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CHARACTERIZABILITY OF $\mathbf{PSU(p+1,q)}$ BY ITS ORDER COMPONENT(S)

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ABSTRACT. Order components of a finite group were introduced by Chen [**5**]. It was proved that some finite groups are characterizable by their order components.

In this paper we prove that $PSU(p+1,q)$ is uniquely determined by its order component(s) if and only if $(q+1) \mid (p+1)$. A main consequence of our results is the validity of Thompson's conjecture for the groups $PSU(p+1,q)$ where $(q+1)|(p+1)$.

1. Introduction. Let $\pi(n)$ be the set of prime divisors of n, where n is a positive integer. If G is a finite group, then $\pi(G)$ is defined to be $\pi(|G|)$. By using the orders of elements in G, we construct the prime graph of G as follows.

The *prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \dots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

Now $|G|$ can be expressed as a product of coprime positive integers m_i, $i = 1, 2, \dots, t(G)$ where $\pi(m_i) = \pi_i$. These integers are called *the order components* of G. The set of order components of G will be denoted by $OC(G)$. Also we call $m_2, \dots, m_{t(G)}$ *the odd order components* of G. The order components of non-abelian simple groups having at least three prime graph components are obtained by Chen [**9**, Tables 1–3]. Similarly the order components of non-abelian simple groups with two order components can be obtained by using the tables

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in [**28, 35**], see [**20**]. By using these tables we know that

$$OC(PSU(p+1,q)) = \left\{ m_1 = q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-(-1)^i),\ m_2 = \frac{q^p+1}{q+1} \right\}.$$

The following groups are uniquely determined by their order components: Suzuki-Ree groups [**7**], sporadic simple groups [**4**], almost sporadic simple groups, except Aut (McL) and Aut (J_2) [**22**], $E_6(q)$ [**27**], ${}^2E_6(q)$ [**26**], $E_8(q)$ [**8**], $G_2(q)$ where $q \equiv 0 \pmod 3$ [**3**], $F_4(q)$ where $q = 2^n$ [**19**], $C_2(q)$ where $q > 5$ [**20**], ${}^2D_n(q)$ where $n = 2^m \geq 4$ [**24, 25**], $PSL(p,q)$ [**9, 14, 15, 23**] and $PSU(p,q)$, where $p = 3,\ 5,\ 7,\ 11$ [**16–18, 21**].

In this paper, we prove the following theorem:

Main theorem. *Let G be a finite group and $M = PSU(p+1,q)$, where p is an odd prime number. Then*

(a) *if $(q+1) \mid (p+1)$, then $G \cong M$ if and only if $OC(G) = OC(M)$,*

(b) *if $(q+1) \nmid (p+1)$ then M is not characterizable by its order component.*

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [**10**], for example. Also frequently we use the results of Williams [**35**] and Kondrat'ev [**28**] about the prime graph of simple groups.

We denote by (a,b) the greatest common divisor of positive integers a and b. Let m be a positive integer and p a prime number. Then $|m|_p$ denotes the $p-$part of m. In other words, $|m|_p = p^k$ if $p^k \| m$, i.e., $p^k | m$ but $p^{k+1} \nmid m$.

We recall that a Mersenne prime is a prime number of the form $2^n - 1$.

2. Preliminary results. The proof of the main theorem depends on the classification of finite simple groups and the following lemmas. We begin with an easy remark.

Remark 2.1 [**22**]. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact, if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

Definition 2.1 [**13**]. A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

We use the following unpublished result of Gruenberg and Kegel [**12**].

Lemma 2.1 [**35**, Theorem A]. *If G is a finite group with its prime graph having more than one component, then G is one of the following groups*:

(a) *a Frobenius or a 2-Frobenius group*;

(b) *a simple group*;

(c) *an extension of a π_1-group by a simple group*;

(d) *an extension of a simple group by a π_1-solvable group*;

(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.2. [**35**, Lemma 3]. *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

The next lemma follows from Theorem 2 in [**2**]:

Lemma 2.3. *Let G be a Frobenius group of even order, and let H and K be the Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures*:

(a) $2 \in \pi(K)$ *and all Sylow subgroups of H are cyclic*;

(b) $2 \in \pi(H)$, *K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the 2-Sylow subgroups of H are cyclic or generalized quaternion groups*;

(c) $2 \in \pi(H)$, *K is an abelian group and there exists* $H_0 \leq H$ *such that* $|H : H_0| \leq 2$, $H_0 = Z \times SL(2,5)$, $(|Z|, 2 \times 3 \times 5) = 1$ *and the Sylow subgroups of* Z *are cyclic.*

The next lemma follows from Theorem 2 in [**2**] and Lemma 2.2:

Lemma 2.4. *Let G be a* 2*-Frobenius group of even order. Then* $t(G) = 2$ *and G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that*

(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi_2 = \pi(K/H)$;

(b) G/K *and* K/H *are cyclic,* $|G/K|$ *divides* $|\mathrm{Aut}\,(K/H)|$, $(|G/K|, |K/H|) = 1$ *and* $|G/K| < |K/H|$;

(c) H *is nilpotent and G is a solvable group.*

Lemma 2.5 [**6**, Lemma 8]. *Let G be a finite group with* $t(G) \geq 2$ *and let N be a normal subgroup of G. If N is a* π_i*−group for some prime graph component of G and* $m_1, m_2, \ldots, m_r$ *are some order components of G but not* π_i*-numbers, then* $m_1 m_2 \cdots m_r$ *is a divisor of* $|N| - 1$.

Lemma 2.6 [**5**, Lemma 1.4]. *Let G and M be two finite groups satisfying* $t(M) \geq 2$, $N(G) = N(M)$, *where* $N(G) = \{n \mid G$ has a conjugacy class of size $n\}$, *and* $Z(G) = 1$. *Then* $|G| = |M|$.

The next lemma follows from Lemma 1.5 in [**5**].

Lemma 2.7. *Let* G_1 *and* G_2 *be finite groups satisfying* $|G_1| = |G_2|$ *and* $N(G_1) = N(G_2)$. *Then* $t(G_1) = t(G_2)$ *and* $OC(G_1) = OC(G_2)$.

