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A SIMPLER FUBINI PROOF

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ABSTRACT. We give a new and simpler proof of the Fubini theorem. The proof uses a different definition of measurability which allows for a more geometric approach than usual.

1. Introduction. The definition of the product outer measure is straightforward. The measure of a set in the product is the infimum of outer approximation by sums of rectangular areas. The difficulties, and they are substantial, arise in the proof of the "obvious" relationship between the integral with respect to the product measure and the iterated integrals with respect to the measures on the two spaces. The basic problem is to show that if $\lambda = \mu \times \nu$, and E is λ-measurable, then

$$\lambda(E) = \iint \chi_E(x,y)\, d\nu(y)\, d\mu(x). \tag{1}$$

Equation (1) is the simplest case of the Fubini theorem, and also its essential core, for the general result follows easily from the special case (1).

The difficulty in verifying (1) lies in showing that the sections of a λ-measurable set E are measurable with respect to μ and ν, and that $\int \chi_E(x,y)\, d\nu(y)$ is a measurable function of x. If there is a suitable topology available, as for instance in $[0,1] \times [0,1]$, then compactness can be used to simplify matters. In the general (non-topological) case, the standard arguments all involve a skein of set theory which effectively hides the geometry. See, e.g., [**1**, pp. 135–147], [**2**, pp. 143–148], [**3**, pp. 303–310], [**4**, pp. 147–151].

We give here a proof for a general product which gives a clearer picture of how close approximation of a product set by rectangles forces a close approximation to sections by measurable sets in the factor spaces. The proof depends on a formally weaker condition for measurability than the usual Carathéodory condition. This condition

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is less mystical than the usual one and allows a more geometric proof that sections are measurable.

2. Product measure. We start with two complete finite measure spaces, $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$, where for simplicity we assume that $\mu(X) = \nu(Y) = 1$. Our approach provides some simplification even in the case $X = Y = [0, 1]$, where compactness is usually used to advantage.

Define an outer measure λ on $X \times Y$ by

$$\lambda(E) = \inf \left\{ \sum \mu(A_i)\nu(B_i) : E \subset \bigcup A_i \times B_i \right\}, \tag{2}$$

where the A_i and B_i are measurable sets in X and Y. We will refer to a set $A \times B$ with $A \in \mathcal{S}$ and $B \in \mathcal{T}$ as a *rectangle*.

The essence of the Fubini theorem is the identity (1). A function $f(x, y)$ which is λ-measurable can be approximated above and below by countable-valued simple functions

$$u(x, y) = \sum M_i \chi_{E_i}(x, y); \quad l(x, y) = \sum m_i \chi_{E_i}(x, y), \tag{3}$$

where the $\{E_i\}$ partition $X \times Y$ into disjoint sets, and there are partitions which make $u(x, y) - l(x, y)$ uniformly small. The λ-integrals of $u(x, y)$ and $l(x, y)$ are upper and lower Darboux sums for the integral of f:

$$\int u \, d\lambda = \sum M_i \lambda(E_i); \quad \int l \, d\lambda = \sum m_i \lambda(E_i). \tag{4}$$

If (1) holds, then the iterated integrals of $u(x, y)$ and $l(x, y)$ also equal the upper and lower Darboux sums (4). Hence the Fubini formula

$$\int f(x, y) \, d\lambda = \iint f(x, y) \, d\nu(y) \, d\mu(x) \tag{5}$$

follows easily from the special case (1).

3. Properties of λ. The function λ defined on subsets of $X \times Y$ by (2) is clearly an outer measure. To see that $\lambda(A \times B) = \mu(A)\nu(B)$, let $\{A_i \times B_i\}$ be a covering of $E = A \times B$ by rectangles. Then

$$\chi_E(x, y) = \chi_A(x)\chi_B(y) \leq \sum \chi_{A_i}(x)\chi_{B_i}(y). \tag{6}$$

Integrating first with respect to y and using the monotone convergence theorem, we get

$$\chi_A(x)\nu(B) \leq \sum \chi_{A_i}(x)\nu(B_i). \tag{7}$$

Now integrate with respect to x and get

$$\mu(A)\nu(B) \leq \sum \mu(A_i)\nu(B_i). \tag{8}$$

Hence the one-rectangle covering of $A \times B$ by itself is optimal, and $\lambda(A \times B) = \mu(A)\nu(B)$. The same integration argument shows that if $A \times B = \cup A_i \times B_i$, where the $A_i \times B_i$ are disjoint, then $\lambda(A \times B) = \sum \mu(A_i)\nu(B_i)$. Thus λ is finitely and countably additive on rectangles whose union is a rectangle. We will need that fact later.

