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BOUNDARY VALUE PROBLEM FOR SECOND-ORDER DIFFERENTIAL OPERATORS WITH NONREGULAR INTEGRAL BOUNDARY CONDITIONS

M. DENCHE AND A. KOURTA

ABSTRACT. In this paper, we study a nonregular second order differential operator with weighted integral boundary conditions. Under certain conditions on the weighting functions which occur in the integral boundary conditions expressed in terms of the values at the interval endpoints, we prove that the resolvent has no minimal growth. Furthermore, the studied operator generates an analytic semi-group with singularities in $L^1(0,1)$. The obtained results are then used to show the correct solvability of a mixed problem for a parabolic partial differential equation with nonregular integral boundary conditions.

1. Introduction. In space $L^1(0,1)$, we consider the boundary value problem

$$(1)\qquad \begin{cases} L(u) = u''(x) = f(x), \\ B_i(u) = \displaystyle\int_0^1 R_i(t)u(t)\,dt + \int_0^1 S_i(t)u'(t)\,dt = 0 \quad i = 1, 2, \end{cases}$$

where functions R_i, $S_i \in C([0,1], \mathbf{C})$, $i = 1, 2$. We associate to problem (1) in space $L^1(0,1)$ the operator

$$L_1(u) = u'',$$

with domain $D(L_1) = \{u \in W^{2,1}(0,1) : B_i(u) = 0,\ i = 1, 2\}$.

Many papers and books give the full spectral theory of Birkhoff regular differential operators with two point linearly independent boundary conditions, in terms of coefficients of boundary conditions. The reader should refer to [**7, 11, 21–23, 30, 33, 35**] and references therein. Few works were devoted to the study of a nonregular situation. The

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case of separated nonregular boundary conditions was studied by Eberhard, Hopkins, Jakson, Keldysh, Khromov, Seifert, Stone, Ward (see S. Yakubov and Y. Yakubov [**35**] for exact references). A situation of nonregular nonseparated boundary conditions was considered by Benzinger [**2**], Denche [**4**], Eberhard and Freiling [**8**], Gasumov and Magerramov [**13, 14**], Khromov [**19**], Mamedov [**20**], Shkalikov [**26**], Silchenko [**27**], Tretter [**31**], Vagabov [**32**], S. Yakubov [**34**] and Y. Yakubov [**36**]. As was mentioned in [**35**], some nonstandard results exist for nonregular boundary value problems in contrast to the classical regular situation. The latter motivates more the study of such problems.

A mathematical model with integral boundary conditions was derived by [**9, 24**] in the context of optical physics. The importance of this kind of problems have been also pointed out by Samarskii [**25**].

In this paper, we study a problem for second order ordinary differential equations with nonregular integral boundary conditions, the regular case was studied in [**12**]. In the case $S_i\,(t) = 0$ and nonregular boundary conditions, more general problems were studied by Yu. T. Silchenko [**28**] and not only in $L_1\,(0,1)$ space, but also in $L_p\,(0,1)$ space. This work of Yu. T. Silchenko was published before Gallardo's work [**12**]. In [**10, 29**], a situation of a variable coefficient of $u''\,(x)$ in the equation has been treated. The integral boundary conditions are again nonregular but they assume less restrictions on the functions $R_i\,(t)$ (here again $S_i\,(t) = 0$). In particular the corresponding estimate in $L_2\,(0,1)$ has been established.

Following the technique in [**12, 21**–**23**], we should bound the resolvent in the space $L^1\,(0,1)$ by means of a suitable formula for Green's function. We show, in particular, that the resolvent decreases with respect to the spectral parameter, but there is no maximal decreasing at infinity in contrast to the regular case. Furthermore, the studied operator generates an analytic semi-group with singularities [**27**] in $L^1\,(0,1)$. The obtained results are then used to show the correct solvability of a mixed problem for a parabolic partial differential equation with non regular integral boundary conditions.

2. Green's function.

Let $\lambda \in \mathbf{C}$, $u_1(x) = u_1(x,\lambda)$, and let $u_2(x) = u_2(x,\lambda)$ be a fundamental system of solutions of equation

$$L\,(u) - \lambda u = 0.$$

Following [**21**], the Green's function of problem (1) is given by:

$$G(x,s,\lambda)=\frac{N(x,s,\lambda)}{\Delta(\lambda)}, \tag{2}$$

where $\Delta(\lambda)$ is the characteristic determinant of the considered problem defined by

$$\Delta(\lambda)=\begin{vmatrix} B_1(u_1) & B_1(u_2)\\ B_2(u_1) & B_2(u_2)\end{vmatrix} \tag{3}$$

and

$$N(x,s,\lambda)=\begin{vmatrix} u_1(x) & u_2(x) & g(x,\ s,\ \lambda)\\ B_1(u_1) & B_1(u_2) & B_1(g)_x\\ B_2(u_1) & B_2(u_2) & B_2(g)_x\end{vmatrix} \tag{4}$$

for $x,\ s\in[0,\ 1]$. The function $g(x,\ s,\ \lambda)$ is defined as follows

$$g(x,s,\lambda)=\pm\,\frac{1}{2}\,\frac{u_1(x)u_2(s)-u_1(s)u_2(x)}{u_1'(s)u_2(s)-u_1(s)u_2'(s)} \tag{5}$$

where it takes the plus sign for $x>s$ and the minus sign for $x<s$.