Lemma 2.8 [**22**]. *Let G be a finite group, and let M be a non-abelian finite group with* $t(M) = 2$ *satisfying* $OC(G) = OC(M)$. *Let* $|M| = m_1 m_2$, $OC(M) = \{m_1, m_2\}$ *and* $\pi(m_i) = \pi_i$ *for* $i = 1, 2$. *Then* $|G| = m_1 m_2$, *and one of the following holds*:

(a) *G is a Frobenius or a* 2*-Frobenius group*;

(b) *G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* G/K *is a* π_1*-group, H is a nilpotent* π_1*-group, and* K/H *is a non-abelian sim-*

ple group. Moreover, $OC(K/H) = \{m'_1, m'_2, \dots, m'_s, m_2\}$, $|K/H| = m'_1 m'_2 \cdots m'_s m_2$ *and* $m'_1 m'_2 \cdots m'_s | m_1$ *where* $\pi(m'_j) = \pi_j(K/H)$, $1 \leq j \leq s$. *Also* $|G/K|$ *divides* $|Out(K/H)|$.

3. Some related results. As corollaries of the main theorem we prove a conjecture which was put forward by Thompson and another conjecture which arose by Shi and Bi, for the group $PSU(p+1,q)$, where $(q+1)|(p+1)$.

Thompson's conjecture. *If* G *is a finite group with* $Z(G) = 1$ *and* M *is a non-abelian simple group satisfying* $N(G) = N(M)$, *then* $G \cong M$.

We can give a positive answer to this conjecture for the groups $PSU(p+1,q)$, where $(q+1)|(p+1)$, by our characterization of these groups.

Theorem 3.1. *Let* $M = PSU(p+1,q)$, *where* $(q+1)|(p+1)$. *If* G *is a finite group with* $Z(G) = 1$ *and* $N(G) = N(M)$, *then* $G \cong M$.

Proof. By using Lemmas 2.6 and 2.7 we conclude that the order components of G and M are the same. So the result follows by using the main theorem. □

Also Shi and Bi in [**32**] put forward the following conjecture:

Conjecture. *Let* G *be a group and* M *a finite simple group. Then* $G \cong M$ *if and only if*

(i) $|G| = |M|$,

(ii) $\pi_e(G) = \pi_e(M)$, *where* $\pi_e(G)$ *denotes the set of orders of elements in* G.

This conjecture is valid for sporadic simple groups [**29**], alternating groups [**33**], some simple groups of Lie type [**30**–**32**] and some almost simple groups [**22**]. As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.2. *Let G be a finite group and $M = PSU(p+1, q)$, where $(q+1) \mid (p+1)$. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.*

Proof. By assumption the prime graphs of G and H are the same and also we have $OC(G) = OC(M)$. Thus the result follows by the main theorem. □

4. Number theoretic lemmas. For the proof of the main theorem we need some results about the numbers and specially about the greatest common divisor of numbers. Hence in this section we state a few number theoretical lemmas.

Lemma 4.1. *Let p be a prime number and q a prime power. If $(q+1) \mid (p+1)$, then $m_2 - 1 = (q^p+1)/(q+1) - 1$ is not a power of 2.*

Proof. If $(q^p+1)/(q+1) = 2^t + 1$, for some $t > 0$, then $q(q^{p-1} - 1)/(q+1) = 2^t$. But $(q, (q^{p-1} - 1)/(q+1)) = 1$, which implies that $q = 2^t$ and $q^{p-1} - 1 = q + 1$. Therefore $q = 2$ and $p = 3$, which is a contradiction. □

Lemma 4.2. *If n is an integer and x is a prime number, then $|n!|_x$ divides $x^{[(n-1)/(x-1)]}$.*

Proof. Let $x^t \le n < x^{t+1}$. Then $|n!|_x = x^k$ where

$$k = \left[\frac{n}{x}\right] + \left[\frac{n}{x^2}\right] + \cdots + \left[\frac{n}{x^t}\right] \le \frac{n}{x} + \frac{n}{x^2} + \cdots + \frac{n}{x^t}$$
$$= \frac{n}{x} \times \frac{1-(1/x^t)}{1-(1/x)} \le \frac{n(1-(1/n))}{x(1-(1/x))} = \frac{n-1}{x-1}. \qquad \square$$

The following result of Kondrat'ev will be used several times.

Lemma 4.3 [**28**, Lemma 3]. *Let a, m and n be natural numbers. Then*

(a) $(a^n - 1, a^m - 1) = a^{(m,n)} - 1$;

(b) $(a-1,\, (a^n-1)/(a-1)) = (n, a-1)$;

(c) $((a^n-1)/(a^{(m,n)}-1),\, a^m-1) = (n/(m,n), a^{(m,n)}-1)$.

Lemma 4.4. *Let i and $q>1$ be natural numbers.*

(a) *If i is odd, then $(q+1,\, (q^i+1)/(q+1))$ divides $(i, q+1)$;*

(b) *if i is even, then $(q+1,\, (q^i-1)/(q^2-1))$ divides $(i/2, q+1)$;*

(c) *if i is odd, then $(q+1,\, (q^i-1)/(q-1)) = 1$.*

Proof. (a) We know that $(q+1) \mid (q^2-1)$ and $(q^i+1)/(q+1)$ is a divisor of $(q^{2i}-1)/(q^2-1)$. Therefore, if $k = (q+1, (q^i+1)/(q+1))$, then k divides $(q^2-1, (q^{2i}-1)/(q^2-1))$. Hence $k \mid (i, q^2-1)$, by Lemma 4.3. So $k \mid (q+1)$, $k \mid i$ and $k \mid (q^2-1)$, which implies that $k \mid (i, q+1)$, since i is odd.

(b) and (c). The proofs are similar to (a) and we omit them for convenience. □

Similarly we can prove the following lemma.

Lemma 4.5. *Let i and $q>1$ be natural numbers.*

(a) *If i is odd, then $(q-1,\, (q^i+1)/(q+1))$ divides $(i, q-1)$;*

(b) *if i is even, then $(q-1,\, (q^i-1)/(q^2-1))$ divides $(i/2, q-1)$.*

Lemma 4.6. *Let x be an odd prime number and n, $q>1$ positive integers.*

(a) *If $x \mid (q-1)$, then $|(q^n-1)/(q-1)|_x$ divides $|n|_x$;*

(b) *if $x \mid (q+1)$ and $2 \mid n$, then $|(q^n-1)/(q^2-1)|_x$ divides $|n/2|_x$;*

(c) *if $x \mid (q^s-1)$ and $s \mid n$, then $|(q^n-1)/(q^s-1)|_x$ divides $|n/s|_x$.*

Proof. (a) By using Lemma 4.3, we have $x \mid n$. Let $q = kx+1$, for some $k>0$. Then $(q^n-1)/(q-1) = \sum_{r=1}^{n} \binom{n}{r}(kx)^{r-1}$. Now we claim that if $x^m \mid (q^n-1)/(q-1)$ then $x^m \mid n$. It is true for $m=1$. Now we use induction on m. Let $x^{m+1} \mid (q^n-1)/(q-1)$ and so $x^m \mid n$. If $r \leq 2$ or $r > m+1$, then $x^{m+1} \mid \binom{n}{r}(kx)^{r-1}$ and it is sufficient to prove this statement for $3 \leq r \leq m+1$. In fact, we must prove that $x^{m+1} \mid x^{m+r}/(r+1)!$. By Lemma 4.2, we must prove that $m+1 \leq m+r-r/(x-1)$, and it is true for $r \geq 3$. Therefore $|(q^n-1)/(q-1)|_x \mid |n|_x$.