We define a set $E \subset X \times Y$ to be *measurable*, with respect to λ, provided

$$\lambda(E) + \lambda(E') = 1, \tag{9}$$

where E' is the complement of E in $X \times Y$. This is essentially Lebesgue's original definition for sets on the line, with some transliteration and the topology rinsed off [**1**, p. 41]. We will use this formally weaker definition in all three spaces, X, Y, $X \times Y$, to show that sections of λ-measurable sets are measurable.

Lemma. *Rectangles are measurable.*

Proof. For the rectangle $A \times B$ we have

$$(A \times B)' = (A \times B') \cup (A' \times B) \cup (A' \times B'), \tag{10}$$

and the rectangles on the right are disjoint. By subadditivity of λ,

$$\begin{aligned}
\lambda(A \times B)' &\leq \lambda(A \times B') + \lambda(A' \times B) + \lambda(A' \times B') \\
&= \mu(A)(1-\nu(B)) + (1-\mu(A))\nu(B) + (1-\mu(A))(1-\nu(B)) \\
&= 1 - \mu(A)\nu(B).
\end{aligned}$$

Thus

$$\lambda(A \times B) + \lambda(A \times B)' \leq 1, \tag{11}$$

and $A \times B$ is measurable. $\square$

Lemma. *If E is a measurable subset of $X \times Y$, then for every set $T \subset X \times Y$,*

$$\lambda(E \cap T) + \lambda(E' \cap T) = \lambda(T). \tag{12}$$

Proof. We show first that measurable sets must split rectangles additively, and use this to show that they split every set additively. The following argument is the two dimensional version of the argument for subsets of $[0, 1]$. (See [**1**, pp. 28–31].)

Let $R_1 = A \times B$ be any rectangle, and let $\{r_i\} = \{a_i \times b_i\}$ be a covering of the measurable set E by rectangles, with

$$\sum \lambda(r_i) = \sum \mu(a_i)\nu(b_i) < \lambda(E) + \varepsilon. \tag{13}$$

Let $R_1 = A \times B$, $R_2 = A \times B'$, $R_3 = A' \times B$, $R_4 = A' \times B'$, so the rectangle $X \times Y$ is the disjoint union of the rectangles R_1, R_2, R_3, R_4. For $j = 1, 2, 3, 4$, let $r_{ij} = r_i \cap R_j$, so for fixed j, the rectangles r_{ij} cover $E \cap R_j$. Moreover, for each i,

$$r_{i1} \cup r_{i2} \cup r_{i3} \cup r_{i4} = r_i, \tag{14}$$

and the r_{ij} are disjoint, so

$$\lambda(r_{i1}) + \lambda(r_{i2}) + \lambda(r_{i3}) + \lambda(r_{i4}) = \lambda(r_i). \tag{15}$$

Hence

$$\begin{aligned} \lambda(E) + \varepsilon &> \sum_i \lambda(r_i) \\ &= \sum_i \lambda(r_{i1}) + \cdots + \sum_i \lambda(r_{i4}) \\ &\geq \lambda(E \cap R_1) + \cdots + \lambda(E \cap R_4) \\ &\geq \lambda(E). \end{aligned} \tag{16}$$

Since ε is arbitrary,

$$\lambda(E) = \lambda(E \cap R_1) + \lambda(E \cap R_2) + \lambda(E \cap R_3) + \lambda(E \cap R_4). \tag{17}$$

By the same argument,

$$\lambda(E') = \lambda(E' \cap R_1) + \lambda(E' \cap R_2) + \lambda(E' \cap R_3) + \lambda(E' \cap R_4). \tag{18}$$

Combining (17) and (18) and using the fact that E is measurable, we have

(19)
$$\begin{aligned} 1 &= \lambda(E) + \lambda(E') \\ &= \big[\lambda(E \cap R_1) + \lambda(E' \cap R_1)\big] + \cdots + \big[\lambda(E \cap R_4) + \lambda(E' \cap R_4)\big] \\ &\geq \lambda(R_1) + \cdots + \lambda(R_4) \\ &= 1. \end{aligned}$$

Hence equality holds in (19), and for each $j = 1, 2, 3, 4$,

$$\lambda(E \cap R_j) + \lambda(E' \cap R_j) = \lambda(R_j). \tag{20}$$

Since R_1 was an arbitrary rectangle, E splits all rectangles additively.

