ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 3, 2006

POINT X-RAYS OF CONVEX BODIES IN PLANES OF CONSTANT CURVATURE

PAOLO DULIO AND CARLA PERI

ABSTRACT. Let K be a convex body in a plane of constant curvature $\mathcal{R}^2$. The X-ray of K at a point $p \in \mathcal{R}^2$ gives the lengths of all the intersections of K with geodetics through p. Hammer's X-ray problem queries how many points must be taken to permit the exact reconstruction of K from the corresponding X-rays. The known answer in the Euclidean plane is here extended by proving that a convex body in $\mathcal{R}^2$ is distinguished from others by its X-rays from four point sources in general position, up to a reflection in the origin on the sphere. As a further generalization, we prove that three points in general position suffice when we work with the class of spherical lunes.

1. Introduction. In this paper we continue the discussion of Hammer's X-ray problem in non-Euclidean spaces, which was begun in [**2**] and continued in [**3**].

Our setting here is a complete simply connected Riemannian surface of constant curvature k which is, as is well known, either a Euclidean plane E^2, $k = 0$, or sphere S^2_k, $k > 0$, or hyperbolic plane H^2_k, $k < 0$. We shall scale the distance function induced by the Riemannian metric so that $k = 0, 1, -1$ and denote the spaces E^2, $S^2 := S^2_1$ and $H^2 := H^2_{-1}$ simply by $\mathcal{R}^2$.

Let $K \subset \mathcal{R}^2$ be a compact convex set with nonempty interior. The X-ray of K at a point $p \in \mathcal{R}^2$ gives the lengths of all the intersections of K with lines through p. Hammer's X-ray problem [**10**] (and its non-Euclidean generalization) queries how many points must be taken to permit the exact reconstruction of K from the corresponding X-rays.

In E^2 the answer is due to Volčič [**16**], who proved that four points, no three collinear, are enough to distinguish K among all other planar convex bodies. He was the first who employed the idea of using

2000 AMS *Mathematics Subject Classification.* Primary 52A55, 52A30.
Key words and phrases. Geometric tomography, point X-rays, spherical and hyperbolic convexity.
Received by the editors on August 1, 2003.

Copyright ©2006 Rocky Mountain Mathematics Consortium

a measure having appropriate invariance properties for X-rays, see Lemma 5.

On the other hand, when the sources are chosen in particular position, the number of points can be reduced, as was shown in earlier papers. In [**4**] and [**5**] Falconer proved that a convex body can be distinguished from others by X-rays at two inner points p_1 and p_2, and that the same holds, with some exceptions, if p_1 and p_2 are exterior. This was obtained through a version of the stable manifold theorem of differentiable dynamics, that also provided a method for the reconstruction of the body. A simplified approach was presented by Gardner in [**7**], who generalized the measure introduced by Volčič. Three noncollinear point X-rays determine a convex body in the interior of the triangle whose vertices are the points, [**6**]. If directed X-rays are considered, then three noncollinear point X-rays can distinguish a convex body among all planar convex bodies, [**16**] (see also Kurusa [**12**]).

Concerning the determination of a convex body K by X-rays in non-Euclidean planes, we proved in [**2**] that K is uniquely determined by its X-rays at two points p and q, provided the line through p and q is known to intersect the interior of K and p and q are either both interior or exterior to K. In [**3**] we added the case when the line supports K and showed that these results cannot be improved without restrictions on the curvature of the space.

The main result of the present paper is the generalization of Volčič's four points theorem. Namely, we prove that X-rays from four points, with no three collinear, distinguish between all convex bodies in $\mathcal{R}^2$, up to a reflection in the origin when $\mathcal{R}^2 = S^2$, see Theorem 8. Note that such an exception in the spherical setting cannot be avoided, since a convex body in S^2 and its reflection in the origin have equal X-ray at each point $p \in S^2$. We first show that X-rays at three noncollinear points determine uniquely a convex body, provided some a priori knowledge about the position of the body relative to the points is available, see Theorem 7. On the other hand, examples in the sphere show that three points may not suffice for determining uniquely a convex body, see [**3**, Example 8], while this holds true for a suitable class of spherical sets, namely the class of spherical lunes, see Theorem 14.

A comprehensive introduction to this and related subjects, together with updated references, can be found in Gardner's book [**8**].

2. Definitions and preliminary results. In this section we introduce some basic definitions and technical tools which will be employed in the next section where the main results are presented.

We first reformulate the notion of X-ray in terms of the exponential map. For notions and results from Riemannian geometry that are used without explanation, we refer to [**13**] and [**17**]. For $p \in \mathcal{R}^2$, let $T_p\mathcal{R}^2$ denote the tangent space to $\mathcal{R}^2$ at p. Each nonzero vector $v \in T_p\mathcal{R}^2$ can be written uniquely as ru, where u is a unit vector and r is a positive real number. The *exponential map* at p is the map $\exp_p : T_p\mathcal{R}^2 \to \mathcal{R}^2$ that sends the origin to p and the vector ru to the end point of the segment of length r on the geodesic γ_u issuing from p with initial vector u.