Part (b) is a special case of part (c).

For the proof of (c), let $q' = q^s$ and then use part (a). ◻

Lemma 4.7 [**11**]. *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$, has only one solution, namely $3^2 - 2^3 = 1$.*

5. Proof of the main theorem. First we prove part (a) of the main theorem and then we discuss part (b).

If $q = 2^n$, then $(q-1, q+1) = 1$ but if $q = p_0^n$, where p_0 is an odd prime number, then $(q-1, q+1) = 2$. Therefore we have to consider two cases. First, let q be an odd prime power and $(p,q) \neq (5,2)$, $(3,3)$. Hence, in the following lemmas and in the proof of the main theorem we suppose that p is a prime number and q is an odd prime power. The proof of the other case, i.e., $q = 2^n$, is similar and is not so complicated. Hence we omit the proof for convenience.

Lemma 5.1. *Let $q \neq 5$ be an odd prime power which is not a Mersenne prime and $M = PSU(p+1, q)$, where $(q+1) \mid (p+1)$ and $(p,q) \neq (3,3)$, $(5,2)$. Then the following holds*:

(a) *if $x \in \pi_1(M)$, then $|S_x| \leq q^{p(p+1)/2}$ where $S_x \in \mathrm{Syl}_x(M)$*;

(b) *if $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha + 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{(2k+1)p}$, where $1 \leq 2k+1 \leq (p+1)/2$*;

(c) *if $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha - 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$.*

Proof. We will prove (a), (b) and (c), simultaneously. By an easy calculation we can see that the results hold for $p \leq 19$. So in the proof of this lemma we let $p > 19$.

Also, since we want to use the last lemmas, we need to factor m_1 as follows:

$$\begin{aligned} m_1 &= q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-(-1)^i) \\ &= q^{p(p+1)/2}(q-1)^{(p+1)/2}(q+1)^{p-1}\times\frac{q^{p+1}-1}{q^2-1} \\ &\quad\times\prod_{i \text{ odd},\, i=2}^{p-1}\frac{q^i+1}{q+1}\times\prod_{i \text{ even},\, i=2}^{p-1}\frac{q^i-1}{q^2-1}. \end{aligned}$$

Now let x be a prime number and x^α a divisor of m_1. For the purpose of using the above lemmas, we consider the following steps:

Step 1. If $x=2$ and $4|(q-1)$, then $2\|(q+1)$, and hence $(q+1,q-1)=2$. If k is odd, then $|(q-1,(q^k+1)/(q+1))|_2=1$, by using Lemma 4.5. Also $|(q-1,(q^{2k}-1)/(q^2-1))|_2$ divides $|k|_2$. Therefore 2^α is a divisor of

$$\begin{aligned} 2^{p-1}\times\left|\frac{p+1}{2}\right|_2\times\left|\frac{p-1}{2}\right|_2\times\cdots\times 1\times(q-1)^{(p+1)/2} \\ = 2^{p-1}\times|((p+1)/2)!|_2\times(q-1)^{(p+1)/2}. \end{aligned}$$

Then by using Lemma 4.2, we have $2^\alpha \mid 2^{3(p-1)/2}(q-1)^{(p+1)/2}$.

If $q-1=2^\beta A$, where $A\geq 3$ is odd, then obviously we have

$$2^\alpha \leq \frac{8^{(p-1)/2}(q-1)^{(p+1)/2}}{3^{(p+1)/2}} < m_2-1,$$

which implies that $2^\alpha\pm 1\not\equiv 0 \pmod{m_2}$. So let $q-1=2^\beta$ and note that $q\geq 9$. If $q=9$, then $2^\alpha=8^p\leq m_2=(9^p+1)/10$, since $p\geq 1/\log(9/8)$. If $q>9$, then $q\geq 17$ and hence $8\leq(q-1)/2$. But then

$$(q+1)(q-1)^p\leq 2^{(p-1)/2}(q^p+1),$$

which implies that $2^\alpha<m_2-1$ and hence $2^\alpha\pm 1\not\equiv 0 \pmod{m_2}$.

Step 2. If $x=2$ and $4|(q+1)$, then $2\|(q-1)$. Similar to Step 1 and by using Lemma 4.4, we have $|(q+1,(q^{2t+1}+1)/(q+1))|_2=1$

and $|(q+1,(q^{2k}-1)/(q^2-1))|_2$ divides $|k|_2$, where $1 \leq k \leq (p+1)/2$. Therefore 2^α is a divisor of $2^{(p+1)/2} \times |((p+1)/2)!|_2 \times (q+1)^{p-1}$. Hence, by using Lemma 4.2, we have $2^\alpha \mid 2^p(q+1)^{p-1}$.

Since q is not a Mersenne prime, $q+1 = 2^\beta A$, where $A \geq 3$ is an odd number. Then $2^\alpha \mid 2^p(2^\beta)^{p-1} = 2^p((q+1)/A)^{p-1}$, which implies that

$$2^\alpha \leq 2^p\left(\frac{q+1}{A}\right)^{p-1} \leq 2^p\left(\frac{q+1}{3}\right)^{p-1} \leq m_2 - 1.$$

Therefore, $2^\alpha \pm 1 \not\equiv 0 \pmod{m_2}$.

Step 3. Let $x^\alpha | q^{p(p+1)/2}$. Since q is a prime power, we have $q = x^n$, for some $n > 0$.

Now let $x^\alpha | q^p$ and $x^\alpha + 1 \equiv 0 \pmod{m_2}$, which implies that $(x^\alpha+1)(q+1) = t(q^p+1)$, for some $t > 0$. Also $m_2 \leq x^\alpha + 1 \leq q^p + 1$ and $q+1 < m_2$, which implies that $q \mid x^\alpha$. Therefore $q \mid (t-1)$, and so $q+1 \leq t$. On the other hand, $x^\alpha + 1 = t(q^p+1)/(q+1) \leq q^p + 1$ and so $1 \leq t \leq q+1$. Therefore, $t = q+1$ and $x^\alpha = q^p$.

