ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR $2n$-ORDER DIFFERENTIAL EQUATIONS

BINGGEN ZHANG AND YUJI LIU

ABSTRACT. In this paper we consider the higher order differential equation

$$(0.1)\qquad (-1)^n x^{(2n)}(t) = f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)),\quad 0 < t < 1,$$

subject to one of the following boundary value conditions

$$(0.2)\qquad \begin{aligned} x^{(2i)}(1) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1,\\ x^{(2i+1)}(0) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1, \end{aligned}$$

or

$$(0.3)\qquad \begin{aligned} x^{(i)}(1) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1,\\ x^{(i)}(0) &= 0 \quad \text{for} \quad i = n, \dots, 2n-1, \end{aligned}$$

where $f(t, x_0, x_1, \dots, x_{2n-1})$ is continuous. Sufficient conditions for the existence of at least one solution or positive solution of the BVP (1) and (2) and BVP (1) and (3) are established, respectively. The emphasis in this paper is that f depends on all higher-order derivatives and we allow that the variables $x_0, \dots, x_{2n-1}$ in f have the degrees greater than 1. Examples are given to illustrate the main results.

1. Introduction. Recently, there has been increasing interest in the study of the existence of positive solutions of boundary value problems for second order or higher order ordinary differential equations, we refer the reader to [**5, 7, 9, 12**–**15, 17**–**19**] and the monographs [**1**–**3**].

For the second order case, the existence of positive solutions of boundary value problems for nonlinear differential equations has been

Key words and phrases. Solvability, two-point boundary value problem, higher order differential equation, solution, positive solution.
The first author is supported by NNSF of China (10371103), and the second author is supported by NSF of Educational Committee of Hunan Province of China.
Received by the editors on December 15, 2003, and in revised form on June 9, 2004.

studied by many authors. Especially, the study of the following differential equation

$$x''(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \tag{1}$$

subjected to different boundary value conditions, has received much attention in seeking conditions on the nonlinearity f for which there are either at least one, at least two or at least three positive solutions, one may see [**4, 8, 10, 11, 22**], for examples.

However, the existence of positive solutions of the following differential equation

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{2}$$

associated with different boundary value conditions has not many studies, since the presence of x' in the nonlinearity f causes some considerable difficulties. We name a few, see [**6, 14, 16, 20**] for examples.

Very recently, Chyan and Henderson, in [**7**], studied the following $2m$th-order differential equation

$$x^{(2m)}(t) = f(t, x(t), x''(t), \dots, x^{(2(m-1))}(t)), \quad 0 < t < 1, \tag{3}$$

with either the Lidstone boundary value condition

$$x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad \text{for} \quad i = 0, 1, \dots, m-1, \tag{4}$$

or the focal boundary value condition

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \quad \text{for} \quad i = 0, 1, \dots, m-1. \tag{5}$$

They proved the existence of at least one positive solution in the case that either f is super-linear or f is sub-linear.

Similar problems were also investigated in [**19**] by Palamides by using an analysis of the corresponding field on the face-plane and the well known Sperner's lemma. The method there is different from that in [**7, 14**]. In the papers mentioned above, the nonlinearity f depends on $x, x'', \dots, x^{(2(m-1))}$.

In this paper, we consider the existence of the solutions or positive solutions of the higher order differential equation

$$(-1)^n x^{(2n)}(t) = f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)), \quad 0 < t < 1, \tag{6}$$

subject to one of the following boundary value conditions

$$\begin{aligned} x^{(2i+1)}(0) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1, \\ x^{(2i)}(1) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{7}$$

$$\begin{aligned} x^{(i)}(1) &= 0 \quad \text{for} \quad i = 0, 1, \dots, n-1, \\ x^{(i)}(0) &= 0 \quad \text{for} \quad i = n, \dots, 2n-1, \end{aligned} \tag{8}$$

where $f(t, x_0, x_1, \dots, x_{2n-1})$ is continuous. For the existence of solutions of equation (6) subject to different boundary conditions, such as focal boundary value problems, conjugate boundary value problems and (n,p) boundary value problems, there have been many studies in recent years. In [**1–3**], the existence results were established. One of the main conditions imposed on f is as follows:

$$|f(t, x_0, x_1, \dots, x_{2n-1})| \le L + \sum_{i=0}^{2n-1} L_i |x_i|, \tag{$*$}$$

where L_i, $i = 0, \dots, 2n-1$, are constants. We note that the degree of variable x_i at the right of equation $(*)$ is 1. However, when $(*)$ is not valid, the existence problems for equation (6) have not been enough investigated till now, [**23**]. This paper will establish existence results for equation (6) when $(*)$ is not valid.