Let θ denote the angle between u and a fixed direction. The area element at p in terms of geodesic polar coordinates (r, θ) centered at p is then represented by

$$dq = \mathrm{sn}_k(r)\, dr\, d\theta,$$

with

$$\mathrm{sn}_k(r) := \begin{cases} r & \text{if } k = 0 \\ \sin r & \text{if } k = 1 \\ \sinh r & \text{if } k = -1. \end{cases} \tag{1}$$

If A is a set in $\mathcal{R}^2$, then $\mathrm{cl}\, A$, $\mathrm{bd}\, A$, $\mathrm{int}\, A$ and $\mathrm{conv}\, A$ denote the *closure*, the *boundary*, the *interior* and the *convex hull* of A, respectively. For properties of convex sets, see [**14**].

If $k = 1$ then $-A$ denotes the reflection of A in the origin. Further the characteristic function of a set $A \in \mathcal{R}^2$ will be denoted by 1_A.

Definition 1. Let A be a bounded measurable set in $\mathcal{R}^2$. The X-ray of set A at point p is defined by

$$X_pA(u) := \int_{-\infty}^{+\infty} 1_A(\exp_p(tu))\, dt, \tag{2}$$

for all unit vectors $u \in T_p\mathcal{R}^2$ for which the integral exists.

Remark 2. In $S^2 \subset \mathbf{E}^3$ the tangent planes at two antipodal points $p, -p \in S^2$ can both be identified with the subspace $p^\perp \subset \mathbf{E}^3$ orthogonal to p. Then for each bounded measurable set $A \subset S^2$ we have $X_{-p}A(u) = X_pA(u)$, for all unit vectors $u \in p^\perp$ for which the integral (2) exists.

In Euclidean spaces X-rays have been studied mainly within the class of star sets or that of convex bodies. Here our main results are stated for convex bodies. A compact set K with nonempty interior, which is contained in an open hemisphere when $k = 1$, is a *convex body* if any two points of K can be connected by a geodesic segment inside K. As for star sets, we refer to the notion given in [**2**], which extends that introduced by Gardner and Volčič in [**9**], see also [**8**, p. 18].

It should be noted that a convex body in Euclidean and hyperbolic plane is always star-shaped at any given point p, whereas this is not the case in the sphere. In fact, a convex body K in S^2 which contains a segment having the point $-p$ in its relative interior is not star-shaped at p. However, in this case $-K$ is star-shaped at p, so that the uniqueness results obtained in [**2**] for X-rays of star sets hold true even for convex bodies, up to a reflection in the origin when $k = 1$.

We also need the following trigonometric formulas. Let $\Delta \subset \mathcal{R}^2$ be a geodesic triangle with side lengths a, b, c, where $a, b, c < \pi$ in the spherical case and opposite angles α, β, γ. Then the Law of Cosines [**13**, p. 324] states

$$\begin{cases} c^2 = a^2 + b^2 - 2ab\cos\gamma & \text{if } k = 0 \\ \cos c = \cos a \cos b + \sin a \sin b \cos\gamma & \text{if } k = 1 \\ \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma & \text{if } k = -1. \end{cases}$$

Furthermore, if $\gamma = \pi/2$, then

$$\mathrm{sn}_k(a) = \mathrm{sn}_k(c)\sin\alpha, \quad \mathrm{sn}_k(b) = \mathrm{sn}_k(c)\sin\beta \tag{4}$$

with the conventions (1).

In [**2**] we introduced the following measure which provides a useful tool for testing whether two sets have the same X-rays at a given point.

Definition 3. Let γ be a given geodesic in $\mathcal{R}^2$. For a bounded measurable set A in $\mathcal{R}^2$, the measure $\nu_\gamma(A)$ is given by

$$\nu_\gamma(A) := \int_A |\mathrm{sn}_k(d(q,\gamma))|^{-1}\, dq,$$

where $d(q,\gamma)$ denotes the distance of q from γ. The geodesic γ is called the base line of ν_γ.

The measure ν_γ is finite on sets bounded away from γ and can be finite even on sets intersecting γ as the next lemma shows.

Lemma 4. *Let γ be a geodesic through a given point $p \in \mathcal{R}^2$ and let (r,θ) be polar coordinates centered at p, θ being evaluated with respect to the base line γ. For $0 < \alpha < \beta < \pi$ and $r_0 > 0$, where $r_0 < \pi$ in the spherical case, the sector defined by*

$$T(\alpha,\beta,r_0) := \{(r,\theta) : \, 0 \le r \le r_0, \ \alpha \le \theta \le \beta\}$$

has finite ν_γ measure.