If $x^\alpha \mid q^p$ and $x^\alpha - 1 \equiv 0 \pmod{m_2}$, then $(x^\alpha-1)(q+1) = t(q^p+1)$, for some $t > 0$. Also $q \mid x^\alpha$ and hence $q \mid (t+1)$, which implies that $t \geq q-1$. Since $t < q+1$, we conclude that $t = q-1$ or $t = q$. Obviously $t \neq q$, and hence $t = q-1$. Then $x^\alpha(q+1) = q(q^p - q^{p-1} + 2)$, which is a contradiction, since $q < x^\alpha$.

It follows that, if x^α is a divisor of q^p, then $x^\alpha - 1 \not\equiv 0 \pmod{m_2}$, but if $x^\alpha + 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^p$.

Let $q^p < x^\alpha \leq q^{2p}$ and $x^\alpha \mid q^{2p}$. Similarly we can prove that $x^\alpha + 1 \not\equiv 0 \pmod{m_2}$. Let $x^\alpha - 1 \equiv 0 \pmod{m_2}$. Since $q^p < x^\alpha$, we have $x^\alpha = q^p x^t$, for some $t > 0$, where $0 < x^t \leq q^p$. Therefore,

$$x^\alpha - 1 = q^p x^t + x^t - x^t - 1 = x^t(q+1)m_2 - x^t - 1,$$

which implies that $m_2 \mid (x^t+1)$. Hence $x^t = q^p$, and so $x^\alpha = q^{2p}$.

By using this method and by induction on k, it is proved that if $q^{2kp} < x^\alpha \leq q^{(2k+1)p}$, then $x^\alpha - 1 \not\equiv 0 \pmod{m_2}$, and if $x^\alpha + 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{(2k+1)p}$. Also if $q^{(2k-1)p} < x^\alpha \leq q^{2kp}$, then $x^\alpha + 1 \not\equiv 0 \pmod{m_2}$, and if $x^\alpha - 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{2kp}$.

Step 4. If x is an odd prime and $x|(q+1)$, then let $q+1=x^{\beta}A$, where $x\nmid A$ and $\beta\geq 1$. Then $A\geq 2$ is even, because q is an odd prime power, so let $A=2k$, for some $k>0$. Again by using Lemmas 4.4 and 4.5, we have $(q+1,q)=1$, $|(q+1,q-1)|_x=1$, $|(q+1,(q^i+1)/(q+1))|_x=|i|_x$, where i is odd, and $|(q+1,(q^i-1)/(q^2-1))|_x=|i/2|_x=|i|_x$, where i is even. Hence x^{α} divides $|(p+1)!|_x\times((q+1)/2)^{p-1}$. Now, by using Lemma 4.2, we have $x^{\alpha}|x^{[p/(x-1)]}((q+1)/2)^{p-1}$. Suppose $q+1=2x^{\beta}k$, where $\beta\geq 1$, $k\geq 1$ and $x\nmid k$. Now we consider two cases:

Case I. Let $k=1$. Then $x^{\beta}=(q+1)/2$. Note that $q\neq 5$. Hence we split the proof into two subcases according to the following possibilities for q:

(1) $x=3$ and $\beta\geq 2$,

(2) $x\geq 5$ and $\beta\geq 1$.

(I) If $x=3$ and $\beta\geq 2$, then $q\geq 17$, since $q\neq 5$. Hence $16(q+1)/17\leq q$. Also, since $p\geq q\geq 17$, we have $2^{1/p}\sqrt{3}\leq 2^{1/17}\sqrt{3}\leq 2\times 16/17$, which implies that

$$\frac{\sqrt{3}^{\,p}}{2^{p-1}}\leq\left(\frac{16}{17}\right)^{p}\Longrightarrow\frac{3^{p/2}(q+1)^{p-1}}{2^{p-1}}<\frac{q^p+1}{q+1}-1\Longrightarrow 3^{\alpha}<m_2-1,$$

and hence $3^{\alpha}\pm 1\not\equiv 0\pmod{m_2}$.

(II) If $x\geq 5$ and $\beta\geq 1$, then $q\geq 9$ and hence $8(q+1)/9\leq q$. If $f(t)=t^{1/(t-1)}$ and $g(t)=2^{1/t}$, where $t\geq 3$, then $f(t)$ and $g(t)$ are decreasing functions. So

$$2^{1/p}x^{1/(x-1)}\leq 2^{1/9}\times 5^{1/4}\leq\frac{16}{9}\Longrightarrow\frac{x^{p/(x-1)}}{2^{p-1}}\leq\left(\frac{8}{9}\right)^{p}\Longrightarrow x^{\alpha}<m_2-1,$$

which implies that $x^{\alpha}\pm 1\not\equiv 0\pmod{m_2}$.

Case II. Let $k\geq 2$. Then $p\geq q\geq 11$ and, similar to Case (I), we have $x^{1/(x-1)}\times 2^{-2/p}\leq 2$. Therefore

$$x^{\alpha}\leq\frac{x^{p/(x-1)}(q+1)^{p-1}}{2^{p-1}}\leq m_2-1,$$

which implies that $x^{\alpha}\pm 1\not\equiv 0\pmod{m_2}$.

Step 5. If x is an odd prime and $x|(q-1)$, then let $q-1=x^{\beta}A$, where x does not divide A and $\beta \geq 1$. Similar to the last steps $A \geq 2$ is even, since q is odd and by using Lemmas 4.3 and 4.6 we conclude that x^{α} is a divisor of $|((p+1)/2)!|_x \times ((q-1)/2)^{(p+1)/2}$. Now by using Lemma 4.2, we have

$$x^{\alpha} \;\Big|\; x^{[(p-1)/2(x-1)]}\left(\frac{q-1}{2}\right)^{(p+1)/2}.$$

But easily we can see that

$$x^{[(p-1)/2(x-1)]}\left(\frac{q-1}{2}\right)^{(p+1)/2} < \frac{q^p+1}{q+1}-1=m_2-1,$$

which implies that $x^{\alpha} \pm 1 \not\equiv 0 \pmod{m_2}$.