2. Existence results for BVPs. In this section, we will establish sufficient conditions for the existence of at least one positive solution of BVP (6) and (7) and BVP (6) and (8), and then we give some examples to illustrate the main results.

We choose Banach space $C[0,1]$ with the maximum norm $\|\cdot\|_\infty$, and we define the condition (H): $f \in C([0,1] \times R^{2n}, R)$, and there exist functions $h \in C([0,1] \times R^{2n}, R)$, $e \in C([0,1], R)$, $g_i \in C([0,1], [0, \infty))$,

$i = 0, 1, 2, \ldots, 2n-1$, and real numbers $\beta > 0$ and $m > 0$ such that, for $(t, x_0, x_1, \ldots, x_{2n-1}) \in [0,1] \times R^{2n}$, f satisfies

$$f(t, x_0, x_1, \ldots, x_{2n-1}) = e(t) + h(t, x_0, x_1, \ldots, x_{2n-1}) + \sum_{i=0}^{2n-1} g_i(t) x_i^m,$$
$$x_{2n-1} h(t, x_0, x_1, \ldots, x_{2n-1}) \le -\beta |x_{2n-1}|^{m+1},$$

and

$$\sum_{i=0}^{2n-1} ||g_i||_\infty < \beta. \tag{9}$$

The following well-known fixed point theorem is crucial in our reasoning.

Lemma 2.1 [**21**, Theorem 4.3.2]. *Let X be a real Banach Space and $T : X \to X$ a compact operator. If the set $\Omega = \{x \in X \mid x = \lambda T x,$ for some $\lambda \in (0,1)\}$ is bounded, then the operator T has a fixed point in X.*

Then we obtain the following main result.

Theorem 2.1. *We assume the nonlinear term f in* (6) *satisfies the condition* (H). *Then* BVP (6)–(7) *has at least one solution.*

Proof. Let $X = C^{2n-1}[0,1]$ be endowed with the norm

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(2n-1)}\|_\infty\}.$$

Let $G(t,s)$ be the Green function [**1**–**3**] of the corresponding problem

$$\begin{aligned} x^{(2n)}(t) &= 0, \quad t \in (0,1), \\ x^{(2i)}(1) &= 0 \quad \text{for} \quad i = 0, 1, \ldots, n-1, \\ x^{(2i+1)}(0) &= 0 \quad \text{for} \quad i = 0, 1, \ldots, n-1. \end{aligned} \tag{10}$$

Define an operator T by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))\, ds, \quad t \in [0,1],$$

for $x \in X$. It is easy to check that the operator $T : X \to X$ is compact and x is a solution of BVP (6) and (7) if and only if x is a fixed point of the operator T. Let

$$\Omega = \{x \in X \mid x = \lambda T x, \text{ for some } \lambda \in (0,1)\}.$$

It suffices to prove that Ω is bounded according to Lemma 2.1. We need to prove that there is a constant $B > 0$ such that

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(2n-1)}\|_\infty\} \le B.$$

For $x \in \Omega$, it is easy to show that

(11)
$$\begin{aligned}
|x^{(2n-2)}(t)| &= \left|x^{(2n-2)}(1) + \int_1^t x^{(2n-1)}(s)\,ds\right| \le \int_0^1 |x^{(2n-1)}(s)|\,ds,\\
|x^{(2n-3)}(t)| &= \left|x^{(2n-3)}(0) + \int_0^t x^{(2n-2)}(s)\,ds\right| \le \int_0^1 |x^{(2n-2)}(s)|\,ds\\
&\le \int_0^1 |x^{(2n-1)}(s)|\,ds,\\
&\cdots\cdots\cdots\cdots,\\
|x(t)| &\le \int_0^1 |x^{(2n-1)}(s)|\,ds.
\end{aligned}$$

We divide our reasoning into two steps.

Step 1. We claim that there is a constant $\overline{M} > 0$ such that $\int_0^1 |x^{(2n-1)}(s)|^{m+1}\,ds \le \overline{M}$.