Proof. This is easily proved by computing the measure ν_γ in the coordinates (r,θ). □

We now consider the behavior of the measure ν_γ for sets having the same X-rays at a given point.

The invariance property of the measure ν_γ for X-rays at points belonging to γ is stated by the following Lemma which easily follows from (4) by computing ν_γ in polar coordinates centered at p, see also [**2**].

Lemma 5. *Let γ be a geodesic through a given point $p \in \mathcal{R}^2$. Suppose that A_1 and A_2 are measurable sets in $\mathcal{R}^2$ with finite measure ν_γ and the same X-rays at p. Then $\nu_\gamma(A_1) = \nu_\gamma(A_2)$.*

The following lemma illustrates the behavior of the measure ν_γ when the point p does not belong to γ.

Lemma 6. *Let $p \in \mathcal{R}^2$ and let (r,θ) be polar coordinates centered at p. For $0<\alpha<\beta<\pi$, let*

$$E_j = \{(r,\theta) :\ 0 < r_j(\theta) \le r \le s_j(\theta),\ \alpha \le \theta \le \beta\},$$

for $j=1,2$, be star-shaped at p, with equal X-rays at p. Suppose also that $s_1(\theta) < r_2(\theta)$ for $\alpha \le \theta \le \beta$. Let γ be a given geodesic such that $p \notin \gamma$. Then

(i) *if $\mathcal{R}^2 = S^2$ and p, E_1, E_2 are contained in the same open hemisphere bounded by γ, then $\nu_\gamma(E_1) < \nu_\gamma(E_2)$;*

(ii) *if $\mathcal{R}^2 = H^2$ and γ separates p from E_1 and E_2, then $\nu_\gamma(E_1) > \nu_\gamma(E_2)$.*

Proof. (i) Let c be the center of the open hemisphere bounded by γ and containing p, E_1 and E_2. If $p=c$ then $\mathrm{sn}_k(d(q,\gamma)) = \cos r$ and

$$\nu_\gamma(E_j) := \int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} \tan r \, dr\, d\theta.$$

Since the integrand increases with r and $s_1(\theta)-r_1(\theta) = s_2(\theta) - r_2(\theta)$, with $s_1(\theta) < r_2(\theta)$, we have $\nu_\gamma(E_1) < \nu_\gamma(E_2)$, as required. If $p \ne c$, let (ρ,φ) denote the polar coordinates centered at c, where both θ and φ are evaluated with respect to the geodesic joining p and c. Then by (3) we get $\mathrm{sn}_k(d(q,\gamma)) = \cos\rho = \cos r \cos a + \sin r \sin a \cos\theta$, where a denotes the distance of p from c. Therefore, we have

$$\nu_\gamma(E_j) := \int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} \frac{1}{\cot r \cos a + \sin a \cos\theta}\, dr\, d\theta.$$

Since the integrand increases with r and $s_1(\theta)-r_1(\theta) = s_2(\theta) - r_2(\theta)$, with $s_1(\theta) < r_2(\theta)$, we have $\nu_\gamma(E_1) < \nu_\gamma(E_2)$, as required.

(ii) Let a denote the distance of p from γ, and let n be the foot of the perpendicular from p to γ. Denote by (ρ,φ) the polar coordinates centered at n, where φ is evaluated with respect to the geodesic line γ. Suppose also that θ is evaluated with respect to the geodesic joining p and n. Then by (3) we get $\mathrm{sn}_k(d(q,\gamma)) = \sinh\rho\sin\varphi = \cosh a \sinh r \cos\theta - \sinh a \cosh r$, so that

$$\nu_\gamma(E_j) := \int_\alpha^\beta \int_{r_j(\theta)}^{s_j(\theta)} \frac{1}{\cosh a \cos\theta - \sinh a \coth r}\, dr\, d\theta.$$

Since the integrand decreases with r and $s_1(\theta) - r_1(\theta) = s_2(\theta) - r_2(\theta)$, with $s_1(\theta) < r_2(\theta)$, we have $\nu_\gamma(E_1) > \nu_\gamma(E_2)$ as required. □

We also need the following notations. Let K and K' be convex bodies with $\operatorname{int}(K \triangle K') \neq \varnothing$, where $K \triangle K'$ denotes the symmetric difference of K and K', and let C be a connected component of $\operatorname{int}(K \triangle K')$. An *endpoint* of C is a point belonging to $\operatorname{cl} C \cap \operatorname{bd} K \cap \operatorname{bd} K'$. If K and K' have the same X-rays at a given point $p \in \mathcal{R}^2$ and $F \subset C$, let

$$F(p) := \cup(\delta \cap (\operatorname{int} K \triangle K')) \setminus F \tag{5}$$

where the union is taken over all the geodesics δ issuing from p such that $\delta \cap F \neq \varnothing$. If F is nearer to p than $F(p)$ we write $pF := F(p)$, whereas if $F(p)$ is nearer to p, we write $p^{-1}F := F(p)$. Thus either pF or $p^{-1}F$ is well defined.