Given an arbitrary $\delta\in(0,(\pi/2))$, we consider the sector

$$\sum\nolimits_\delta=\left\{\lambda\in C\ :\ |\arg(\lambda)|\le\frac{\pi}{2}+\delta,\ \lambda\neq 0\right\}.$$

For $\lambda\in\sum_\delta$, define ρ as the square root of λ with positive real part. For $\lambda\neq 0$, we can consider a fundamental system of solutions of equation $u''=\lambda u=\rho^2u$ given by $u_1(t)=e^{-\rho t}$ and $u_2(t)=e^{\rho t}$.

In the following we are going to deduce adequate formulae for $\Delta\left(\lambda\right)$ and $G\left(x,s;\lambda\right)$. First of all, for $i,j=1,2$, we have

$$B_i(u_j)=\int_0^1 R_i(t)e^{(-1)^j\rho t}\,dt+(-1)^j\rho\int_0^1 S_i(t)e^{(-1)^j\rho t}\,dt,$$

so we obtain from (3)

$$\begin{aligned}\Delta(\lambda)=&\left(\int_0^1(R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\left(\int_0^1(R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\\&-\left(\int_0^1(R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_0^1(R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right),\end{aligned} \tag{6}$$

and $g(x,s;\lambda)$ has the form

$$g(x,s;\lambda)=\begin{cases}1/4\,\rho\,(e^{\rho(x-s)}-e^{\rho(s-x)}) & \text{if } x>s,\\ 1/4\,\rho\,(e^{\rho(s-x)}-e^{\rho(x-s)}) & \text{if } x<s.\end{cases}$$

Thus we have

$$\begin{aligned}&B_i(g)\\ &=\frac{e^{\rho s}}{4\rho}\left(\int_0^s(R_i(t)-\rho S_i(t))e^{-\rho t}\,dt+\int_s^1(-R_i(t)+\rho S_i(t))e^{-\rho t}dt\right)\\ &\quad+\frac{e^{-\rho s}}{4\rho}\left(-\int_0^s(R_i(t)+\rho S_i(t))e^{\rho t}\,dt+\int_s^1(R_i(t)+\rho S_i(t))e^{\rho t}\,dt\right).\end{aligned}$$

After a long calculation, formula (4) can be written as

$$N(x,s;\lambda)=\varphi(x,s;\lambda)+\varphi_i(x,s;\lambda), \tag{7}$$

where

(8)

$$\begin{aligned}&\varphi(x,s;\lambda)\\ &=\frac{e^{\rho(x+s)}}{2\rho}\left[\left(\int_0^s(R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\left(\int_s^1(R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\right.\\ &\qquad\left.\left(\int_0^s(R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\left(\int_s^1(R_1(t)-\rho S_1(t))e^{-\rho t}dt\right)\right]\\ &\quad+\frac{e^{-\rho(x+s)}}{2\rho}\left[\left(\int_0^s(R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_s^1(R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\right.\\ &\qquad\left.-\left(\int_s^1(R_1(t)+\rho S_1(t))e^{\rho t}dt\right)\left(\int_0^s(R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\right],\end{aligned}$$

and the function $\varphi_i(x,s;\lambda)$ is given by

$$\varphi_i(x,s;\lambda)=\begin{cases}\varphi_1(x,s;\lambda) & \text{if } x>s,\\ \varphi_2(x,s;\lambda) & \text{if } x<s,\end{cases}, \tag{9}$$

with

(10)

$$
\begin{aligned}
&\varphi_1(x,s;\lambda)\\
&= \frac{e^{\rho(s-x)}}{2\rho}\Bigg[\left(\int_0^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_0^s (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\\
&\qquad -\left(\int_0^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\left(\int_0^s (R_1(t)-\rho S_1(t))e^{-\rho t}dt\right)\Bigg]\\
&\quad + \frac{e^{\rho(x-s)}}{2\rho}\Bigg[\left(\int_0^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\left(\int_0^s (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\\
&\qquad -\left(\int_0^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\left(\int_0^s (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\Bigg],
\end{aligned}
$$

and

(11)

$$
\begin{aligned}
&\varphi_2(x,s;\lambda)\\
&= \frac{e^{\rho(s-x)}}{2\rho}\Bigg[\left(\int_0^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\left(\int_s^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\\
&\qquad -\left(\int_0^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_s^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\Bigg]\\
&\quad + \frac{e^{\rho(x-s)}}{2\rho}\Bigg[\left(\int_s^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_0^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\\
&\qquad -\left(\int_0^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\left(\int_s^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\Bigg].
\end{aligned}
$$

3. Bounds on the resolvent. Every $\lambda \in \mathbf{C}$ such that $\Delta(\lambda) \neq 0$ belongs to $\rho(L_1)$, and the associated resolvent operator $R(\lambda, L_1)$ can be expressed as a Hilbert-Schmidt operator

(12)

$$
(\lambda I - L_1)^{-1} f = R(\lambda : L_1) f = -\int_0^1 G(.\,, s;\lambda) f(s)\,ds, \quad f \in L^1(0,1).
$$

Then, for every $f \in L^1(0,1)$ we estimate (12)

$$\|R(\lambda, L_1)f\|_{L^1(0,1)} \leq \left(\sup_{0\leq s\leq 1}\int_0^1 |G(x,s;\lambda)|\ dx\right)\|f\|_{L^1(0,1)}\,, \tag{13}$$

and so we need to bound

$$\sup_{0\leq s\leq 1}\int_0^1 |G(x,s;\lambda)|\ dx = \frac{1}{|\Delta(\lambda)|}\sup_{0\leq s\leq 1}\int_0^1 |N(x,s;\lambda)|\ dx.$$