For $x \in \Omega$, we have

$$x^{(2n)}(t) = \lambda f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)). \tag{12}$$

Multiplying both sides of (12) by $x^{(2n-1)}(t)$ and integrating from 0 to 1, by the condition (H), we get

$$\begin{aligned}
&\frac{1}{2}\left(x^{(2n-1)}(1)\right)^2 - \frac{1}{2}\left(x^{(2n-1)}(0)\right)^2\\
&\quad = \lambda \int_0^1 f(s, x(s), x'(s), \dots, x^{(2n-1)}(s))x^{(2n-1)}(s)\,ds\\
&\quad = \lambda \bigg(\int_0^1 h(s, x(s), x'(s), \dots, x^{(2n-1)}(s))x^{(2n-1)}(s)\,ds\\
&\qquad\quad + \sum_{i=0}^{2n-1}\int_0^1 g_i(s)[x^{(i)}(s)]^m x^{(2n-1)}(s)\,ds + \int_0^1 e(s)x^{(2n-1)}(s)\,ds\bigg).
\end{aligned}$$

Thus, from the second part of condition (H),

$$\begin{aligned}
\lambda\,\beta \int_0^1 \left|x^{(2n-1)}(s)\right|^{m+1} ds
&\le -\lambda \int_0^1 h(s, x(s), x'(s), \dots , x^{(2n-1)}(s))x^{(2n-1)}(s)\, ds \\
&= -\frac{1}{2}\left(x^{(2n-1)}(1)\right)^2 + \lambda \sum_{i=0}^{2n-1} \int_0^1 g_i(s)[x^{(i)}(s)]^m x^{(2n-1)}(s)\, ds \\
&\quad + \lambda \int_0^1 e(s)x^{(2n-1)}(s)\, ds \\
&\le \lambda \sum_{i=0}^{2n-1} \int_0^1 g_i(s)[|x^{(i)}(s)|]^m \,|x^{(2n-1)}(s)|\, ds \\
&\quad + \lambda \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|\, ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\beta \int_0^1 \left|x^{(2n-1)}(s)\right|^{m+1} ds \le \sum_{i=0}^{2n-1} \int_0^1 g_i(s)[|x^{(i)}(s)|]^m |x^{(2n-1)}(s)|\, ds \\
+ \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|\, ds.
\end{aligned}$$

So, for $i = 0, 1, \dots , 2n-2$, we have

$$\begin{aligned}
\int_0^1 g_i(s)|x^{(i)}(s)|^m \,|x^{(2n-1)}(s)|\, ds
&\le \left(\int_0^1 |x^{(2n-1)}(s)|\, ds\right)^m \int_0^1 g_i(s)|x^{(2n-1)}(s)|\, ds \\
&\le ||g_i||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|\, ds\right)^{m+1}.
\end{aligned}$$

Since

$$\int_0^1 |x^{(2n-1)}(s)|\, ds \le \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds\right)^{1/(m+1)},$$

we get

$$
\begin{aligned}
&\beta \int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \\
&\quad \le \sum_{i=0}^{2n-2} ||g_i||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|\, ds \right)^{m+1} \\
&\qquad + ||g_{2n-1}||_\infty \int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds + ||e||_\infty \int_0^1 |x^{(2n-1)}(s)|\, ds \\
&\quad \le \sum_{i=0}^{2n-2} ||g_i||_\infty \int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \\
&\qquad + ||g_{2n-1}||_\infty \int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \\
&\qquad + ||e||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \right)^{1/(m+1)}.
\end{aligned}
$$

Hence

$$
\left(\beta - \sum_{i=0}^{2n-1} \|g_i\|_\infty \right) \int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \le \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \right)^{1/(m+1)}.
$$

Since

$$
\beta > \sum_{i=0}^{2n-1} \|g_i\|_\infty,
$$

we get there is an $\overline{M} > 0$ such that

$$
\int_0^1 |x^{(2n-1)}(s)|^{m+1}\, ds \le \overline{M}.
$$

Step 2. We claim that there exists $B > 0$ such that $\|x\| \le B$. From (11), we have, for $i = 0, 1, \dots, 2n-2$,

$$\begin{aligned}\|x^{(i)}\|_\infty &\le \int_0^1 |x^{(2n-1)}(s)|\,ds \\ &\le \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}\,ds\right)^{1/(m+1)} \\ &\le \overline{M}^{1/(m+1)}.\end{aligned}$$