Step 6. Let $x|(q^s+1)/(q+1)$, where $3 \leq s \leq p-1$ is an odd prime number. Also let $x \nmid (q^2-1)$, since the divisors of q^2-1 were discussed in the last steps. Obviously, $(q,(q^s+1)/(q+1))=1$. If n is even and $s \nmid n$, then $|((q^s+1)/(q+1),(q^n-1)/(q^2-1))|_x=1$, since

$$\left(\frac{q^s+1}{q+1},\frac{q^n-1}{q^2-1}\right) \Bigg| \left(q^{2s}-1,\frac{q^n-1}{q^2-1}\right)=\left(q^2-1,\frac{n}{2}\right).$$

If n is even and $s \mid n$, then $(q^s+1)/(q+1)$ divides $(q^n-1)/(q^2-1)$. Hence

$$\left(q^s+1,\frac{q^n-1}{q^s+1}\right) \;\Big|\; \frac{n}{2s} \Longrightarrow \left|\left(q^s+1,\frac{q^n-1}{q^s+1}\right)\right|_x \;\Big|\; \left|\frac{n}{2s}\right|_x.$$

If i is odd and $s \nmid i$, then by using Lemma 4.3, we have

$$\left(q^s+1,\frac{q^i+1}{q+1}\right) \;\Big|\; \left(q^{2s}-1,\frac{q^{2i}-1}{q^2-1}\right)=(i,q^2-1),$$

since s is an odd prime number and $(s,i)=1$. Therefore,

$$\left|\left(q^s+1,\frac{q^i+1}{q+1}\right)\right|_x=\left|\left(q^s+1,\frac{q^{2i}-1}{q^2-1}\right)\right|_x=1.$$

If i is odd and $s \mid i$, then $(q^s+1)/(q+1)$ divides $(q^i+1)/(q+1)$. So by using Lemma 4.6, we have $|(q^i+1)/(q^s+1)|_x \mid |i/s|_x$.

The above relations show that

$$x^\alpha \;\Bigg|\; \left|\left[\frac{p-1}{s}\right]!\right|_x \left(\frac{q^s+1}{q+1}\right)^{[(p-1)/s]} \Longrightarrow x^\alpha \mid x^{[(p-1)/s(x-1)]} \left(\frac{q^s+1}{q+1}\right)^{[(p-1)/s]},$$

by Lemma 4.2. But then $x^\alpha < m_2 - 1$, which implies that $x^\alpha \pm 1 \not\equiv 0 \pmod{m_2}$.

Step 7. Let $x|(q^s+1)/(q+1)$, where s is odd but s is not a prime number; or $x|(q^s-1)/(q-1)$, where s is odd; or $x|(q^{2^t}+1)$, where $t \geq 1$.

Then, similar to Step 6, we conclude that $x^\alpha \pm 1 \not\equiv 0 \pmod{m_2}$. For convenience we omit the proof of this step.

Now the proof of this lemma is completed. □

Remark 5.1. The proof of Lemma 5.1 shows that if $q = 5$ or if q is a Mersenne prime and x is an odd prime number, then again Lemma 5.1 holds. There might be some $\alpha > 0$ such that $2^\alpha | m_1$ and $2^\alpha + 1 \equiv 0 \pmod{m_2}$ or $2^\alpha - 1 \equiv 0 \pmod{m_2}$, for these q's (but we strongly guess that no such α exists). Therefore,

(a) If $x^\alpha \mid m_1$ and $x^\alpha - 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$ or $x = 2$;

(b) if $x^\alpha | \, m_1$ and $x^\alpha + 1 \equiv 0 \pmod{m_2}$, then $x^\alpha = q^{(2k+1)p}$, where $1 \leq 2k+1 \leq (p+1)/2$ or $x = 2$.

(c) Let q be a Mersenne prime. If $2^\alpha \mid |M|$ and $2^\alpha + \varepsilon \equiv 0 \pmod{m_2}$, where $\varepsilon = 1$ or $\varepsilon = -1$, then $2^\alpha \mid 2^p(q+1)^{p-1}$. But since $2^p(q+1)^{p-1} < m_2^2 < 2^{2\alpha}$, we have $2^{2\alpha} \nmid |M|$.

(d) If $q = 5$, $2^\alpha \mid |M|$ and $2^\alpha + \varepsilon \equiv 0 \pmod{m_2}$, where $\varepsilon = 1$ or $\varepsilon = -1$, then $2^\alpha \mid 2^{5(p-1)/2}$. But then $2^{2\alpha} \nmid |M|$.

Lemma 5.2. *Let G be a finite group and $M = PSU(p+1, q)$, where $(q+1) \mid (p+1)$. If $OC(G) = OC(M)$, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. First let G be a Frobenius group, where H and K are the Frobenius complement and Frobenius kernel of G, respectively. Then $OC(G) = \{|H|, |K|\}$, by Lemma 2.3. Also $|H|$ is a divisor of $|K| - 1$ and hence $|H| < |K|$. Since $m_1 > m_2$, we conclude that $|H| = m_2$ and $|K| = m_1$. We know that $\pi(m_1) \geq 3$. So let p_0 be an odd prime number which divides m_1 and $p_0 \nmid q$. Let P_0 be a Sylow p_0-subgroup of K. Then $P_0 \lhd G$, since K is nilpotent. Hence m_2 divides $|P_0| - 1$, by Lemma 2.5. Therefore $|P_0| = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$, or $|P_0| = 2^{\alpha}$, by Lemma 5.1 and Remark 5.1, which is a contradiction. It follows that G is not a Frobenius group.

Now let G be a 2-Frobenius group. So there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K is a Frobenius group with kernel H, and G/H is a Frobenius group with kernel K/H. By using Lemma 2.4, we have $|K/H| = m_2$ and $|G/K| < m_2$. Therefore $|H| \neq 1$, since $|G| = |G/K| \cdot |K/H| \cdot |H|$. Now let $p_0 \in \pi_1$ be an odd prime number such that p_0 does not divide q. Also we can choose p_0 such that $p_0 \mid (q-1)^{(p+1)/2}(q+1)^{p-1}$, since $m_2 < (q-1)^{(p+1)/2}(q+1)^{p-1}$. If P_0 is a Sylow p_0-subgroup of H, then $P_0 \lhd K$, since H is nilpotent. Therefore m_2 is a divisor of $|P_0| - 1$, by Lemma 2.5, which is a contradiction. Therefore G is not a 2-Frobenius group. □

Lemma 5.3. *Let G be a finite group and $M = PSU(p+1, q)$, where $(q+1) \mid (p+1)$. If $OC(G) = OC(M)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd order component of M is equal to an odd order component of K/H. In particular, $t(K/H) \geq 2$. Also $|G/H|$ divides $|\mathrm{Aut}\,(K/H)|$, and in fact $G/H \leq \mathrm{Aut}\,(K/H)$.*

Proof. The proof is similar to the proof of Lemma 3.2 in [**22**]. □

Proof of the main theorem. We know that $OC(G) = OC(PSU(p+1, q))$, so by using Lemma 5.3, there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a non-abelian simple group with non-connected prime graph, $\pi(H) \cup \pi(G/K) \subset \pi_1$ and the odd order component of G, i.e., $(q^p+1)/(q+1)$, is an odd order component of K/H. Obviously $\{2, m_2\} \subseteq \pi(K/H)$, since non-abelian simple groups have even order. Now by using the classification of finite simple groups, the possibilities for K/H are:

(a) sporadic simple groups,

(b) the alternating groups (A_n, $n \geq 5$),

(c) simple groups of Lie type.