3.1 Estimation of $N(x,s,\lambda)$. We will denote by $\|.\|$ the supremum norm. From (7) and (8) it follows

$$\begin{aligned}
&\int_0^1 |N(x,s;\lambda)|\ dx \\
&\quad= \int_0^1 |\varphi_i(x,s;\lambda)|\ dx + \frac{e^{Re(\rho)}-1}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right) \\
&\qquad\times\left(\|R_2\|+|\rho|\,\|S_2\|\right)\left(e^{sRe(\rho)}-1\right)\left(e^{-sRe(\rho)}-e^{-Re(\rho)}\right) \\
&\qquad+\frac{1-e^{-Re(\rho)}}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right) \\
&\qquad\times\left(\|R_2\|+|\rho|\,\|S_2\|\right)\left(1-e^{-sRe(\rho)}\right)\left(e^{Re(\rho)}-e^{sRe(\rho)}\right).
\end{aligned}$$

Using (9) we write $\int_0^1 |\varphi_i(x,s;\lambda)|\ dx$ as follows

$$\int_0^1 |\varphi_i(x,s;\lambda)|\ dx = \int_0^s |\varphi_2(x,s;\lambda)|\ dx + \int_s^1 |\varphi_1(x,s;\lambda)|\ dx,$$

from (11), we have

$$\begin{aligned}
&\int_0^s |\varphi_2(x,s;\lambda)|\ dx \\
&\leq \frac{2}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right) \\
&\quad\times\left(e^{Re(\rho)}-e^{-Re(\rho)}-e^{sRe(\rho)}+e^{-sRe(\rho)}-e^{(1-s)Re(\rho)}+e^{(s-1)Re(\rho)}\right),
\end{aligned}$$

and from (10), we have

$$\begin{aligned}
&\int_s^1 |\varphi_1(x,s;\lambda)|\, dx \\
&\le \frac{2}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right) \\
&\quad \times (e^{Re(\rho)}-e^{-Re(\rho)}-e^{sRe(\rho)}+e^{-sRe(\rho)}-e^{(1-s)Re(\rho)}+e^{(s-1)Re(\rho)}).
\end{aligned}$$

From the previous inequalities we have

$$\begin{aligned}
&\int_0^1 |N(x,s;\lambda)|\, dx \\
&\le \frac{6}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right) \\
&\quad \times (e^{Re(\rho)}-e^{-Re(\rho)}-e^{sRe(\rho)}+e^{-sRe(\rho)}-e^{(1-s)Re(\rho)}+e^{(s-1)Re(\rho)}).
\end{aligned}$$

Since $\mathrm{Re}(\rho)>0$, and $0\le s\le 1$, then

$$\begin{aligned}
\sup_{0\le s\le 1}&\int_0^1 |N(x,s;\lambda)|\, dx \\
&\le \frac{6e^{Re(\rho)}}{|\rho|\,(Re(\rho))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right).
\end{aligned}$$

Note that $|\arg\rho|<(\delta/2)+(\pi/4)$, so $Re(\rho)\ge|\rho|\cos((\delta/2)+(\pi/4))$ >0, and so the following inequality holds

$$\begin{aligned}
\sup_{0\le s\le 1}&\int_0^1 |N(x,s;\lambda)|\, dx \\
&\le \frac{6e^{Re(\rho)}}{|\rho|^4\,(\cos((\delta/2)+(\pi/4)))^3}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right).
\end{aligned}$$

Then from (13), we obtain the following estimate

$$\begin{aligned}
&\|R(\lambda;L_1)\| \\
&\le \frac{6e^{Re(\rho)}}{(\cos((\delta/2)+(\pi/4)))^3\,|\rho|^4\,|\Delta(\lambda)|}\left(\|R_1\|+|\rho|\,\|S_1\|\right)\left(\|R_2\|+|\rho|\,\|S_2\|\right).
\end{aligned} \tag{14}$$

3.2 Estimation of the characteristic determinant. The next step is to determine the cases for which $|\Delta(\lambda)|$ remains bounded below. It will then be necessary to bound $|\Delta(\lambda)|$ appropriately. However, formula (6) is not useful for this purpose, it will be then necessary to make some additional regularity hypotheses on the functions R_i and S_i. We note that, in the regular case [**12**], the functions R_i and S_i are assumed only in $C^1([0,1],\mathbf{C})$. In our case, we suppose that R_i, $S_i \in C^2([0,1],\mathbf{C})$.

Integrating twice by parts in (6) to get

(15)

$$\begin{aligned}\Delta(\lambda) = e^{\rho}\Big[&\big(S_2(0)S_1(1) - S_1(0)S_2(1)\big) + \frac{1}{\rho}\big(R_1(0)S_2(1) - R_2(0)S_1(1)\\ &+ S_2(0)R_1(1) - S_1(0)R_2(1) + S_1(1)S_2'(0) - S_1'(1)S_2(0)\\ &+ S_1(0)S_2'(1) - S_2(1)S_1'(0)\big) + \frac{1}{\rho^2}\big(R_1(0)R_2(1) - R_1(1)R_2(0)\\ &- R_2(0)S_1'(1) - S_2(0)R_1'(1) - R_1(0)S_2'(1) + S_1(0)R_2'(1)\\ &- R_2(1)S_1'(0) + S_2(1)R_1'(0) + R_1(1)S_2'(0) - S_1(1)R_2'(0)\big)\\ &+ \frac{1}{\rho^3}\big(R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0)\\ &- R_1(1)R_2'(0)\big) + \frac{\Phi(\rho)}{\rho^2}\Big],\end{aligned}$$

