aligned} \tag{23}$$

Because u_i, $i = 1,2$, are periodic in t with period ω, the difference of the first two terms on the left-hand side of (23) is zero. Moreover, using the Poincaré inequality to estimate the last term on the left-hand side we get

$$\mu_1\|u_1 - u_2\|^2_{L^2(\Omega,L^2(\omega,2\omega))} \leq \omega^{1/2}\,L\|u_1 - u_2\|_{L^2(\Omega,L^2(\omega,2\omega))}\|v_1 - v_2\|_B.$$

Thus

$$\|u_1 - u_2\|_{L^2(\Omega,L^2(\omega,2\omega))} \leq \frac{L\omega^{1/2}}{\mu_1}\,\|v_1 - v_2\|_B. \tag{24}$$

We also get from (23), using equivalent norms on the space $H_0^1(\Omega)$, that

$$\|u_1 - u_2\|_{L^2(\omega,2\omega;H^1(\Omega))} \le LR\omega^{1/2}\,\|v_1 - v_2\|_B. \tag{25}$$

If we now multiply (22) by $\partial/\partial t(u_1 - u_2)$ and integrate over Ω, we get

$$\int_\Omega \left[\frac{\partial(u_1-u_2)}{\partial t}\right]^2 d - \frac{\partial}{\partial t}\int_\Omega [\nabla(u_1-u_2)]^2\,dx \\ \le |\sigma| \int_\Omega |\mathcal{F}(v_1) - \mathcal{F}(v_2)| \left|\frac{\partial(u_1-u_2)}{\partial t}\right| dx.$$

After integrating in t over $[\omega, 2\omega]$, using estimates similar to those used above, we get

$$\left\|\frac{\partial}{\partial t}(u_1 - u_2)\right\|_{L^2(\Omega,L^2(\omega,2\omega))} \le L\omega^{1/2}\,\|v_1 - v_2\|_B. \tag{26}$$

It follows from (24), (25) and (26) that

$$\|u_1 - u_2\|_{H^1(Q)} \le R_1\|v_1 - v_2\|_B \tag{27}$$

and that T is compact with respect to v because of the compact imbedding (21).

Now, for all fixed $v \in B$, $\sigma_1, \sigma_2 \in [0,1]$, let

$$T(v,\sigma_1) = u^1$$
$$T(v,\sigma_2) = u^2.$$

We have

$$\frac{\partial}{\partial t}(u^1 - u^2) - \triangle(u^1 - u^2) = (\sigma_1 - \sigma_2)[f - \mathcal{F}(v)], \quad \forall\, t \ge \omega.$$

By Lemma 3 and the compact imbedding (21), we get the estimate

$$\|u^1 - u^2\|_B \le |\sigma_1 - \sigma_2|\,\widetilde{K}\left[\|f\|_{L^2(\Omega,L^2(\omega,2\omega))} + \|\mathcal{F}(v)\|_{L^2(\Omega,L^2(\omega,2\omega))}\right].$$

Hence, T is uniformly continuous with respect to σ for any fixed $v \in B$. Now, T is compact with respect to v for fixed σ and uniformly continuous with respect to σ for fixed $v \in B$, and thus T is a compact mapping of $B \times [0,1] \to B$.

To show (ii): For all $\sigma \in [0,1]$, let $T(u,\sigma) = u$, i.e.,

$$\frac{\partial u}{\partial t} - \triangle u = -\sigma \mathcal{F}(u) + \sigma f, \quad \forall\, t \geq \omega.$$

Multiplying by u and integrating over Ω, we get

$$\frac{1}{2}\,\frac{\partial}{\partial t}\,\|u\|^2_{L^2(\Omega)} + \int_\Omega (\nabla u)^2\,dx \leq \int_\Omega \mathcal{F}(u)u\,dx + \int_\Omega fu\,dx.$$

After integrating the last inequality in t over $[\omega, 2\omega]$, and using the periodicity of u in t and the Poincaré inequality, we get