3. Uniqueness theorems. We next prove a uniqueness result for X-rays from three point sources.

Theorem 7. *Let p_1, p_2, p_3 be three noncollinear points in $\mathcal{R}^2$, and let K be a convex body contained in the interior of the triangle formed by the points. If K' has the same X-rays as K at p_j, for $1 \leq j \leq 3$, then $K' = K$ when $k = 0, -1$, and $K' = \pm K$ when $k = 1$.*

Proof. In the Euclidean and hyperbolic plane we can repeat the same proof given in [**8**, Theorem 5.3.5]. In the spherical setting we need a different argument, as the area of two star sets with the same X-ray at a point p does not always increase with the distance from p. We use here the measure ν_γ. Suppose $K' \neq \pm K$ and denote the triangle formed by the points p_j, $1 \leq j \leq 3$, by T. Let us first suppose that $K' \subset \operatorname{int} T$. Then K and K' intersect and $\operatorname{int} K \triangle K' \neq \varnothing$. For $1 \leq i < j \leq 3$, let $\nu_{i,j}$ denote the measure with the geodesic through p_i and p_j as base line. Let μ be the measure defined by

$$\mu(A) := \nu_{1,2}(A) + \nu_{1,3}(A) + \nu_{2,3}(A),$$

for measurable sets $A \subset S^2$. Suppose C is a connected component of $\operatorname{int} K \triangle K'$ of maximum μ-measure. For some j, the set $p_j C$ is well defined and we may suppose $j = 1$, without loss of generality. Then by Lemma 5 we have $\nu_{1,2}(C) = \nu_{1,2}(p_1C)$, $\nu_{1,3}(C) = \nu_{1,3}(p_1C)$, while by Lemma 6 we have $\nu_{2,3}(C) < \nu_{2,3}(p_1C)$ so that $\mu(p_1C) > \mu(C)$, which is a contradiction. Now if $K' \nsubseteq \operatorname{int} T$ then $K' \subset \operatorname{int}(-T)$, since K' has the same X-rays as K at p_j, for $1 \le j \le 3$. Hence the set $K'' = -K'$ is contained in $\operatorname{int} T$ and by applying the previous argument to K'' we get $K'' = K$, so that $K' = -K$, as required. □

We now show that X-rays at four points in general position suffice for determining a convex body.

Theorem 8. *Convex bodies in $\mathcal{R}^2$ are determined by X-rays at any set of four points, with no three collinear, up to a reflection in the origin when $\mathcal{R}^2 = S^2$.*

Proof. Let K and K' be two convex bodies with the same X-rays at p_j, $1 \le j \le 4$. We first consider the Euclidean and hyperbolic cases, since the spherical case is slightly different as we show below. If two of the points p_j belong to $\operatorname{int} K$, then $K = K'$ by [**2**, Proposition 22]. Therefore we can assume that at least three points, say $p_j, j = 1, 2, 3$, do not belong to $\operatorname{int} K$. If the geodesic through two of the points, say p_1 and p_2, intersects the interior of K, then $p_1\ p_2 \notin \operatorname{int} K'$, and K, K' intersect the same component of such a geodesic, since they also have the same X-ray at p_3, p_4. Thus the result follows from [**2**, Theorem 28]. If the geodesic through two of the points supports K then $K = K'$ by [**3**, Theorem 1]. If K is contained in the interior of the triangle T whose vertices are the points p_j, $1 \le j \le 3$, then the result follows from Theorem 7. Then by permuting the indices if necessary, we may assume that K is contained in the cone determined by the geodesics through p_1 and p_j, $j = 2, 3$, and containing the triangle T. In this case we can apply the same argument used in [**8**, Theorem 5.3.6], where in the hyperbolic case Lemma 5.2.5 and Lemma 5.2.7 (ii) have to be replaced by Lemma 5 and Lemma 6 (ii), to get $K' = K$.

We now consider the spherical case.