where

$$\begin{aligned}\Phi(\rho) = &\left(\int_0^1 (R_2'(t) - \rho S_2'(t))e^{-\rho t}\,dt\right)\left(\int_0^1 (R_1'(t) + \rho S_1'(t))e^{\rho(t-1)}\,dt\right)\\ &- \left(\int_0^1 (R_1'(t) - \rho S_1'(t))e^{-\rho t}\,dt\right)\left(\int_0^1 (R_2'(t) + \rho S_2'(t))e^{\rho(t-1)}\,dt\right)\\ &+ \frac{1}{\rho}\Big\{(R_2(0) - \rho S_2(0))\left(\int_0^1 (R_1''(t) + \rho S_1''(t))e^{\rho(t-1)}\,dt\right)\\ &\quad + (R_2(1) - \rho S_2(1))\left(\int_0^1 (R_1''(t) + \rho S_1''(t))e^{\rho(t-2)}\,dt\right)\\ &\quad + (R_1(0) - \rho S_1(0))\left(\int_0^1 (R_2''(t) + \rho S_2''(t))e^{\rho(t-1)}\,dt\right)\end{aligned}$$

$$
\begin{aligned}
&- (R_1(1) - \rho S_1(1)) \left(\int_0^1 (R_2''(t) + \rho S_2''(t)) e^{\rho\,(t-2)}\,dt\right)\\
&+ (R_2(1) + \rho S_2(1)) \left(\int_0^1 (R_1''(t) - \rho S_1''(t)) e^{-\rho\,t}\,dt\right)\\
&- (R_1(1) + \rho S_1(1)) \left(\int_0^1 (R_2''(t) - \rho S_2''(t)) e^{-\rho\,t}\,dt\right)\\
&+ (R_1(0) + \rho S_1(0)) \left(\int_0^1 (R_2''(t) - \rho S_2''(t)) e^{-\rho\,(t+1)}\,dt\right)\\
&- (R_2(0) + \rho S_2(0)) \left(\int_0^1 (R_1''(t) - \rho S_1''(t)) e^{-\rho\,(t+1)}\,dt\right)\Bigg\}\\
+ 2e^{-\rho}\Bigg\{&\rho\big(R_2(1)S_1(1) - R_1(1)S_2(1) + R_1(0)S_2(0)\\
&- R_2(0)S_1(0) + S_2(0)S_1'(0) - S_1(0)S_2^{'}(0)\\
&- S_1(1)S_2'(1) + S_1'(1)S_2(1)\big) + \frac{1}{\rho}\big(- R_2(0)R_1'(0)\\
&- R_2(1)R_1^{'}(1) + R_1(1)R_2'(1) + R_1(0)R_2'(0)\big)\Bigg\}\\
+ e^{-2\rho}\Bigg\{&\frac{1}{\rho}\Big[(-R_2'(0)+\rho S_2'(0))\,(R_1(1)-\rho S_1(1))+((R_2(1)-\rho S_2(1))\\
&\times (R_1'(0)+\rho S_1'(0))+(R_2(0)+\rho S_2(0))\,(R_1'(1)-\rho S_1'(1))\\
&- (R_1(0) + \rho S_1(0))\,(R_2'(1) - \rho S_2'(1))\Big]\\
&+ (R_1(1) - \rho S_1(1))(R_2(0) + \rho S_2(0))\\
&- (R_1(0) + \rho S_1(0))(R_2(1) - \rho S_2(1))\Bigg\}.
\end{aligned}
$$

After a straightforward calculation, we obtain the following inequality valid for ρ, where $\rho \in \Sigma_\rho = \{\rho \in \mathbf{C} : |\arg\rho| \le \pi/4 + \delta/2,\ \rho \neq 0\}$, $0 < \delta < \pi/2$, with $Re(\rho)$ sufficiently large.

$$
\begin{aligned}
|\Phi(\rho)| \le &\frac{2}{(\cos((\delta/2) + (\pi/4)))^2}\left(\frac{1}{|\rho|}\,\|R_1'\| + \|S_1'\|\right)\\
&\times \left(\frac{1}{|\rho|}\,\|R_2'\| + \|S_2'\|\right) + \frac{1}{\cos((\delta/2) + (\pi/4))}
\end{aligned}
$$

$$
\begin{aligned}
(16)\qquad &\times\Bigg[\left(\frac{1}{|\rho|}|R_2(0)|+|S_2(0)|\right)\left(\frac{1}{|\rho|}\|R_1''\|+\|S_1''\|\right)\\
&\quad+\left(\frac{1}{|\rho|}|R_2(1)|+|S_2(1)|\right)\left(\frac{1}{|\rho|}\|R_1''\|+\|S_1''\|\right)\\
&\quad+\left(\frac{1}{|\rho|}|R_1(0)|+|S_1(0)|\right)\left(\frac{1}{|\rho|}\|R_2''\|+\|S_2''\|\right)\\
&\quad+\left(\frac{1}{|\rho|}|R_1(1)|+|S_1(1)|\right)\left(\frac{1}{|\rho|}\|R_2''\|+\|S_2''\|\right)\Bigg]\\
&+\frac{2}{(\cos((\delta/2)+(\pi/4)))^2}\Bigg[\frac{1}{|\rho|}\big|R_2(1)S_1(1)-R_1(1)S_2(1)\\
&\quad+R_1(0)S_2(0)-R_2(0)S_1(0)+S_2(0)S_1'(0)-S_1(0)S_2'(0)\\
&\qquad-S_1(1)S_2'(1)+S_2(1)S_1'(1)\big|+\frac{1}{||\rho|^2}\big|-R_2(0)R_1'(0)\\
&\qquad\qquad-R_2(1)R_1'(1)+R_1(1)R_2'(1)+R_1(0)R_2'(0)\big|\Bigg]\\
&+\frac{1}{2(\cos((\delta/2)+(\pi/4)))^2}\\
&\times\Bigg[\left(\frac{1}{|\rho|}\|R_2'\|+\|S_2'\|\right)\left(\frac{1}{|\rho|^2}\|R_1\|+\frac{1}{|\rho|}\|S_1\|\right)\\
&\quad+\left(\frac{1}{|\rho|}\|R_1'\|+\|S_1'\|\right)\left(\frac{1}{|\rho|^2}\|R_2\|+\frac{1}{|\rho|}\|S_2\|\right)\\
&\quad+\left(\frac{1}{|\rho|}\|R_2\|+\|S_2\|\right)\left(\frac{1}{|\rho|}\|R_1\|+\|S_1\|\right)\Bigg],
\end{aligned}
$$