Multiplying both sides of (12) by $x^{(2n-1)}(t)$, integrating from 0 to t and by condition (H), we get

$$\begin{aligned}&\frac{1}{2}\left(x^{(2n-1)}(t)\right)^2 - \frac{1}{2}\left(x^{(2n-1)}(0)\right)^2 \\ &\quad = \lambda \int_0^t f(s, x(s), x'(s), \dots, x^{(2n-1)}(s))x^{(2n-1)}(s)\,ds \\ &\quad = \lambda \int_0^t h(s, x(s), x'(s), \dots, x^{(2n-1)}(s))x^{(2n-1)}(s)\,ds \\ &\qquad + \lambda \int_0^t g_0(s)[x(s)]^m x^{(2n-1)}(s)\,ds \\ &\qquad + \lambda \sum_{i=1}^{2n-1} \int_0^t g_i(s)[x^{(i)}(s)]^m x^{(2n-1)}(s)\,ds + \lambda \int_0^t e(s)x^{(2n-1)}(s)\,ds \\ &\quad \le -\lambda\beta \int_0^t |x^{(2n-1)}(s)|^{m+1}\,ds + \int_0^1 g_0(s)[|x(s)|]^m |x^{(2n-1)}(s)|\,ds \\ &\qquad + \sum_{i=1}^{2n-1} \int_0^1 g_i(s)[|x^{(i)}(s)|]^m |x^{(2n-1)}(s)|\,ds \\ &\qquad + \int_0^1 |e(s)||x^{(2n-1)}(s)|\,ds \\ &\quad \le \sum_{i=0}^{2n-2} ||g_i||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|\,ds\right)^m \int_0^1 |x^{(2n-1)}(s)|\,ds \\ &\qquad + ||e||_\infty \int_0^1 |x^{(2n-1)}(s)|\,ds + ||g_{2n-1}||_\infty \int_0^1 |x^{(2n-1)}(s)|^{m+1}\,ds.\end{aligned}$$

Similarly to Step 1, we can get

$$\begin{aligned}\frac{1}{2}\,|x^{(2n-1)}(t)|^2 &\le \left(\sum_{i=0}^{2n-1}||g_i||_\infty\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}\,ds \\ &\quad + ||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}\,ds\right)^{1/(m+1)} \\ &\le \sum_{i=0}^{2n-1}||g_i||_\infty\overline{M} + ||e||_\infty\overline{M}^{1/(m+1)}.\end{aligned}$$

So there exists $\overline{M}' > 0$ such that $\|x^{(n-1)}\|_\infty \le \overline{M}'$. It follows that

$$\|x\| \le \max\left\{\overline{M},\,\overline{M}'\right\} =: B.$$

It follows from Steps 1 and 2 that Ω is bounded. Hence from Lemma 2.1, T has at least one fixed point, which is a solution of BVP (6) and (7). This completes the proof of Theorem 2.1. $\square$

Corollary 2.1. *Suppose the nonlinear term f in* (6) *satisfies the condition* (H) *and f is nonnegative. If $f(t,0,\dots,0)\not\equiv 0$ on any subinterval $[\alpha,\beta]\subset[0,1]$, where $0\le\alpha<\beta\le 1$, then* BVP (6) *and* (7) *has at least one positive solution.*

Proof. From Theorem 2.1, BVP (6) and (7) has at least one solution x, so it suffices to prove that $x(t) > 0$ for all $t \in (0,1)$. From [**1**–**3**], $G(t,s)\ge 0$ for all $(t,s)\in[0,1]\times[0,1]$, then we have

$$x(t) = \int_0^1 G(t,s)f(s,x(s),\dots,x^{(2n-1)}(s))\,ds \ge 0, \quad t\in[0,1].$$

If $x(t_0) = 0$ for some $t_0 \in (0,1)$, it follows from the boundary value conditions (7) and

$$(-1)^n x^{(2n)}(t) = f(t,x(t),\dots,x^{(2n-1)}(t)) \ge 0$$

that $(-1)^n x^{(2n-1)}(t) \ge 0$ for all $t\in[0,1]$, and so $(-1)^n x^{(2n-2)}(t)\ge 0$ for all $t\in[0,1]$ since $x^{(2n-2)}(1) = 0$. By similar analogy, we get $x'(t)$ is monotone on $[0,1]$, and so $x(t)\equiv 0$ for all $t\in[t_0,1]$. Thus

$f(t, 0, \cdots, 0) \equiv 0$ for $t \in [t_0, 1]$, and we get a contradiction. Thus $x(t)$ is positive on [0,1]. So $x(t)$ is a positive solution of BVP (6) and (7).