If two of the points in $P := \{p_j, -p_j, 1 \le j \le 4\}$ belong to $\operatorname{int} K$, then the result follows from [**2**, Proposition 22], by Remark 2. Hence at most one of the points in P belongs to $\operatorname{int} K$, so that we can suppose $p_j, -p_j \notin \operatorname{int} K$, where $1 \leq j \leq 3$. If K does not intersect any geodesic through two of the points p_j, for $1 \leq j \leq 3$, then K is contained in the interior of one of the triangles with vertices belonging to $\{p_j, -p_j, 1 \leq j \leq 3\}$, so that the result follows from Theorem 7, by Remark 2. If the geodesic through two of the points p_j, for $1 \leq j \leq 3$, supports K then the result follows from [**3**, Theorem 1]. Thus, the case that remains is when the geodesic through two of the points p_j, for $1 \leq j \leq 3$, say p_1 and p_2, intersects the interior of K. This case has been considered in [**2**, Theorem 28], where nevertheless some additional assumptions on the mutual positions of K and the points p_1 and p_2 were required. However, our assumptions here enable us to get the result, as is shown below. Let us consider a hemisphere H such that $K \subset \operatorname{int} H$. Since the points p_j, for $1 \leq j \leq 3$, are noncollinear, then $p_i \neq -p_j$, for $i.j = 1, 2$, so that we can suppose $p_j \in \operatorname{int} H$, for $j = 1, 2$, and $\operatorname{int} H \cap \gamma \subseteq [p_1, p_2]$ ($[p_1, p_2]$ being the geodesic segment joining p_1 and p_2), up to exchanging the roles of p_j and $-p_j$, thanks to Remark 2. Denote the geodesic through p_1 and p_2 by γ. If γ supports K', then the result follows again from [**3**, Theorem 1]. Hence we have $\gamma \cap \operatorname{int} K' \neq \varnothing$ and $p_1,p_2 \notin \operatorname{int} K'$, since K and K' have the same X-rays at p_1 and p_2. Further, since K and K' also have the same X-rays at p_3 and p_4, then we have $\operatorname{int} K' \cap \gamma \subseteq [p_1, p_2]$, up to a reflection in the origin. This also implies $K' \subset H$, so that we can apply Theorem 28 proved in [**2**] to get $K' = K$, up to a reflection in the origin. □

We end this section by considering the case when the point sources are collinear. In the Euclidean plane it is unknown if X-rays at n collinear points suffice for determining a convex body, see [**8**, Problem 5.4]. Next we show that this question has a negative answer in the sphere.

Proposition 9. *For each $n \in \mathbf{N}$, there exists a set of n collinear points in S^2 such that there are different convex spherical polygons with the same X-rays at these points.*

Proof. Let $p \in S^2$ and let (r, θ) be geodesic polar coordinates centered at p. For $0 < \alpha < \pi/2$, consider the spherical regular n-gon Q with vertices $a_j := (\alpha, 2\pi j/n)$, for $1 \leq j \leq n$. Let Q' be obtained from Q by a rotation of π/n about p. Denote the vertices of Q' by a'_j, for $1 \leq j \leq n$. The points a_j, a'_j, for $1 \leq j \leq n$, are the vertices of a spherical regular $2n$-gon P. Denote by p_j, $-p_j$, for $1 \leq j \leq n$, the intersections of $p^\perp \cap S^2$ with the geodesic joining two consecutive vertices of P. By symmetry Q and Q' have the same X-rays at the set of collinear points p_j, for $1 \leq j \leq n$. $\square$

For odd values of n the spherical polygons Q, Q', introduced in the previous proof, have the same X-rays at the point p as well. This implies the following result.

Proposition 10. *For each $n \in \mathbf{N}$, there exists a set of $n+1$ points in S^2, n of them on a same geodesic γ, such that there are different convex polygons with the same X-rays at these points, which miss γ and contain the remaining point.*

For $n = 2$ this provides a negative answer to the spherical version of Problem 5.3 in [**8**] and shows that knowledge of X-rays at three points does not determine, in general, a convex body in a plane of constant curvature, see also [**3**, Example 8]. Hence Theorem 8 cannot be improved without restrictions on the curvature of the space.

4. On spherical lunes. In our definition a convex body in S^2 cannot contain pairs of antipodal points. It is worth remarking that convex sets on a convex surface are sometimes defined to be sets which contain with any two points x and y at least a geodesic segment joining the points x and y [**1**]. Further in some texts, see for instance [**11**], a set $K \subset S^n$ is said to be convex if the cone $o * K$ in $\mathbf{E}^{n+1}$ defined by

$$o * K := \{\lambda u : u \in K \text{ and } \lambda \geq 0\}$$

is convex in $\mathbf{E}^{n+1}$. Thus, according to this definition, in S^2 we did not include the set of lunes (a lune in S^n being the intersection of at most n hemispheres). We prefer to treat this case apart, chiefly because it renders a stronger result as we show below, see Theorem 14.

First let us observe that we cannot give unrestricted uniqueness results. In fact, since the X-ray of a hemisphere H at each point $p \in S^2 \setminus \operatorname{bd} H$ has constant value π, two hemispheres are not distinguished by their X-rays at any set of points which are not in their boundaries.

However, this is the unique exception, as we are going to show. We also note that any point in the interior of a hemisphere H is an equichordal point of H, according to the definition given in [**8**, pp. 225–226]. A family of convex bodies in S^2, different from hemispheres, with two equichordal points is shown in [**15**].