where we have used that $Re(\rho)\geq|\rho|\cos((\delta/2)+(\pi/4))$, $(1-e^{-2Re(\rho)})$ ≤ 1,

$$e^{-Re(\rho)}\leq\frac{2}{|\rho|^2(\cos((\delta/2)+(\pi/4)))^2}$$

and

$$e^{-2Re(\rho)}\leq\frac{1}{2|\rho|^2(\cos((\delta/2)+(\pi/4)))^2}.$$

Then for $|\rho|\geq r_0$ we have $|\Phi(\rho)|\leq C(r_0)$.

There are several cases to analyze depending on the functions R_i and S_i.

Case 1. Suppose that $\|S_1\| \neq 0$, $\|S_2\| \neq 0$, $|S_2(0)S_1(1) - S_1(0)S_2(1)| = 0$ and

$$\begin{aligned} &R_1(0)S_2(1) - R_2(0)S_1(1) + S_2(0)R_1(1) - S_1(0)R_2(1) \\ &\quad + S_1(1)S_2'(0) - S_2(0)S_1'(1) + S_1(0)S_2'(1) - S_2(1)S_1'(0) = K_1 \neq 0. \end{aligned}$$

In this from (15), we have for $|\rho|$ sufficiently large

$$\begin{aligned} |\Delta(\lambda)| \geq e^{Re(\rho)} \Bigg[&\frac{1}{|\rho|} \big| S_2(1)R_1(0) - S_1(1)R_2(0) \\ &\quad + S_2(0)R_1(1) - S_1(0)R_2(1) + S_1(1)S_2'(0) \\ &\quad - S_2(0)S_1'(1) + S_1(0)S_2'(1) - S_2(1)S_1'(0) \big| \\ &- \frac{1}{|\rho|^2} \big| R_1(0)R_2(1) - R_2(0)R_1(1) \\ &\qquad + R_2(0)S_1'(1) - S_2(0)R_1'(1) - R_1(0)S_2'(1) \\ &\qquad + S_1(0)R_2'(1) - R_2(1)S_1'(0) + S_2(1)R_1'(0) \\ &\qquad + R_1(1)S_2'(0) - S_1(1)R_2'(0) - S_1(0)R_2'(0) \big| \\ &- \frac{1}{|\rho|^3} \big| R_2(0)R_1'(1) - R_1(0)R_2'(1) \\ &\qquad + R_2(1)R_1'(0) - R_1(1)R_2'(0) \big| - \frac{|\Phi(\rho)|}{|\rho|^2} \Bigg]. \end{aligned}$$

Take $Re(\rho) > r_0$, where $r_0 > 0$, using (16), we get

$$\begin{aligned} |\Delta(\lambda)| \geq \frac{e^{Re(\rho)}}{|\rho|} \Bigg[&\big| S_2(1)R_1(0) - S_1(1)R_2(0) + S_2(0)R_1(1) \\ &\quad - S_1(0)R_2(1) + S_1(1)S_2'(0) - S_2(0)S_1'(1) \\ &\quad + S_1(0)S_2'(1) - S_2(1)S_1'(0) \big| \\ &- \frac{1}{r_0} \big| R_1(0)R_2(1) - R_2(0)R_1(1) + R_2(0)S_1'(1) \\ &\qquad - S_2(0)R_1'(1) - R_1(0)S_2'(1) + S_1(0)R_2'(1) \\ &\qquad - R_2(1)S_1'(0) + S_2(1)R_1'(0) + R_1(1)S_2'(0) \\ &\qquad - S_1(1)R_2'(0) - S_1(0)R_2'(0) \big| \end{aligned}$$

$$- \frac{1}{r_0^2} \big| R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0) - R_1(1)R_2'(0) \big| - \frac{C\,(r_0)}{r_0} \bigg].$$

We can now choose $r_0 > 0$ such that

$$\begin{aligned} &\frac{1}{r_0} \big| R_1(0)R_2(1) - R_2(0)R_1(1) + R_2(0)S_1'(1) - S_2(0)R_1'(1) \\ &\quad - R_1(0)S_2'(1) + S_1(0)R_2'(1) - S_2(0)R_1'(1) - S_2(0)R_1'(1) \\ &\quad + S_2(1)R_1'(0) + R_1(1)S_2'(0) - S_1(1)R_2'(0) - S_1(1)R_2'(0) \big| \\ &+ \frac{1}{r_0^2} \big| R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0) - R_1(1)R_2'(0) \big| \\ &+ \frac{C(r_{0)}}{r_0} \leq \frac{1}{2} \, |K_1|, \end{aligned}$$

then, for $Re(\rho) > r_0$, we have

$$|\Delta(\lambda)| \geq \frac{e^{Re(\rho)}}{2\,|\rho|} \, |K_1| \, . \tag{17}$$

From (14) we deduce the following bound, valid for every $Re(\rho) > r_0$ with $|\arg \rho| \leq (\pi/4) + (\delta/2)$

$$\begin{aligned} &\|R(\lambda, L_1)\| \\ &\leq \frac{1}{|\rho|} \left[\frac{12}{|K_1| \, (\cos((\delta/2) + (\pi/4)))^2} \left(\frac{\|R_1\|}{|\rho|} + \|S_1\| \right) \left(\frac{\|R_2\|}{|\rho|} + \|S_2\| \right) \right]. \end{aligned}$$

This proves for $|\rho| \to \infty$, that

$$\|R(\lambda, L_1)\| \leq \frac{C}{|\lambda|^{1/2}}.$$

where $C = 12 \, \|S_1\| \, \|S_2\| / |K_1| \, (\cos((\delta/2) + (\pi/4)))^2$.