In the sequel, by using the results in Tables 1–3 in [**20**] we prove that the only possibility for K/H is $PSU(p+1, q)$.

Step 1. Let $K/H \cong S$, where S is a sporadic simple group.

Since $(p, q) \neq (3, 3)$ and $(5, 2)$, we have $m_2 \geq 521$. This is a contradiction, since the odd order components of sporadic simple groups are less than 521 and hence $m_2 < 521$.

Therefore K/H is not a sporadic simple group.

Step 2. Let $K/H \cong A_n$, where $n = p'$, $p'+1$, $p'+2$ and $p' \geq 5$ is a prime number.

Then $m_2 = p'$ or $p'-2$, where $p' \geq 5$ is an odd prime number.

If $2^t \| \, |A_{p'}|$, then $t > [p'/2] + [p'/4] + [p'/8]$, where $p' \geq m_2$, which implies that $t > 3m_2/4$. Hence $2^{3m_2/4} | \, |K/H|$. As we can see from the proof of Lemma 5.1, if $2^\alpha \| \, |G|$, then $2^\alpha \leq 2^p(q+1)^{p-1}$ or $2^\alpha \leq 2^{3(p-1)/2}(q-1)^{(p+1)/2}$. But it is obvious that

$$\alpha \leq p(1 + \log_2(q+1)) < \frac{3}{4} \times \frac{q^p+1}{q+1},$$

which is a contradiction.

Therefore K/H is not an alternating group.

Step 3. If K/H is a simple group of Lie type, then K/H can be isomorphic to one of the groups listed in Tables 1–3 in [**21**]. Since the proofs are similar, we only do a few of them. For convenience let $X = \{5\} \cup \{x \mid x \text{ is a Mersenne prime}\}$. In the sequel p' will be an odd prime number and q' will be a prime power.

• If K/H is isomorphic to $A_1(q')$ where $4 \mid (q'+1)$, then $m_2 = (q'-1)/2$ or $m_2 = q'$.

If $m_2 = q'$, then we consider two cases:

If $|A_1(m_2)|$ does not divide $|G|$, then obviously we get a contradiction.

If $|A_1(m_2)| \mid |G|$, then

$$\frac{|G|}{|K/H|} = \frac{|G|}{|A_1(m_2)|} = |H| \cdot |G/K| \neq 1.$$

By Lemma 5.3, $|G/K| \mid |\mathrm{Out}\,(K/H)|$, and if $q = p_0^n$, then

$$|\mathrm{Out}\,(A_1(m_2))| \mid 2n,$$

which implies that $|H| \neq 1$. Now let x be an odd prime number such that x does not divide q and $x \mid |H|$. Then let T be a Sylow $x-$subgroup of H. Since H is nilpotent, $T \lhd G$. Hence $m_2 \mid (|T|-1)$, by Lemma 2.5. Therefore $|T| = q^{2kp}$, by Lemma 5.1, which is a contradiction.

If $m_2 = (q'-1)/2$, then $q' = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$, because q' is odd. Therefore

$$(q+1)(q^{2kp}-1)/(q^p+1) = 2,$$

which is impossible, since $q+1 \geq 3$.

• If K/H is isomorphic to $A_1(q')$ where $4|(q'-1)$, then $m_2 = (q'+1)/2$ or $m_2 = q'$. Again we get a contradiction, similar to the last case.

• If K/H is isomorphic to $A_1(q')$ where $4|q'$, then $m_2 = q'+1$or $m_2 = q'-1$. Obviously $m_2 \neq q'+1$, by Lemma 4.1. If $m_2 = q'+1$, then we can proceed similar to the above case and get a contradiction.

• Also K/H is not isomorphic to $A_{p'}(q')$, where $(q'-1)|(p'+1)$, or $A_{p'-1}(q')$. For example, if K/H is isomorphic to $A_{p'-1}(q')$, then

$$\frac{q'^{p'}-1}{(q'-1)(p',q'-1)} = \frac{q^p+1}{q+1}, \tag{1}$$

which implies that $q'^{p'} = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$, or $q'^{p'} = 2^{\alpha}$, for some $\alpha > 0$.

If ${q'}^{p'} = q^{2kp}$, where $1 \leq 2k \leq (p+1)/2$, then by using Lemma 5.1(c), we have $(q^{2kp})^{(p'-1)/2} \leq q^{p(p+1)/2}$ which implies that $2k(p'-1) \leq p+1$ and hence $p' < p$. Also (1) implies that

$$(q'-1)(p', q'-1) = (q^p-1)(q+1)(q^{2p(k-1)} + \cdots + q^{2p} + 1).$$

But $(q'-1)(p', q'-1) \leq (q'-1)^2 < q'^2 \leq q^{4kp/3}$, because $q'^3 \leq {q'}^{p'} = q^{2kp}$. If $k \geq 3$, then $q'^2 \leq q^{4kp/3} \leq q^{2p(k-1)} < (q+1)(q^{2kp}-1)/(q^p+1)$, which is a contradiction by (1). Also, for $k = 1, 2$ we can easily get a contradiction. For example, let $k = 1$. Suppose $q = x^s$ and $q' = x^t$, where x is a prime number. Then $tp' = 2ps$, which implies that $s = p'\alpha$ and $t = 2p\alpha$, for some $\alpha > 0$, since $p \neq p'$. Therefore $q = x^{p'\alpha}$ and $q' = x^{2p\alpha}$. Hence

$$\frac{x^{pp'\alpha} - 1}{(x^{2p\alpha}-1)(p', x^{2p\alpha}-1)} = \frac{1}{x^{p'\alpha}+1}. \tag{2}$$

But it is straightforward to see that

$$(x^{pp'\alpha}-1)(x^{p'\alpha}+1) > (x^{2p\alpha}-1)(p', x^{2p\alpha}-1),$$

and it is a contradiction.