For the rest of this section, a *lune* in S^2 will always mean a set with nonempty interior which is the intersection of two distinct hemispheres. Therefore, a lune L contains exactly one pair of antipodal points $q, -q$, called the *poles* of L, and it will be denoted by $L(q)$ when its poles have to be specified. The length of the arc $L(q) \cap q^{\perp}$ will be called the *girth* of L.

For $x, y \in S^2$, we denote by $d(x, y)$ the spherical distance between x and y. By $[x, y]$ we also denote the geodesic segment with endpoints x and y, where $y \neq -x$. Further $d(x, A)$ will denote the distance of a point x from a set $A \subset S^2$.

We also need the following technical result.

Lemma 11. *Let H be a hemisphere and $q \in \operatorname{bd} H$. Consider a lune $L(q) \subset H$ with girth α and a geodesic γ through $y \in \operatorname{bd} H \cap q^{\perp}$ such that $\gamma \nsubseteq \operatorname{bd} H \cup q^{\perp}$. Let $\{a, b\} = \gamma \cap \operatorname{bd} L(q)$. Then $d(a, b)$ decreases with $\delta := \min\{d(y, L(q)), d(-y, L(q))\}$, when $L(q)$ varies within the set of the lunes with poles $\pm q$ and girth α.*

Proof. Consider two lunes $L_1(q), L_2(q) \subset H$, with the same girth α.

Let $\delta_i := \min\{d(y, L_i(q)), d(-y, L_i(q))\}$ and $\{a_i, b_i\} = \gamma \cap \operatorname{bd} L_i(q)$, where $i = 1, 2$. We have to prove that $\delta_1 < \delta_2$ implies $d(a_1, b_1) > d(a_2, b_2)$. Up to a reflection in the center of H, we can suppose $\delta_i = d(y, L_i(q))$, for $i = 1, 2$.

Suppose further that y, a_1, b_1, a_2, b_2 are in that order on γ. Let $n \in \gamma \cap H$ be the point having distance $\pi/2$ from y and suppose $d(q, n) = \min\{d(q, n), d(-q, n)\}$. For $i = 1, 2$, we have

$$\sin d(a_i, b_i) = \frac{\sin \alpha}{\sin d(q, n)} \sin d(q, a_i) \sin d(q, b_i).$$

Since $d(q, a_1) > d(q, a_2)$ and $d(q, b_1) > d(q, b_2)$, we obtain the result. The other cases can be treated similarly. $\square$

In [**2**, Proposition 25] it was proved that a lune is uniquely determined by its X-rays at two distinct inner points, up to reflection in the origin. This is not the case when the two points do not belong to the lune, as the following examples show, see also [**3**]. The first example considers the case when the geodesic through the points meets the interior of the lune, whereas in the second one the geodesic through the two points supports the lune.

Example 12. Let $q \in S^2$, and let (r, θ) be geodesic polar coordinates centered at q. For $0 < \alpha < \beta < \pi/2$, let L and L' be the lunes with poles $\pm q$ which are defined by

$$\begin{aligned} L &:= \{(r,\theta) :\ 0 \le r \le \pi,\ \alpha \le \theta \le \beta\}\,, \\ L' &:= \{(r,\theta) :\ 0 \le r \le \pi,\ \pi - \beta \le \theta \le \pi - \alpha\}\,. \end{aligned}$$

By symmetry L and L' have the same X-rays at $p_1 := (\pi/2, 0)$ and $p_2 := (\pi/2, \pi/2)$.

Example 13. Let $q \in S^2$ and (r, θ) be geodesic polar coordinates centered at q. Consider the lune L with poles $\pm q$, defined by

$$L := \{(r,\theta) :\ 0 \le r \le \pi,\ \pi/4 \le \theta \le 3\pi/4\}\,,$$

and let L' be obtained from L by a rotation of $\pi/2$ about the point $p := (\pi/2, \pi/2)$. By symmetry L and L' have the same X-rays at $p_1 := (\pi/4, 0)$ and $p_2 := (\pi/4, \pi)$.

The last example shows that Theorem 1 in [**3**] does not hold for lunes. Furthermore, in the previous examples the lunes L and L' also have the same X-rays at the points $-p_j$, $j = 1, 2$, by Remark 2. Therefore, three collinear points containing a pair of antipodal points in general do not distinguish a lune from the others. However, in contrast to the case of convex bodies, see Proposition 9, this is the unique exception, as we are going to show.

Theorem 14. *A lune in S^2 is determined by X-rays at any set of three points, noncontaining antipodal points, up to a reflection in the origin.*

Proof. Let $P := \{p_j,\, 1 \leq j \leq 3\}$ be a set of distinct points in S^2 noncontaining pairs of antipodal points, and let $L \subset S^2$ be a lune with poles $q, -q$. We can assume $p_j \neq q, -q$, for each j, with $1 \leq j \leq 3$, since otherwise L is uniquely determined by the X-rays at q, up to a reflection in the origin. Denote by γ_j the geodesic through q and p_j, for $1 \leq j \leq 3$.