Case 2. Suppose that $\|S_1\| = 0$, $\|S_2\| \neq 0$ and $R_1(0)S_2(1) + R_1(1)S_2(0) = 0$ with

$$\begin{aligned} &R_1(0)R_2(1) - R_2(0)R_1(1) - S_2(0)R_1'(1) \\ &\qquad - R_1(0)S_2'(1) + R_1'(0)S_2(1) + R_1(1)S_2'(0) = K_2 \neq 0, \end{aligned}$$

we have the following bound, valid for $Re(\rho) > r_0$ and $|\arg \rho| \leq (\pi/4) + (\delta/2)$

$$\|R(\lambda, L_1)\| \leq \frac{1}{|\rho|} \left[\frac{12\,\|R_1\|}{|K_2|\,(\cos((\delta/2) + (\pi/4)))^2} \left(\frac{\|R_2\|}{|\rho|} + \|S_2\| \right) \right].$$

Then, we have

$$\|R(\lambda, L_1)\| \leq \frac{C}{|\lambda|^{1/2}},$$

as $|\rho| \to \infty$, where $C = 12\,\|R_1\|\,\|S_2\|/|K_2|\,(\cos((\delta/2) + (\pi/4)))^2$.

Case 3. Suppose that $\|S_1\| \neq 0$, $\|S_2\| = 0$, $R_2(0)S_1(1) + R_2(1)S_1(0) = 0$ and

$$\begin{aligned} R_1(0)R_2(1) - R_2(0)R_1(1) + R_2(0)S_1'(1) \\ + S_1(0)R_2'(1) - R_2(1)S_1'(0) - S_1(1)R_2'(0) = K_3 \neq 0. \end{aligned}$$

Similarly, we get

$$\|R(\lambda, L_1)\| \leq \frac{1}{|\rho|} \left[\frac{12\,\|R_2\|}{|K_3|\,(\cos((\delta/2) + (\pi/4)))^2} \left(\frac{\|R_1\|}{|\rho|} + \|S_1\| \right) \right],$$

then, we have

$$\|R(\lambda, L_1)\| \leq \frac{C}{|\lambda|^{\frac{1}{2}}},$$

as $|\rho| \to \infty$, where $C = 12\,\|R_2\|\,\|S_1\|/|K_3|\,(\cos((\delta/2) + (\pi/4)))^2$.

Case 4. $\|S_1\| = \|S_2\| = 0$, $\|R_1\|\,\|R_2\| \neq 0$, $R_1(0)R_2(1) - R_1(1)R_2(0) = 0$ and

$$R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0) - R_1(1)R_2'(0) = K_4 \neq 0.$$

Again in this case, we have

$$\|R(\lambda, L_1)\| \leq \frac{1}{|\rho|} \left[\frac{12\,\|R_2\|\,\|R_1\|}{|K_4|\,(\cos((\delta/2) + (\pi/4)))^2} \right],$$

then
$$\|R(\lambda, L_1)\| \le \frac{C}{|\lambda|^{1/2}},$$
as $|\rho| \to \infty$, where $C = \left[12\,\|R_2\|\,\|R_1\|/|K_4|\,(\cos((\delta/2)+(\pi/4)))^2\right]$.

Definition 1. The boundary value conditions in (1) are called nonregular if the functions $R_i, S_i \in C^2([0,1], \mathbf{C})$, $i = \overline{1,2}$, and if and only if one of the following conditions holds:

1. $S_2(0)S_1(1) - S_1(0)S_2(1) = 0$, $\|S_1\| \, . \, \|S_2\| \ne 0$ and

$$\begin{aligned} R_1(0)S_2(1) - R_2(0)S_1(1) + S_2(0)R_1(1) - S_1(0)R_2(1) \\ + S_1(1)S_2'(0) - S_2(0)S_1'(1) + S_1(0)S_2'(1) - S_2(1)S_1'(0) \ne 0 \end{aligned}$$

2. $\|S_1\| = 0$, $\|S_2\| \ne 0$, $R_1(0)S_2(1) + R_1(1)S_2(0) = 0$ with

$$\begin{aligned} R_1(0)R_2(1) - R_2(0)R_1(1) - S_2(0)R_1'(1) \\ - R_1(0)S_2'(1) + R_1'(0)S_2(1) + R_1(1)S_2'(0) \ne 0 \end{aligned}$$

3. $\|S_2\| = 0$, $\|S_1\| \ne 0$, $R_2(0)S_1(1) + R_2(1)S_1(0) = 0$ with

$$\begin{aligned} R_1(0)R_2(1) - R_2(0)R_1(1) + R_2(0)S_1'(1) \\ + S_1(0)R_2'(1) - R_2(1)S_1'(0) - S_1(1)R_2'(0) \ne 0 \end{aligned}$$

4. $\|S_1\| \, . \, \|S_2\| = 0$, $\|R_1\| \, . \, \|R_2\| \ne 0$, $R_1(0)R_2(1) - R_1(1)R_2(0) = 0$ with

$$R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0) - R_1(1)R_2'(0) \ne 0.$$

This proves the following theorem.