(28)

$$\left(\frac{\omega_n}{|\Omega|}\right)^{2n} \|u\|^2_{L^2(\Omega,L^2(\omega,2\omega))} \leq \|f\|_{L^2(\Omega,L^2(\omega,2\omega))}\|u\|_{L^2(\Omega,L^2(\omega,2\omega))} + \|u\|_{L^2(\Omega,L^2(\omega,2\omega))}\|\mathcal{F}(u)\|_{L^2(\Omega,L^2(\omega,2\omega))}.$$

The last term in (28) can now be estimated by assumption (A4) as follows

$$\|u\|_{L^2(\Omega,L^2(\omega,2\omega))}\|\mathcal{F}(u)\|_{L^2(\Omega,L^2(\omega,2\omega))} \leq C_1\|u\|_{L^2(\Omega,L^2(\omega,2\omega))}.$$

Then we have altogether that

$$\|u\|_{L^2(\Omega,L^2(\omega,2\omega))} \leq \widetilde{C} = \text{constant},$$

so by Lemma 3 and by the compact imbedding (21) also $\|u\|_B \leq C_2$. So all assumptions of the Leray-Schauder fixed point theorem are satisfied, thus there exists $u \in D$, such that $T(u,1) = u$. This is the ω-periodic solution of (20).

If instead of (A4) we assume (A3), as was done by Longfeng in [**5**], then we get the same result, estimating the last term in (28) by assumption (A3), but we need to assume also that the constant $K_1 < K$ in (A3). This was done in [**5**] for a special kind of hysteresis operator. □

Remark 3. Existence of a periodic solution of (1) can be proved alternatively under slightly different assumptions on the hysteresis operator, using a variation of an approach used by Krejčí in [**3**], based on the classical Galerkin method.

4. An asymptotic result. We consider the model equation (1) coupled with initial and boundary conditions, where $\mathcal{F}$ is a continuous operator with memory, and f is a given function. Here we do not require $\mathcal{F}$ to be rate independent, but applications to hysteresis are our main concern. We suppose, however, that the operator $\mathcal{F}$ is piecewise monotone. This is a property often satisfied by hysteresis operators.

Definition 1. Piecewise monotonicity preservation property (or more briefly, piecewise monotonicity):

$$\begin{cases} \forall\,(u,w_0)\in \mathrm{Dom}\,(\mathcal{F}),\ \forall\,[t_1,t_2]\subset[0,T], \\ \text{if } u \text{ is nondecreasing (resp. nonincreasing) in } [t_1,t_2], \\ \text{then so is } \mathcal{F}(u,w_0). \end{cases} \tag{29}$$

If $u,\mathcal{F}(u,w_0)\in W^{1,1}(0,T)$, then this can be described by a simple inequality:

$$\frac{du}{dt}\left[\frac{d}{dt}\,\mathcal{F}(u,w_0)\right]\geq 0, \quad \text{a.e. in} \quad (0,T). \tag{30}$$

Theorem 5. *Let all the assumptions of Theorem* 1 *including the global Lipschitz continuity property be satisfied for any* $T\in(0,\infty)$ *and* $f\equiv$ *constant in* $\mathbf{R}^+\times\Omega$. *Suppose also that*

$$w_0,\ \triangle u_0\in L^2(\Omega),$$

and $\mathcal{F}$ *is piecewise monotonicity preserving* (*or, more briefly, piecewise monotone*).

Then there exist positive constants C_1, C_2, K_1 *such that, for any solution* u *of* (1) *with zero Dirichlet boundary data, we have*

$$\int_\Omega \left\{\left(\frac{\partial u}{\partial t}\right)^2+\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right\}(x,t)\,dx\leq C_1\,e^{-C_2 t}. \tag{31}$$

This then implies that

$$u_\infty = \lim_{t\to\infty} u(.,t) \tag{32}$$

exists and that the following estimate holds:

$$\|u_\infty - u(.,t)\|_{L^1} \le \frac{K_1}{C_2}\, e^{-C_2 t}. \tag{33}$$

Moreover,

$$w_\infty = \lim_{t\to\infty} w(x,t) \tag{34}$$

also exists weakly in $L^2(\Omega)$ *and* w_∞ *is the solution of the equation*

$$-\triangle u_\infty + w_\infty = f \quad \textit{weakly in} \quad W^{1,2}(\Omega). \tag{35}$$

