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A SIMPLE PROOF THAT A LINEARLY ORDERED SPACE IS HEREDITARILY AND COMPLETELY COLLECTIONWISE NORMAL

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It is known [**1**] that a linearly ordered space is hereditarily collectionwise normal. In this note we give a simpler proof that a linearly ordered space is both hereditarily and completely collectionwise normal [**3**, p. 168].

Let X be a linearly ordered set endowed with the usual open interval topology. We denote intervals in the usual way by (a,b), $(a,b]$, $[a,b)$ or $[a,b]$. We prove

Theorem I. *Let $\{A_i\}$ be a family of subsets of X such that each A_i is disjoint from the closure of $\cup_{j\neq i}A_j$. Then there is a family $\{U_i\}$ of mutually disjoint open sets such that $A_i \subset U_i$ for each index i.*

Proof. For convenience, put $P = \cup_i A_i$. We say that points $a, b \in X\backslash P$ are equivalent if the interval joining a to b is a subset of $X\backslash P$. Then $X\backslash P$ is partitioned into equivalence classes we call the *components* of $X\backslash P$. Use the Axiom of Choice to select a point $f(C)$ in each component C.

Fix an index i. For each $x \in A_i$ that is not the greatest point in X we select a point $t_x > x$ as follows:

Case (1). If x is a right accumulation point of A_i, select $t_x \in A_i$ so that $t_x > x$ and the interval (x, t_x) is disjoint from $P\backslash A_i$.

Case (2). If x has an immediate successor, we designate it by t_x.

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Case (3). If x is a right accumulation point of X but not of A_i, then the set $\{p \in P : p > x\}$ is bounded away from x, and hence x is the greatest lower bound of a unique component C of $X\backslash P$. Let $t_x = f(C)$.

In all the Cases (1), (2) and (3), the interval $[x, t_x)$ is disjoint from the set $P\backslash A_i$.

For each $x \in A_i$ that is not the least point of X, select $s_x < x$ in the analogous way with the order reversed. We define the interval I_x for each $x \in A_i$ to be (s_x, t_x) if both s_x and t_x are defined, to be $[x, t_x)$ if x is the least point of X and $(s_x, x]$ if x is the greatest point of X. Thus I_x is an open neighborhood of x disjoint from $P\backslash A_i$ for each $x \in A_i$.

It remains only to prove that if $u \in A_i$, $y \in A_j$, $i \neq j$, then I_u and I_y are disjoint. Assume to the contrary that I_u and I_y intersect. Say $u < y$ for definiteness. Then $y \notin I_u$ because I_u can contain no point of A_j. Likewise $u \notin I_y$. It follows that $s_y \in I_u$ and $t_u \in I_y$. Then Case (1) does not govern the definition of t_u because I_y contains no point of A_i. Likewise Case (1) does not govern the definition of s_y. Clearly Case (2) does not govern the definition of t_u or s_y.

So Case (3) governs the definitions of t_u and s_y. Say $(u, t_u) \subset C_1$ and $(s_y, y) \subset C_2$ for appropriate components C_1 and C_2 of $X\backslash P$. It follows that

$$I_u \cap I_y = (u, t_u) \cap (s_y, y) \subset C_1 \cap C_2, \tag{0.1}$$

and $C_1 \cap C_2$ is nonvoid. We deduce from this that $C_1 = C_2$ and

$$t_u = f(C_1) = f(C_2) = s_y.$$

At that point the intervals I_u and I_y must abut, so they are disjoint. This contradiction completes the proof.

Observe that the analogue of Theorem I is true when X is replaced by any subspace of X. We leave the proof.

From Theorem I it follows that X and all its subspaces are completely collectionwise normal. Then X is hereditarily collectionwise normal as well.

Remarks. Our hypothesis in Theorem I need not require that the family $\{A_i\}$ be discrete [**1**, p. 35]. Consider for example the family of singleton sets $\{1/i\}$ in the real line as i assumes all nonzero integer values. This family is not discrete, nor is it even locally finite.

It is easy to find completely collectionwise normal spaces that are not homeomorphic to any subspace of any linearly ordered space. For example, the Euclidean plane (or any metric space) is trivially completely collectionwise normal [**3**, p. 168]. Use the x and y-axes to find four connected subsets of the plane, the intersection of any two of which is the same singleton set. No subspace of any linearly ordered space can contain four such connected subsets.

On the other hand, there are linearly ordered spaces that are not metrizable. Consider the set $\{(r, v) : r \text{ real}, \ v = 0 \text{ or } 1\}$ with the lexicographic order. This ordered space is separable but has no countable base. Hence it is not metrizable.

Finally, for a completely collectionwise normal space that is not homeomorphic to any subspace of any linearly ordered space or metrizable space, take the sum of the Euclidean plane and the linearly ordered space in the preceding paragraph.

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