Theorem 2.2. *Under the same assumption on f in* (6) *as in Theorem* 2.1*,* BVP (6) *and* (8) *also have at least one solution.*

Proof. Let X be defined as the one in the proof of the Theorem 2.1. Define the operator T by

$$Tx(t) = (-1)^n \int_t^1 \frac{(s-t)^{n-1}}{(n-1)!} \int_0^s \frac{(s-u)^{n-1}}{(n-1)!} \times f(u, x(u), \dots, x^{(2n-1)}(u))\, du\, ds,$$

for $t \in [0,1]$ and $x \in X$. It is easy to check that T is compact and x is a solution of BVP (6) and (8) if and only if $x(t)$ is a solution of the operator equation $Tx = x$ in X. The remainder of the proof is similar to that of Theorem 2.1 and is omitted. □

Corollary 2.2. *Under the same assumption on f in* (6) *as in Corollary* 2.1*,* BVP (6) *and* (8) *also have at least one positive solution.*

Proof. From Theorem 2.2, BVP (6) and (8) have at least one solution x, it suffices to prove that $x(t) > 0$ for all $t \in (0,1)$. Since

$$x(t) = (-1)^n \int_t^1 \frac{(s-t)^{n-1}}{(n-1)!} \int_0^s \frac{(s-u)^{n-1}}{(n-1)!} \times f(u, x(u), \dots, x^{(2n-1)}(u))\, du\, ds,$$

we know x is nonnegative. If $x(t_0) = 0$ for some $t_0 \in (0,1)$, it follows from the boundary value conditions (8) it is easy to have $x(t) \equiv 0$ for all $t \in [t_0, 1]$. Thus $f(t, 0, \dots, 0) \equiv 0$ for $t \in [t_0, 1]$, and we get a contradiction. So x is a positive solution of BVP (6) and (8).

Remark 1. By a similar method, we can establish the existence results for the following boundary value problem

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) & t \in (0,1), \\ x^{(i)}(1) = 0 & i = 0, 1, \dots, p-1, \\ x^{(i)}(0) = 0 & i = p, \dots, n-1, \end{cases}$$

and the multi-point boundary value problem

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) & t \in (0,1), \\ x^{(i_k)}(\xi_k) = 0 & k = 1, \dots, n-1, \\ x^{(n-1)}(0) = 0, \end{cases}$$

where $1 \le p \le n-1$, $0 \le \xi_0 \le \cdots \le \xi_{n-2} \le 1$ and $\{i_0, i_2, \dots, i_{n-2}\} = \{0, 1, 2, \dots, n-2\}$. We omitted the details.

3. Examples. Now, we give some examples to illustrate the main results.

Example 1. Consider the following boundary value problem

$$(18)\quad \begin{cases} x^{(4)}(t) = \bar{e}(t) - \beta\big(1 + |\sin(x(t))|\big)\,[x'''(t)]^3 + \exp(t)\,[x''(t)]^3, \\ x(1) = x''(1) = x'(0) = x'''(0) = 0. \end{cases}$$

It is easy to see that

$$\begin{gathered} f(t, x_0, x_1, x_2, x_3) = \bar{e}(t) + h(t, x_0, x_1, x_2, x_3) + \exp(t)\,[x_2]^3, \\ h(t, x_0, x_1, x_2, x_3) = -\,\beta(1 + |\sin x_0|)x_3^3, \\ g_0 = g_1 = g_3 = 0, \quad g_2(t) = \exp(t), \\ \sum_{i=0}^{3} ||g_i||_\infty = e, \quad m = 3, \quad \beta > 0, \\ x_3 h(t, x_0, x_1, x_2, x_3) = -\,\beta(1 + |\sin x_0|)\,x_3^4 \le -\,\beta\,x_3^4. \end{gathered}$$

It follows from Theorem 2.1 that BVP (13) has at least one solution for every continuous function $\bar{e} \in C[0,1]$ provided $\beta > e$.

Example 2. Consider the following boundary value problem

(14)

$$\begin{cases} x^{(6)}(t) = e(t) - \beta[x^{(5)}(t)]^3 + t[x(t)]^3 + t^3[x''(t)]^3 + t^6[x^{(5)}(t)]^3, \\ x(1) = x'(1) = x''(1) = x'''(0) = x^{4p}(0) = x^{(5)}(0) = 0. \end{cases}$$

It is easy to see that

$$
\begin{gathered}
f(t,x_0,x_1,\dots,x_5) = e(t) + h(t,x_0,x_1,\dots,x_5) + tx_0^3 + t^3x_2^3 + t^6x_5^3,\\
h(t,x_0,x_1,\dots,x_5) = -\beta\, x_5^3,\\
g_0(t) = t,\quad g_1 = g_3 = g_4 = 0,\quad g_2(t) = t^3,\quad g_5(t) = t^6,\\
\sum_{i=0}^{3} ||g_i||_\infty = 3,\quad m = 3,\quad \beta > 0,\\
x_5 h(t,x_0,x_1,\dots,x_5) = -\beta\, x_5^4 \le -\beta\, x_5^4.
\end{gathered}
$$

It follows from Theorem 2.2 that BVP (14) has at least one solution for every continuous function $e \in C[0,1]$ provided $\beta > 3$.