Now let T be any set and let $\{r_i\}$ be rectangles which cover T with

$$\lambda(T) + \varepsilon > \sum \lambda(r_i). \tag{21}$$

Then $E \cap T \subset \cup(E \cap r_i)$ and $E' \cap T \subset \cup(E' \cap r_i)$, so

$$\begin{aligned} \lambda(E \cap T) &\leq \sum \lambda(E \cap r_i), \\ \lambda(E' \cap T) &\leq \sum \lambda(E' \cap r_i)\,, \qquad (22) \\ \lambda(E \cap T) + \lambda(E' \cap T) &\leq \sum \big[\lambda(E \cap r_i) + \lambda(E' \cap r_i)\big] \\ &= \sum \lambda(r_i) \\ &< \lambda(T) + \varepsilon. \qquad (23) \end{aligned}$$

Since ε is arbitrary, and subadditivity is the other inequality, a measurable set E satisfies the Carathéodory criterion. $\square$

By standard arguments, e.g., [**1**, pp. 31–35], we now know that the λ-measurable sets form a σ-algebra, and λ is countably additive on the measurable sets.

4. The Fubini theorem. We want to show that almost all sections of a λ-measurable set are measurable. To discuss the "measure" of a set in X or Y which is not assumed to be in $\mathcal{S}$ or $\mathcal{T}$ we go backwards and define outer measures μ^* and ν^* in terms of μ and ν. If the measures μ and ν are obtained from outer measures, as for example if $X = Y = [0,1]$ and $\mu = \nu$ is Lebesgue measure, then μ^* and ν^* are the usual outer measures.

For any set $A \subset X$, let

$$\mu^*(A) = \inf\{\mu(B) : A \subset B \in \mathcal{S}\}. \tag{24}$$

If $\{B_n\}$ is a sequence of measurable sets in X which contain A, with $\mu(B_n) \to \mu^*(A)$, then $A \subset \cap B_n$ and $\mu^*(A) = \mu(\cap B_n)$. Thus every set $A \subset X$ is enclosed in a measurable set B with $\mu(B) = \mu^*(A)$.

Notice that μ^* is indeed an outer measure on X : $\mu^*(\phi) = 0$, $\mu^*(A) \geq 0$ for all A, and μ^* is monotone. To verify subadditivity, let $A_n \subset B_n$ with $\mu^*(A_n) = \mu(B_n)$ and B_n measurable. Then

$$\mu^*\Big(\bigcup A_n\Big) \leq \mu^*\Big(\bigcup B_n\Big) = \mu\Big(\bigcup B_n\Big) \leq \sum \mu(B_n) = \sum \mu^*(A_n). \tag{25}$$

Lemma. *A set A in X is measurable, i.e., in $\mathcal{S}$, if and only if*

$$\mu^*(A) + \mu^*(A') = 1. \tag{26}$$

Proof. Let $\mu^*(A) = \mu(B)$ and $\mu^*(A') = \mu(C)$ where $A \subset B$, $A' \subset C$, and B, C are measurable. Assume $\mu^*(A) + \mu^*(A') = 1$. Then since $B \cup C = X$,

$$1 = \mu^*(A) + \mu^*(A') = \mu(B) + \mu(C) \geq \mu(B \cup C) = 1. \tag{27}$$

Therefore $\mu(B \cap C) = 0$, and A, A' differ from the measurable sets B and C by sets of measure zero. Since μ and ν are assumed to be complete measures, A and A' are measurable. $\square$

For any set $E \subset X \times Y$ and any $x \in X$, let $E(x)$ be the vertical section of E through x, i.e.,

$$E(x) = \{y \in Y : (x, y) \in E\}. \tag{28}$$

If $E = A \times B$, then $E(x) = B$ if $x \in A$, and $E(x) = \varnothing$ if $x \notin A$. Hence for $E = A \times B$, with B measurable,

$$\nu(E(x)) = \begin{cases} \nu(B) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{29}$$