If ${q'}^{p'} = 2^\alpha$, then $p' = 3$, by Remark 5.1. Again we get a contradiction, similarly.

• If K/H is isomorphic to one of the following simple groups:

$B_n(q')$ where $n = 2^m \geq 4$, q' odd;

$C_n(q')$ where $n = 2^m \geq 2$, q' odd;

${}^2D_n(q')$ where $n = 2^m \geq 4$, q' odd;

then $m_2 = (q'^n + 1)/2$, which implies that $q'^n = q^{(2k+1)p}$, where $1 \leq 2k+1 \leq (p+1)/2$. Then $2 = (q+1)(q^{(2k+1)p}+1)/(q^p+1)$, which is a contradiction, since $q \geq 2$.

• If K/H is isomorphic to one of the following simple groups:

$C_n(q')$, where $n = 2^m \geq 2$, $q' = 2^t$, $t \geq 1$;

${}^2D_n(q')$, where $n = 2^m \geq 4$, $q' = 2^t$, $t \geq 1$;

${}^2D_n(2)$, where $n = 2^m + 1 \geq 5$, is not a prime number;

${}^2D_{p'+1}(2)$ where $p' = 2^n - 1$, $n \geq 2$,

$F_4(q')$, where $q' = 2^t$, $t > 1$;

then the odd order component(s) of K/H is (are) equal to $2^s + 1$, for some $s > 0$. Hence $2^s+1 = m_2$, which is a contradiction, by Lemma 4.1.

• If K/H is isomorphic to one of the following simple groups:

${}^2D_{p'}(3)$ where $p' = 2^m + 1 \geq 5$ is a prime number;

${}^2D_n(3)$ where $n = 2^m + 1 \geq 5$ is not a prime number;

${}^2D_{p'}(3)$ where $p' \neq 2^m + 1$, $p' \geq 5$ is a prime number,

then similarly we get a contradiction, by using the above lemmas. For example, if $K/H \cong {}^2D_{p'}(3)$ where $p' = 2^m + 1 \geq 5$ is a prime number, then we consider two cases:

If $m_2 = (3^{p'} + 1)/4$, then $3^{p'} = q^{(2k+1)p}$, which implies that $q = 3$, $p = p'$, by Lemma 5.1. Now $3^{p(p-1)}$ divides $|K/H|$, which implies that $p = 3$, since $3^{p(p+1)/2} \| \, |G|$. But $p = p' \geq 5$, which is a contradiction.

If $m_2 = (3^{p'-1} + 1)/2$, then $3^{p'-1} = q^{(2k+1)p}$, where $1 \leq 2k + 1 \leq (p+1)/2$. Hence $2 = (q+1)(3^{p'-1}+1)/(q^p+1)$, which is a contradiction.

Therefore $K/H \ncong {}^2D_{p'}(3)$.

• If $K/H \cong D_{p'}(q')$, where $p' \geq 5$ is a prime number and $q' = 2$, 3 or 5, then ${q'}^{p'} = q^{2kp}$, $1 \leq 2k \leq (p + 1)/2$, or ${q'}^{p'} = 2^\alpha$. By using Remark 5.1, ${q'}^{p'} \neq 2^\alpha$, since $p' - 1 \geq 2$. If ${q'}^{p'} = q^{2kp}$, then $q = q'$ and $p' = 2kp$, which is a contradiction.

Similarly it follows that K/H is not isomorphic to $B_{p'}(3)$; $C_{p'}(q')$, where $q' = 2$, 3; and $D_{p'+1}(q')$, where $q' = 2$, 3.

• If K/H is isomorphic to $E_6(q')$; $F_4(q')$, where q' is odd; ${}^3D_4(q')$; ${}^2E_6(q')$; or $G_2(q')$, then we get a contradiction, similarly. For example, if $K/H \cong E_6(q')$, then ${q'}^9 - 1 \equiv 0 \pmod{m_2}$, which implies that ${q'}^9 = q^{2kp}$, $1 \leq 2k \leq (p + 1)/2$, since ${q'}^{36} \mid |G|$. Also we have

$$(3) \qquad (q^p - 1)(q^{2p(k-1)} + \cdots + q^{2p} + 1)(q + 1) = ({q'}^3 - 1)(3, q' - 1).$$

If $k > 1$, then equality in (3) does not hold, since $3{q'}^3 \leq 3q^{2kp/3} < (q + 1)q^{2p(k-1)}$. If $k = 1$, then ${q'}^9 = q^{2p}$ and again the equality does not hold, since $3{q'}^3 < 3q^{2p/3} < (q + 1)(q^p - 1)$.

• If K/H is isomorphic to ${}^2F_4(2)'$, ${}^2A_5(2)$, ${}^2A_3(2)$, ${}^2A_3(3)$, $A_2(2)$, $A_2(4)$, ${}^2E_6(2)$, $E_7(2)$ or $E_7(3)$ then $m_2 = 3, 5, 7, 9, 11, 13, 17, 19, 757, 1093$ which is a contradiction, since $m_2 > 1093$.

• If K/H is isomorphic to ${}^2G_2(q')$, where $q'=3^{2n+1}$; ${}^2F_4(q')$, where $q'=2^{2n+1}>2$; $E_8(q')$; ${}^2A_{p'-1}(q')$; or ${}^2B_2(q')$, where $q'=2^{2n+1}>2$, then we get a contradiction. Again the proof is similar for each type. Thus we choose one type. For example, let $K/H\cong {}^2G_2(q')$, where $q'=3^{2n+1}$. Then $m_2=q'+\varepsilon\sqrt{3q'}+1$, where $\varepsilon=1$ or $\varepsilon=-1$. Therefore $q'^3+1\equiv 0 \pmod{m_2}$, which implies that $q'=q^{(2k+1)p/3}$. Then $3^{n+1}(3^n\pm 1)=q(q^{p-2}-q^{p-3}+\cdots-1)$. Hence $q=3^{n+1}$, but $q^{p(2k+1)}=3^{(n+1)(2k+1)p}>3^{3(2n+1)}=q'^3$, which is a contradiction.