Theorem 1. *If the boundary value conditions in* (1) *are nonregular, then* $\Sigma_\delta \subset \rho(L_1)$ *for sufficiently large* $|\lambda|$, *and there exists* $C > 0$ *such that*
$$\|R(\lambda, L_1)\| \le \frac{C}{|\lambda|^{1/2}}, \quad |\lambda| \to \infty.$$

Remark 1. From Theorem 1 results that the operator L_1 generates an analytic semi-groups with singularities [**27**] of type $\mathrm{A}(1,3)$.

3.3 Application. In the following, we apply the above obtained results to the study of a class of mixed problem for a parabolic equation with weighted integral boundary conditions of the form

(18)
$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - a\,\dfrac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) \\ L_i(u) = \displaystyle\int_0^1 R_i(\xi)\,u(t,\xi)\,d\xi + \int_0^1 S_i(\xi)\,u'(t,\xi)\,d\xi = 0, \quad i = \overline{1,2}, \\ u(0,x) = u_0(x), \end{cases}$$

where $(t,\xi) \in [0,T] \times [0,1]$.

Boundary value problems for parabolic equations with integral boundary conditions are studied by [**1, 3, 5, 6, 15**–**18, 37**] using various methods. For instance, the potential method in [**3**] and [**18**], Fourier method in [**1, 15**–**17**] and the energy inequalities method has been used in [**5, 6, 37**]. In our case, we apply the method of operator differential equation. The study of the problem is then reduced to a Cauchy problem for a parabolic abstract differential equation, where the operator coefficients has been previously studied.

For this purpose, let E, E_1, and E_2 be Banach spaces. Introduce two Banach spaces

$$C_\mu\left((0,T],E\right) = \left\{ f/f \in C\left((0,T],E\right),\ \ \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| < \infty \right\},$$
$$\mu \geq 0;$$

$$C_\mu^\gamma\left((0,T],E\right) = \left\{ f/f \in C\left((0,T],E\right),\ \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| \right.$$
$$\left. + \sup_{0<t<t+h\leq T} \|f(t+h) - f(t)\|\, h^{-\gamma} t^\mu < \infty \right\},$$
$$\mu \geq 0, \quad \gamma \in (0,1]\,;$$

and a linear space

$$C^1((0,T], E_1, E_2) = \{f/f \in C((0,T], E_1) \cap C^1((0,T], E_2)\}, \quad E_1 \subset E_2,$$

where $C((0,T], E)$ and $C^1((0,T], E)$ are spaces of continuous and continuously differentiable, respectively, vector-function from $(0,T]$ into E. We denote, for a linear operator A in a Banach space E, by

$$E(A) = \left\{u/u \in D(A),\ \|u\|_{E(A)} = \left(\|Au\|^2 + \|u\|^2\right)^{1/2}\right\},$$

and

$$C^1((0,T], E(A), E) = \{f/f \in C((0,T], E(A)),\ f' \in C((0,T], E))\}.$$

Let us derive a theorem which was proved by various methods in [**27**] and [**33, 34**]. Consider, in a Banach space E, the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{19}$$

where A is, generally speaking, unbounded linear operator in E, u_0 is a given element of E, $f(t)$ is a given vector-function and $u(t)$ is an unknown vector-function in E.

Theorem 2. *Let the following conditions be satisfied*:

1. *A is a closed linear operator in a Banach space E and for some $\beta \in (0,1]$, $\alpha > 0$*

$$\|R(\lambda, A)\| \leq C|\lambda|^{-\beta}, \qquad |\arg\lambda| \leq \frac{\pi}{2} + \alpha, \quad |\lambda| \to \infty;$$

2. *$f \in C_\mu^\gamma((0,T], E)$ for some $\gamma \in (1-\beta, 1]$, $\mu \in [0,\beta)$*;
3. *$u_0 \in D(A)$.*

Then, the Cauchy problem (19) *has a unique solution*

$$u \in C([0,T], E) \cap C^1((0,T], E(A), E)$$

and for the solution u the following estimates hold

$$\|u(t)\| \leq C\left(\|Au_0\| + \|u_0\| + \|f\|_{C_\mu((0,t],E)}\right), \quad t \in (0,T],$$

$$\|u'(t)\| + \|Au(t)\| \leq C\left(t^{\beta-1}\left(\|Au_0\| + \|u_0\|\right] + t^{\beta-\mu-1}\|f\|_{C_\mu^\gamma((0,t],E)}\right),$$
$$t \in (0,T].$$

As a result of this, we get the following theorem.

Theorem 3. *Let the following conditions be satisfied*:

1. $a \neq 0$, $|\arg a| < \pi/2$,

2. *The functions* $R_i(t)$, $S_i(t) \in C^2([0,1], \mathbf{C})$, $i = \overline{1,2}$, *and one of the following conditions are satisfied* $|S_2(0)S_1(1) - S_1(0)S_2(1)| = 0$, $\|S_1\| . \|S_2\| \neq 0$ *and*

$$R_1(0)S_2(1) - R_2(0)S_1(1) + S_2(0)R_1(1) - S_1(0)R_2(1) \\ + S_1(1)S_2'(0) - S_2(0)S_1'(1) + S_1(0)S_2'(1) - S_2(1)S_1'(0) \neq 0,$$

or $R_1(0)S_2(1) + R_1(1)S_2(0) = 0$ *with* $\|S_1\| = 0$, $\|S_2\| \neq 0$ *and*

$$R_1(0)R_2(1) - R_2(0)R_1(1) - S_2(0)R_1'(1) \\ - R_1(0)S_2'(1) + R_1'(0)S_2(1) + R_1(1)S_2'(0) \neq 0,$$

or $R_2(0)S_1(1) + R_2(1)S_1(0) = 0$ *with* $\|S_2\| = 0$, $\|S_1\| \neq 0$ *and*

$$R_1(0)R_2(1) - R_2(0)R_1(1) + R_2(0)S_1'(1) \\ + S_1(0)R_2'(1) - R_2(1)S_1'(0) - S_1(1)R_2'(0) \neq 0$$

or $R_1(0)R_2(1) - R_1(1)R_2(0) = 0$ *with* $\|S_1\| = \|S_2\| = 0$, $\|R_1\| . \|R_2\| \neq 0$ *and*

$$R_2(0)R_1'(1) - R_1(0)R_2'(1) + R_2(1)R_1'(0) - R_1(1)R_2'(0) \neq 0.$$

3. $f \in C_\mu^\gamma\left((0,T], L^1(0,1)\right)$ *for some* $\gamma \in ((1/2), 1]$ *and some* $\mu \in [0, (1/2))$,

4. $u_0 \in W_1^2\left((0,1), L_i u = 0, i = \overline{1,2}\right)$.

Then, the problem (18) *has a unique solution*

$$u \in C\left((0,T], L^1(0,1)\right) \cap C^1\left((0,T], W_1^2(0,1), L^1(0,1)\right)$$

and, for this solution, we have the estimates

$$\|u(t,.)\|_{L^1(0,1)} \leq c\left(\|u_0\|_{W_1^2(0,1)} + \|f\|_{C_\mu((0,t],L^1(0,1))}\right), \quad t \in (0,T], \tag{20}$$

$$\begin{aligned} &\|u''(t,.)\|_{L^1(0,1)} + \|u'(t,.)\|_{L^1(0,1)} \\ &\quad \leq c\left(t^{-1/2}\|u_0\|_{W_1^2(0,1)} + t^{-(1/2)-\mu}\|f\|_{C_\mu^\gamma((0,t],L^1(0,1))}\right), \quad t \in (0,T]. \end{aligned} \tag{21}$$

Proof. We consider in the space $L^1(0,1)$, the operator A defined by $A(u) = au''(x)$, $D(A) = \left\{u \in W_1^2(0,1), L_i(u) = 0, i = \overline{1,2}\right\}$.

Then, problem (18) can be written as

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = u_0, \end{cases}$$

where $u(t) = u(t,.)$, $f(t) = f(t,.)$ and $u_0 = u_0(.)$ are functions with values in the Banach space $L^1(0,1)$. From Theorem 1, we conclude that

$$\|R(\lambda, A)\| \leq c\,|\lambda|^{-1/2}, \quad \text{for} \quad |\arg\lambda| \leq \frac{\pi}{2} + \alpha, \quad \text{as} \quad |\lambda| \to \infty.$$

Then, from Theorem 2, the problem (18) has a unique solution $u \in C\left((0,T], L^1(0,1)\right) \cap C^1\left((0,T], W_1^2(0,1), L^1(0,1)\right)$ and we have the following estimates

$$\|u(t,.)\|_{L^1(0,1)} \leq c\left(\|Au_0\|_{L^1(0,1)} + \|u_0\|_{L^1(0,1)} + \|f\|_{C_\mu((0,t],L^1(0,1))}\right), \tag{22}$$

$$\begin{aligned}\|u'(t,.)\| + \|Au(t,.)\| \le c\Big(&t^{-1/2}\big(\,\|Au_0\|_{L^1(0,1)} + \|u_0\|_{L^1(0,1)}\,\big)\\ &+ t^{-(1/2)-\mu}\,\|f\|_{C_\mu^\gamma((0,t],L^1(0,1))}\Big),\end{aligned} \tag{23}$$

where $t \in (0,T]$. From (22) we get

$$\begin{aligned}\|u(t,.)\|_{L_1(0,1)} &\le c\left(\|u_0''\|_{L^1(0,1)} + \|u_0\|_{L^1(0,1)} + \|f\|_{C_\mu((0,t],L^1(0,1))}\right)\\ &\le c\left(\|u_0\|_{W_1^2(0,1)} + \|f\|_{C_\mu((0,t],L^1(0,1))}\right), \quad t \in (0,T]\,,\end{aligned}$$

and from (23) we get

$$\begin{aligned}&\|u''(t,.)\|_{L_1(0,1)} + \|u'(t,.)\|_{L_1(0,1)}\\ &\le c\left(t^{-1/2}\left(\|u_0''\|_{L^1(0,1)} + \|u_0\|_{L^1(0,1)}\right) + t^{-(1/2)-\mu}\,\|f\|_{C_\mu^\gamma((0,t],L^1(0,1))}\right)\\ &\le c\left(t^{-1/2}\,\|u_0\|_{W_1^2(0,1)} + t^{-(1/2)-\mu}\,\|f\|_{C_\mu^\gamma((0,t],L^1(0,1))}\right), \quad t \in (0,T]\,,\end{aligned}$$

which gives the desired result. □

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Laboratoire Equations Differentielles, Departement de Mathematiques, Faculte des Sciences, Universite Mentouri, 25000 Constantine, Algeria
E-mail address: `denech@Wissal.dz`

Laboratoire Equations Differentielles, Departement de Mathematiques, Faculte des Sciences, Universite Mentouri, 25000 Constantine, Algeria