If A and B are measurable, then $\nu(E(x))$ is a measurable step function on X, and

$$\int \nu(E(x))\, d\mu(x) = \mu(A)\nu(B) = \lambda(A \times B). \tag{30}$$

Theorem. *If E is a measurable subset of $X \times Y$, then $E(x)$ is a measurable subset of Y for almost all x, and $\nu(E(x))$ is a measurable function on X, and*

$$\int \nu(E(x))\, d\mu(x) = \lambda(E). \tag{31}$$

Proof. Assume E is measurable, so $\lambda(E) + \lambda(E') = 1$. For each n, let $\{r_{ni}\}$ be a covering of E by rectangles with

$$\lambda(E) + \frac{1}{n} > \sum_i \lambda(r_{ni}). \tag{32}$$

Let $\{s_{ni}\}$ be a covering of E' by rectangles with

$$\lambda(E') + \frac{1}{n} > \sum_i \lambda(s_{ni}). \tag{33}$$

For each n and each $x \in X$, the sets $\{r_{ni}(x)\}$ cover $E(x)$ and the sets $\{s_{ni}(x)\}$ cover $E'(x)$. Therefore, for each x,

$$\begin{aligned} \nu^*(E(x)) &\le \sum_i \nu(r_{ni}(x)) \equiv \rho_n(x), \\ \nu^*(E'(x)) &\le \sum_i \nu(s_{ni}(x)) \equiv \sigma_n(x). \end{aligned} \tag{34}$$

Hence, for each x and each n,

$$1 \leq \nu^*(E(x)) + \nu^*(E'(x)) \leq \rho_n(x) + \sigma_n(x). \tag{35}$$

For each n and i, let $r_{ni} = a_{ni} \times b_{ni}$ and $s_{ni} = c_{ni} \times d_{ni}$, where a_{ni}, b_{ni}, c_{ni}, d_{ni} are measurable sets in X or Y. For each (n,i), by (30) we have

$$\int \nu(r_{ni}(x))\, d\mu(x) = \mu(a_{ni})\nu(b_{ni}) = \lambda(r_{ni}). \tag{36}$$

Similarly,

$$\int \nu(s_{ni}(x))\, d\mu(x) = \mu(c_{ni})\nu(d_{ni}) = \lambda(s_{ni}). \tag{37}$$

The functions $\rho_n(x)$, $\sigma_n(x)$ defined in (34) are sums of measurable step functions, so measurable, and

$$\begin{aligned} \int \rho_n(x)\, d\mu(x) &= \sum_i \int \nu(r_{ni}(x)\, d\mu(x)) \\ &= \sum_i \lambda(r_{ni}) \\ &< \lambda(E) + \frac{1}{n}, \qquad (38) \end{aligned}$$

$$\int \sigma_n(x)\, d\mu(x) < \lambda(E') + \frac{1}{n}. \tag{39}$$

Let $\rho'_n(x) = \min\{\rho_1(x), \dots, \rho_n(x)\}$ and $\sigma'_n(x) = \min\{\sigma_1(x), \dots, \sigma_n(x)\}$, so ρ'_n, σ'_n decrease to measurable functions $\rho(x)$, $\sigma(x)$, with

$$\begin{aligned} \int \rho(x)\, d\mu(x) &\leq \lambda(E) \\ \int \sigma(x)\, d\mu(x) &\leq \lambda(E'). \end{aligned} \tag{40}$$

From (35) we have $\rho(x) + \sigma(x) \geq 1$, so

$$1 \leq \int (\rho(x) + \sigma(x))\, d\mu(x) \leq \lambda(E) + \lambda(E') = 1. \tag{41}$$

It follows that $\rho(x) + \sigma(x) = 1$ almost everywhere and that equality holds in (40). From (35) we have

$$1 = \nu^*(E(x)) + \nu^*(E'(x)) \quad \text{a.e.}$$

Therefore $E(x)$ is measurable for almost all x, and

$$\int \nu(E(x))\, d\mu(x) = \int \rho(x)\, d\mu(x) = \lambda(E). \tag{42}$$

Equation (42) is the same as

$$\iint \chi_E(x, y)\, d\nu(y)\, d\mu(x) = \lambda(E).$$

The argument is symmetric in x and y, so the other iterated integral also equals $\lambda(E)$. $\square$

REFERENCES

1. H.S. Bear, *A primer of Lebesgue integration*, 2nd ed., Academic Press, San Diego, 2002.

2. Paul R. Halmos, *Measure theory*, D. Van Nostrand, New York, 1951.

3. H.L. Royden, *Real analysis*, 3rd ed., Macmillan, New York, 1988.

4. Walter Rudin, *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974.

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