• If K/H is isomorphic to ${}^2A_{p'}(q')$ where $(q'+1)\mid(p'+1)$ and $(p',q')\neq(3,3),(5,2)$, then $q'^{p'}=q^{p(2k+1)}$, $1\leq 2k+1\leq(p+1)/2$. By using Lemma 5.1 and Remark 5.1, it follows that $p'\leq p$. Also we have

$$q'+1=(q+1)(q^{2kp}-q^{p(2k-1)}+\cdots-q^p+1). \tag{4}$$

Then $k=0$ and $q=q'$, otherwise $q\mid q'$ and $q'/q=qA+1$, for some $A>0$, which is a contradiction. Since $k=0$ and $q=q'$ it follows that $p=p'$ and hence $K/H\cong PSU(p+1,q)$.

Therefore $K/H\cong PSU(p+1,q)$. Now since $|G|=|PSU(p+1,q)|$, it follows that $|H|=1$, $G=K$ and hence $G\cong PSU(p+1,q)$, as required.

Now we discuss part (b) of the main theorem. In fact it is obvious, since $OC(\mathbf{Z}_{|PSU(p+1,q)|})=OC(PSU(p+1,q))$, where $(q+1)\not|\,(p+1)$, but $\mathbf{Z}_{|PSU(p+1,q)|}\ncong PSU(p+1,q)$.

Therefore $PSU(p+1,q)$, where $(q+1)\nmid(p+1)$, is not characterizable with this method.

The proof of this theorem is now completed. □

Remark 5.2. If $q=2^n$ then the proof is exactly similar to the case $q=x^n$ where x is an odd prime number. Therefore, by a small modification of the above lemmas and the above proof, we can get the result.

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REFERENCES

1. G.Y. Chen, *A new characterization of simple group of Lie type* $^2G_2(q)$, Proc. 2nd Ann. Meeting of Youth of Chin. Sci. and Tech. Assoc., Press of Xinan Jiaotong Univ., 1995 (in Chinese), pp. 221–224.

2. G.Y. Chen, *On Frobenius and* 2*-Frobenius group*, J. Southwest China Normal Univ. **20** (1995), 485–487.

3. ———, *A new characterization of* $G_2(q)$, $[q \equiv 0 \pmod 3]$, J. Southwest China Normal Univ. (special issue dedicated to Prof. Chen Zhongmu's 70th birthday) (1996), 47–51.

4. ———, *A new characterization of sporadic simple groups*, Algebra Colloq. **3** (1996), 49–58.

5. ———, *On Thompson's conjecture*, J. Algebra **185** (1996), 184–193.

6. ———, *Further reflections on Thompson's conjecture*, J. Algebra **218** (1999), 276–285.

7. ———, *A new characterization of Suzuki-Ree groups*, Sci. China Ser. A **27** (1997), 430–433.

8. ———, *A new characterization of* $E_8(q)$, J. Southwest China Normal Univ. **21** (1996), 215–217.

9. ———, *A new characterization of* $PSL(2, q)$, Southeast Asian Bull. Math. **22** (1998), 257–263.

10. J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

11. P. Crescenzo, *A Diophantine equation which arises in the theory of finite groups*, Adv. Math. **17** (1975), 25–29.

12. K.W. Gruenberg and O.H. Kegel, unpublished material, 1975.

13. K.W. Gruenberg and K.W. Roggenkamp, *Decomposition of the augmentation ideal and of the relation modules of a finite group*, Proc. London Math. Soc. **31** (1975), 146–166.

14. A. Iranmanesh, S.H. Alavi and B. Khosravi, *A Characterization of* $PSL(3, q)$ *where* q *is an odd prime power*, J. Pure Appl. Algebra **170** (2002), 243–254.

15. ———, *A characterization of* $PSL(3, q)$ *where* $q = 2^n$, Acta Math. Appl. Sinica (English Ser.) **18** (2002), 463–472.

16. A. Iranmanesh and B. Khosravi, *A characterization of* $PSU(5, q)$, Int. Math. J. **3** (2003), 129–141.

17. ———, *A characterization of* $PSU(7, q)$, Int. J. Appl. Math. **15** (2004), 329–340.

18. ———, *A characterization of* $PSU(11, q)$, Canad. Math. Bull. **47** (2004), 530–539.

19. ———, *A characterization of* $F_4(q)$ *where* $q = 2^n$, $(n > 1)$, Far East J. Math. Sci. **2** (2000), 853–859.

20. ———, *A characterization of* $C_2(q)$ *where* $q > 5$, Comment. Math. Univ. Carolin. **43** (2002), 9–21.

21. A. Iranmanesh, B. Khosravi and S.H. Alavi, *A characterization of* $PSU(3,q)$ *where* $q > 5$, Southeast Asian Bull. Math. **26** (2002), 33–44.

22. A. Khosravi and B. Khosravi, *A new characterization of almost sporadic groups*, J. Algebra Appl. **1** (2002), 267–279.

23. ———, *A new characterization of* $PSL(p,q)$, Comm. Algebra **32** (2004), 2325–2339.

24. A. Khosravi and B. Khosravi, *A characterization of* $^2D_n(q)$*, where* $n = 2^m$, Int. J. Math. Game Theory Algebra **13** (2003), 253–266.

25. Behrooz Khosravi, *A characterization of* $^2D_4(q)$, Pure Math. Appl. **12** (2001), 415–424.

26. Behrooz Khosravi and Bahman Khosravi, *A characterization of* $^2E_6(q)$, Kumamoto J. Math. **16** (2003), 1–11.

27. Behrooz Khosravi and Behnam Khosravi, *A characterization of* $E_6(q)$, Algebras Groups Geom. **19** (2002), 225–243.

28. A.S. Kondrat'ev, *Prime graph components of finite groups*, Math. USSR-sb. **67** (1990), 235–247.

29. W. Shi, *A new characterization of the sporadic simple groups*, in *Group theory*, Proc. of 1987 Singapore Group Theory Conf., Walter de Gruyter, Berlin, 1989, pp. 531–540.

30. ———, *Pure quantitative characterization of finite simple groups* (I), Progr. Natur. Sci. **4** (1994), 316–326.

31. ———, *A new characterization of some simple groups of Lie type*, Contemp. Math., vol. 82, 1989, pp. 171–180.

32. W. Shi and J. Bi, *A characteristic property for each finite projective special linear group*, Lecture Notes in Math., vol. 1456, 1990, pp. 171–180.

33. ———, *A new characterization of the alternating groups*, Southeast Asian Bull. Math. **17** (1992), 81–90.

34. ———, *A characterization of Suzuki-Ree groups*, Sci. China Ser. A **34** (1991), 14–19.

35. J.S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981), 487–513.

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