If L contains two distinct points of the set $\{p_j, -p_j,\, 1 \leq j \leq 3\}$ in its interior, then the result follows from [**2**, Proposition 25].

Suppose that at most one of the points in the set $\{p_j, -p_j,\, 1 \leq j \leq 3\}$ belongs to the interior of L. Let L' be either a convex body or a lune with the same X-rays as L at p_j, for $1 \leq j \leq 3$.

First we show that L' is a lune as well. Suppose it is the contrary. If either $p_j \in L$ or $-p_j \in L$, for some j, then the geodesic joining p_j and q meets L and therefore also L' in a segment of length π, contrary to the assumption that L' is not a lune. Hence we can suppose $p_j, -p_j \notin L$, for $1 \leq j \leq 3$, so that γ_j meets L exactly at q and $-q$, while any other geodesic through p_j meets L in a segment of positive length. The same happens for L', since L' has the same X-rays as L at p_j, for $1 \leq j \leq 3$. For $1 \leq j \leq 3$, the geodesic γ_j is thus a support line of L' and $p_j \in L'$, since L' is not a lune. Further γ_j is the unique support line of L' at p_j. The convexity of L' gives a contradiction.

Hence let $L' \neq \pm L$ be a lune with poles $q', -q'$, which has the same X-rays as L at p_j, for $1 \leq j \leq 3$. We distinguish two cases.

Case 1. First let us suppose $\{q, -q\} \neq \{q', -q'\}$. For each j, with $1 \leq j \leq 3$, the geodesic γ_j meets L either in a segment of length π or at the two points $q, -q$. Since L' is a lune having the same X-rays as L at p_j, then $\{q', -q'\} \subset (\gamma_j \cap L')$.

If $\gamma_i \neq \gamma_j$, for some $i \neq j$, then $\gamma_i \cap \gamma_j = \{q, -q\} = \{q', -q'\}$, contrary to the assumption.

If $\gamma_1 = \gamma_2 = \gamma_3$, then γ_1 supports L and L', since otherwise at least two distinct points of $\{p_j, -p_j,\, 1 \leq j \leq 3\}$ belong to the interior of L or L', so that $L' = \pm L$ by [**2**, Proposition 25], contrary to the assumption. Let H be the hemisphere bounded by γ_1 which contains L. Up to a reflection in the origin we can assume $L' \subset H$ as well. Let us first suppose that $\gamma_1 \cap L = \{q, -q\}$. Then $\gamma_1 \cap L' = \{q,' -q'\}$ so that $L \cap L'$

is a quadrangle. Let a, b, c, d be the vertices of $L \cap L'$ and suppose that they are in that order on $\operatorname{bd}(L \cap L')$. Since L' has the same X-rays as L at p_j, then the points p_j belong either to the geodesic joining a and c or to the geodesic joining b and d. This contradicts the assumption that the set P does not contain a pair of antipodal points. Let us now suppose that γ_1 meets L in a segment of length π. Then $L \cap L'$ is a triangle T having two vertices at the poles of L and L', say q, q'. Let v be the third vertex of T. The points p_j, for $1 \leq j \leq 3$, do not belong to the side $[q, q']$ of T, since each geodesic joining p_j to a point in the relative interior of $[v, q]$ meets L and L' in segments of different lengths. Thus we can suppose $p_j \in [q, -q']$, for $1 \leq j \leq 3$, up to exchange the roles of p_j and $-p_j$, if necessary. For $p \in [q, -q']$, let $\alpha := d(p, q), \delta := d(q, q')$. If $\alpha \neq (\pi - \delta)/2$, then $d(p, q') \neq d(q, -p)$, so that the geodesic joining p to a point $a \in [v, q']$ sufficiently close to q' meets L and L' in segments of different lengths. This gives a contradiction.

Case 2. Let now $\{q, -q\} = \{q', -q'\}$. First suppose that one of the geodesics γ_j, say γ_1, meets L just at q and $-q$. Denote by H the hemisphere bounded by γ_1 which contains L. Since L' has the same X-rays as L at p_1, then $\gamma_1 \cap L' = \{q, -q\}$, so that we can assume $L' \subset H$, up to a reflection in the origin. Further, we can suppose $p_2, p_3 \in H$ by Remark 2. Denote by α, α' the girth of L, L', respectively. We can assume $\alpha \leq \alpha'$, by exchanging the roles of L and L', if necessary. We select a point $x \in q^{\perp} \cap H$ such that $0 < d(x, L') < \min\{d(x, L), d(-x, L)\}$. Let $L''(q) \subseteq L'(q)$ be a lune with girth α such that $d(x, L'') = d(x, L')$. Let us first suppose that the points p_j are noncollinear. Then at least one of them does not belong to $q^{\perp}$, say p_i. Consider the geodesic δ joining x and p_i. By Lemma 11, δ meets L'' in an arc which is longer than the corresponding arc in L. This contradicts the assumption that L and L' have the same X-rays at p_i. Now let us suppose that the points p_j belong to a same geodesic γ. If $\gamma \neq H \cap q^{\perp}$ then again one of the points p_j does not belong to $q^{\perp}$ and we can repeat the previous argument to get a contradiction. If $\gamma = H \cap q^{\perp}$ then $\alpha = \alpha'$, since L and L' have the same X-rays at p_j. By applying Lemma 11 to a geodesic δ_j through p_j such that $\delta_j \neq \gamma, \gamma_j$, we get $\min\{d(p_j, L), d(-p_j, L)\} = \min\{d(p_j, L'), d(-p_j, L')\}$, for $0 \leq j \leq 3$. This gives a contradiction, since P does not contain pairs of antipodal points.