REFERENCES

1. R.P. Agarwal, *Boundary value problems for higher order differential equations*, World Scientific, Singapore, 1986.

2. ———, *Focal boundary value problems for differential and difference equations*, Kluwer, Dordrecht, 1998.

3. R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive solutions of differential, difference and integral equations*, Kluwer Acad. Publ., Dordrecht, 1999.

4. R.I. Avery, C.J. Chyan and J. Henderson, *Twin positive solutions of boundary value problems for ordinary differential and finite difference equations*, Comput. Math. Appl. **42** (2001), 695–704.

5. C.F. Beards, *Vibration analysis with applications to control systems*, Edward Arnold, London, 1995.

6. D. Cao and R. Ma, *Positive solutions to a second-order multi-point boundary value problem*, Electron. J. Differential Equations **2000** (2000), 1–8.

7. C.J. Chyan and J. Henderson, *Positive solutions of* $2m^{th}$*-order boundary value problems*, Appl. Math. Lett. **15** (2002), 767–774.

8. J.M. Davis, P.W. Eloe and J. Henderson, *Triple positive solutions and dependence on higher-order derivatives*, J. Math. Anal. Appl. **237** (1999), 710–720.

9. E. Dulacska, *Soil settlement effects on buildings*, Developments Geotech. Engrg., vol. 69, Elsevier, Amsterdam, 1992.

10. L. Erbe and M. Tang, *Existence and multiplicity of positive solutions to a nonliear boundary value problem*, Differential Equations Dynam. Systems **4** (1996), 313–320.

11. L. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

12. J.R. Graef and B. Yang, *Existence and non-existence of positive solutions of fourth-order nonlinear boundary value problems*, Appl. Anal. **74** (2000), 201–204.

13. ———, *On a nonlinear boundary value problem for fourth-order differential equations*, Appl. Anal. **72** (1999), 439–448.

14. J. Henderson, *Existence of multiple solutions for second order boundary value problems*, J. Differential Equations **166** (2000), 443–454.

15. J. Henderson and H.B. Thomson, *Multiple symmetric positive solutions for a second-order boundary value problem*, Proc. Amer. Math. Soc. **128** (2000), 2373–2379.

16. G.L. Karakostas and P.Ch. Tsamatos, *Sufficient conditions for the existence of nonnegative solutions of a nonlocal boundary value problem*, Appl. Math. Lett. **15** (2002), 401–407.

17. R. Ma and H. Wang, *On the existence of positive solutions of fourth-order differential equations*, Appl. Anal. **59** (1995), 225–231.

18. E.H. Mansfield, *The bending and stretching of plates*, Internat. Ser. Monographs Aeronaut. Astronaut., vol. 6, Pergamon, New York, 1964.

19. P.K. Palamides, *Positive solutions for higher-order Lidstone boundary value problems*: *A new approach via Sperner's lemma*, Comput. Math. Appl. **42** (2001), 75–89.

20. ———, *Positive and monotone solutions of an m-point boundary value problem*, Electron. J. Differential Equations **2002** (2002), 1–16.

21. D.R. Smart, *Fixed point theorem*, Cambridge Univ. Press, Cambridge, 1980.

22. P.J.Y. Wong, *Triple positive solutions of conjugate boundary value problems*, Comput. Math. Appl. **36** (1998), 19–35.

23. Binggen Zhang and Xueyan Liu, *Existence of multiple symmetric positive solutions of higher order Lidstone problems*, J. Math. Anal. Appl. **284** (2003), 672–689.

Department of Mathematics, Ocean University of China, Qingdao, 266071, P.R. China
E-mail address: binggenzhang2006@yahoo.com.cn

Department of Mathematics, Guangdong University of Business Studies, Guangdong, Guangzhou 510000, P.R. China
E-mail address: liuyuji888@sohu.com