Proof. By Theorem 1 we know that there exists a unique solution of (1) coupled with zero Dirichlet boundary data and an initial condition $u(x,0) = u_0(x)$ such that

$$u \in H^1\big(0,T; L^2(\Omega)\big) \cap L^\infty(0,T;V),$$
$$\mathcal{F}(u) \in L^2\big(\Omega; C^0([0,T])\big),$$

for any $T \in [0,\infty)$. Combining results of Proposition X.1.4 and Proposition IX.1.2 in [**7**], we have the following regularity of the solution:

$$u \in H^2\left(0,T; L^2(\Omega)\right) \cap W^{1,\infty}\left(0,T; H_0^1(\Omega)\right), \tag{36}$$

$$\mathcal{F}(u) \in H^1\left(0,T; L^2(\Omega)\right), \tag{37}$$

for any $T \in (0,\infty)$.

We can now differentiate the equation (1) with respect to t (necessary regularity follows from (36) and from the fact that u is a solution) and get

$$\frac{\partial^2 u}{\partial t^2} - \triangle\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\mathcal{F}(u)\right) = 0. \tag{38}$$

Now we do the following things: We multiply (38) by $\partial u/\partial t$ and get after integration over Ω:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx\right] + \int_\Omega \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le 0, \tag{39}$$

where we used the piecewise monotonicity property of the operator $\mathcal{F}$. Let L be the Lipschitz constant for $\mathcal{F}$, and let K denote a constant which will be specified later. Choose $\alpha > L^2K/4$, multiply (39) by α and get

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\Omega \alpha\left(\frac{\partial u}{\partial t}\right)^2 dx\right] + \int_\Omega \alpha\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le 0. \tag{40}$$

We now multiply (38) by $\partial^2 u/\partial t^2$ and again integrate over Ω:

$$\begin{aligned}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}\right)^2 dx + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx &\le \int_\Omega \left|\frac{\partial}{\partial t}\mathcal{F}(u)\right| \left|\frac{\partial^2 u}{\partial t^2}\right| dx \\ &\le L\int_\Omega \left|\frac{\partial u}{\partial t}\right| \left|\frac{\partial^2 u}{\partial t^2}\right| dx \\ &\le \int_\Omega \left(\frac{\partial^2 u}{\partial t^2}\right)^2 dx \\ &\quad + \frac{L^2}{4}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx,\end{aligned}$$

where we used the piecewise Lipschitz continuity of the operator $\mathcal{F}$ with Lipschitz constant L. The last inequality gives us:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 dx \le \frac{L^2}{4}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx. \tag{41}$$

Adding (40) and (41) results in:

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\Omega \left(\alpha\left(\frac{\partial u}{\partial t}\right)^2 + \left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2\right) dx\right] \\ &\qquad\qquad \le -\int_\Omega \left(\alpha\left[\nabla\left(\frac{\partial u}{\partial t}\right)\right]^2 - \frac{L^2}{4}\left|\frac{\partial u}{\partial t}\right|^2\right) dx.\end{aligned}$$

Using the equivalent norm in $H_0^1(\Omega)$, we have the following estimate for some constant K:

$$-\alpha \left\| \nabla \left(\frac{\partial u}{\partial t} \right) \right\|^2_{L^2(\Omega)} \le -\frac{\alpha}{K} \left\| \nabla \left(\frac{\partial u}{\partial t} \right) \right\|^2_{L^2(\Omega)} - \frac{\alpha}{K} \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)}.$$

So we get altogether:

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} &\left\{ \int_\Omega \left\{ \alpha \left(\frac{\partial u}{\partial t} \right)^2 + \left[\nabla \left(\frac{\partial u}{\partial t} \right) \right]^2 \right\} dx \right\} \\ &\le - \int_\Omega \left\{ \frac{\alpha}{K} \left[\nabla \left(\frac{\partial u}{\partial t} \right) \right]^2 + \left(\frac{\alpha}{K} - \frac{L^2}{4} \right) \left| \frac{\partial u}{\partial t} \right|^2 \right\} dx \\ &\le - \min \left\{ \frac{\alpha}{K}, \frac{\big((\alpha/K) - (L^2/4) \big)}{\alpha} \right\} \\ &\quad \times \int_\Omega \left\{ \alpha \left(\frac{\partial u}{\partial t} \right)^2 + \left[\nabla \left(\frac{\partial u}{\partial t} \right) \right]^2 \right\} dx. \end{aligned}$$

Note that $\alpha/K - L^2/4 > 0$, because of our condition on α. Therefore Gronwall's lemma implies that:

$$\begin{aligned} \int_\Omega &\left\{ \alpha \left(\frac{\partial u}{\partial t} \right)^2 + \left[\nabla \left(\frac{\partial u}{\partial t} \right) \right]^2 \right\} (x,t)\, dx \\ &\le e^{-2Ct} \int_\Omega \left\{ \alpha \left(\frac{\partial u}{\partial t} \right)^2 + \left[\nabla \left(\frac{\partial u}{\partial t} \right) \right]^2 \right\} (x,0)\, dx. \end{aligned}$$

The estimate (31) now follows.

To show (32), note first that (31) implies

$$\int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx \le C_1 e^{-C_2 t}$$

and using Hölder's inequality we also have, since Ω is bounded,

$$\int_\Omega \left| \frac{\partial u}{\partial t} \right| dx \le K_1 e^{-C_2 t}. \tag{42}$$

Equation (42) implies that

$$\left|\frac{\partial u}{\partial t}\right| \in L^1\left(0,\infty; L^1(\Omega)\right),$$

and also

$$\left|\frac{\partial}{\partial t}\left(\int_\Omega u(x,t)\,dx\right)\right| \in L^1(0,\infty).$$

Therefore, $\lim_{t\to\infty}\int_\Omega u(x,t)\,dx$ exists.

It also follows from (42) that for $t < s$

$$\int_\Omega |u(x,s) - u(x,t)|\,dx \le \frac{K_1}{C_2}\left(e^{-C_2 t} - e^{-C_2 s}\right). \tag{43}$$

Hence the system $\{u(.,t)\}_{t>0}$ is fundamental in $L^1(\Omega)$, which is a complete space. Therefore we can conclude that $u_\infty = \lim_{t\to\infty} u(.,t)$ exists and it also follows from (43) that

$$\|u_\infty - u(.,t)\|_{L^1} \le \frac{K_1}{C_2}\,e^{-C_2 t}. \tag{44}$$

In the same way we can get from the inequality (31) that

$$p_\infty = \lim_{t\to\infty} \nabla u(.,t) \tag{45}$$

exists in $L^1(\Omega)$ and a similar estimate to (44) holds, namely

$$\|p_\infty - \nabla u(.,t)\|_{L^1} \le \frac{K_1}{C_2}\,e^{-C_2 t}. \tag{46}$$

We want to show next that $p_\infty = \nabla u_\infty$. This can be done in the following way: It follows from Theorem 1 that

$$u(x,t) \text{ is bounded in the space } L^\infty(0,T;W^{1,2}(\Omega)). \tag{47}$$

Therefore it follows that there exists a sequence $s_n \uparrow \infty$ such that

$$u(x,s_n) \longrightarrow \widetilde{u}(x) \quad \text{weakly in} \quad W^{1,2}(\Omega). \tag{48}$$

But then it follows that $\widetilde{u}(x) = u_\infty$ and $\nabla\widetilde{u}(x) = p_\infty$ and this is precisely what we wanted. In equation (1), taking the limit as $t \to \infty$, we get from the combination of previous results that the weak limit $w_\infty = \lim_{t\to\infty} w(x,t)$ exists in $L^2(\Omega)$ and that w_∞ satisfies the equation

$$-\triangle u_\infty + w_\infty = f \tag{49}$$

weakly in $W^{1,2}(\Omega)$. □

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