Let us now suppose that each geodesic γ_j, for $1 \leq j \leq 3$, meets L in a segment of length π. At least one of these geodesics, say γ_1, has empty intersection with $\operatorname{int} L \cup \operatorname{int} L'$, since otherwise one of the lunes L, L' contains at least two points of the set $\{p_j, -p_j, 1 \leq j \leq 3\}$ in its interior, contrary to the assumption. Let H be the hemisphere bounded by γ_1 which contains L. Up to a reflection in the origin, we may assume $L' \subset H$ and $p_2, p_3 \in H$ as well. If $\gamma_1 \cap L' = \gamma_1 \cap L$, then $L = L'$, since L and L' have the same X-rays at p_1. This gives a contradiction. Therefore, let $\gamma_1 \cap L' \neq \gamma_1 \cap L$. We can assume $p_1 \in \gamma_1 \cap L$, by exchanging the roles of L and L', if necessary. If $p_1 \notin q^{\perp}$, then there exists a geodesic η through p_1 which meets $L \cap (L \triangle L')$ in an arc longer than $\pi/2$, so that η meets L' in an arc shorter than $\pi/2$ since p_1 is an equichordal point of H. This contradicts the assumption that L and L' have the same X-rays at p_1. If $p_1 \in q^{\perp}$, then L and L' have the same girth α. By applying Lemma 11 to a geodesic δ_1 through p_1 other than γ_1 and $q^{\perp} \cap H$, we get $\min\{d(p_1, L), d(-p_1, L)\} = \min\{d(p_1, L'), d(-p_1, L')\}$. Further at least one of the points p_2, p_3, say p_2, differs from the center c of H. Thus, we can find a geodesic η through p_2 which cut arcs with different lengths in L and L', contrary to the assumption that L and L' have the same X-rays at p_2. $\square$

REFERENCES

1. H. Busemann, *Convex surfaces*, Interscience, New York, 1958.

2. P. Dulio and C. Peri, *Uniqueness theorems for convex bodies in non-Euclidean spaces*, Mathematika **49** (2002), 13–31.

3. ———, *On Hammer's X-ray problem in spaces of constant curvature*, Rend. Circ. Mat. Palermo (2) Suppl. **70** (2002), 229–236.

4. K.J. Falconer, *Hammer's X-ray problem and the stable manifold theorem*, J. London Math. Soc. **28** (1983), 149–160.

5. ———, *X-ray problem for point sources*, J. London Math. Soc. **46** (1983), 241–262.

6. R.J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. **41** (1983), 183–189.

7. ———, *Chord functions of convex bodies*, J. London Math. Soc. **36** (1987), 314–326.

8. ———, *Geometric tomography*, Cambridge Univ. Press, New York, 1995.

9. R.J. Gardner and A. Volčič, *Tomography of convex and star bodies*, Adv. Math. **108** (1994), 367–399.

10. P.C. Hammer, *"Problem 2"*, in *Convexity*, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, 1963, pp. 498–499.

11. D.A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge Univ. Press, Cambridge, 1997.

12. A. Kurusa, *Generalized X-ray pictures*, Publ. Math. Debrecen **48** (1996), 193–198.

13. P. Petersen, *Riemannian geometry*, Grad. Texts in Math., Springer, Berlin, 1997.

14. R. Schneider, *Convex bodies*: *The Brunn-Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

15. N. Spaltenstein, *A family of curves with two equichordal points on a sphere*, Amer. Math. Monthly **91** (1984), 423.

16. A. Volčič, *A three-point solution to Hammer's X-ray problem*, J. London Math. Soc. **34** (1986), 349–359.

17. T.J. Willmore, *Riemannian geometry*, Clarendon Press, Oxford, 1993.

Dipartimento di Matematica, "F. Brioschi," Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy
E-mail address: paodul@mate.polimi.it

Università Cattolica S.C., Largo Gemelli 1, I-20123 Milano, Italy
E-mail address: carla.peri@unicatt.it