ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

SHARP ESTIMATES FOR SOME ITERATED OPERATORS IN ORLICZ SPACES

ELEONOR HARBOURE, OSCAR SALINAS AND BEATRIZ VIVIANI

ABSTRACT. In [**7**] and [**6**] sharp Orlicz estimates for the maximal and conjugate functions on the one-dimensional torus were given. Starting from their results we describe the pairs of growth functions (ψ, φ) such that modular $L^\psi \to L^\phi$ estimates hold for both, the strong maximal function and the nth-iteration of the Hilbert transform. We also show that our conditions are sharp. These results are achieved in a setting general enough to include both operators.

1. Introduction. The behavior of classical operators in Orlicz spaces has been extensively studied by many authors, see, for instance, [**1**–**4, 8**]. Whenever we have an operator bounded on L^p for p ranging on some interval, usually it is not bounded on some of the end points in the sense of the p-norm, although it satisfies some weaker estimates. In such situation the behavior of the operator near those extreme values is better understood when we refine the L^p-family of spaces by introducing the richer class of Orlicz-spaces.

More precisely, the kind of questions to answer here are the following: given an operator T and some Orlicz space, say L^ψ, which is the optimal local integrability for all the functions in $T(L^\psi)$? Or further, when the underlying measure space has finite measure, which is the optimal space L^ϕ such that T is bounded from L^ψ into L^ϕ?

Results in this direction may be found in [**7**] for the Hardy-Littlewood maximal function on the torus, in [**6**] for fractional maximal and integral operators in any dimension and the conjugate function, and in [**4**] for generalized Hardy operators.

In this paper we shall be concerned with the "iterated" Hilbert transform and the strong maximal function on the n-dimensional torus

2000 AMS *Mathematics Subject Classification.* Primary 42B25.
Key words and phrases. Iterated operators, Orlicz spaces.
The authors were supported by the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina and by the Universidad Nacional del Litoral.
Received by the editors on Jan. 21, 2003, and in revised form on Nov. 4, 2003.

τ^n, defined for functions 2π-periodic in each variable, given respectively by

$$\tilde{f}^n(x) = \frac{1}{\pi^n} \lim_{\varepsilon\to 0} \int_{D(x,\varepsilon)} f(y) \prod_{i=1}^{n} \cot\left(\frac{x_i - y_i}{2}\right) dy, \tag{1.1}$$

where $x=(x_1, \dots , x_n)$, $\varepsilon=(\varepsilon_1, \dots , \varepsilon_n)$ and $D(x, \varepsilon)=\{y=(y_1, \dots , y_n) : \varepsilon_i < |y_i - x_i| < \pi\}$ and

$$M_S f(x) = \sup_{x\in R} \frac{1}{|R|} \int_R |f(y)|\, dy \tag{1.2}$$

where the sup is taken over all intervals $R \subset \tau^n$ containing x.

It is well known that the relation between $\tilde{f}^n$ and the convergence of multiple Fourier series, see, for instance [**10**], and that of $M_S f$ with the differentiation of the integral respect to the n-dimensional intervals. The classical approach to solve each of these problems relies upon boundedness properties of $\tilde{f}^n$ and $M_S f$ respectively, see [**5**].

Our purpose is to find necessary and sufficient conditions on pairs of growth functions, (ψ, ϕ), such that the above operators are bounded from the Orlicz space L^ψ into the Orlicz space L^ϕ.

As it is known, see, for example [**5**, p. 161], the strong maximal function may be equal to infinity almost everywhere for a function in $L^1(\tau^n)$. However, for the smaller space $L(\log^+ L)^{n-1}(\tau^n)$, there is an estimate for the distribution function of $M_S f$, as the Jensen-Marcinkiewicz-Zygmund theorem asserts; namely, there exists some constant C such that

$$|\{M_S f > \lambda\}| \le C \int_{\tau^n} \frac{|f|}{\lambda} \left(1 + \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1}\right), \quad \lambda > 0. \tag{1.3}$$

This kind of inequality is called "extra weak" with respect to the function $\varphi(t) = t(1 + \log^+ t)$ by some authors, see, for example, [**8**]. We have not found in the literature a reference on whether or not an inequality as above holds true for the "iterated" Hilbert transform defined by (1.1) if $n > 1$. Nevertheless, in [**10**] it is shown that $\tilde{f}^n$ is finite almost everywhere for $f \in L(\log^+ L)^{n-1}(\tau^n)$ by proving estimates like

$$\int_{\tau^n} |\tilde{f}^n(x)|^\delta \le C_\delta + C_\delta \int_{\tau^n} |f(x)|(\log^+ |f(x)|)^{n-1}\, dx \tag{1.4}$$

for each $\delta \in (0, 1)$ which, by the way, are weaker than (1.3).

Let us notice that in all the above-mentioned papers, the standard technique to obtain boundedness on the Orlicz spaces is built upon estimates on the "end points," as in the Marcinkiewicz interpolation theorem. However we will not pursue this approach here. Fortunately, the "iterated" nature of our operators, allows us to skip the unfriendly extra-weak inequality, (1.3), using instead the already known Orlicz boundedness results for $n = 1$ given in [**7**] and [**6**]. Nevertheless, as we will show, our technique will lead to sharp Orlicz modular estimates.

We shall use this approach in the general setting of what we will call "iterated operators" whose precise definition is given in the next section.

Now, we turn our attention to the function spaces under consideration, fixing some notation. We recall that a growth function, that is, a nonnegative increasing function ϕ defined on $[0, \infty)$ with $\lim_{t\to 0^+} \phi(t) = 0$, is said to be of lower type p if there exists a constant C such that

$$\phi(st) \leq C s^p \phi(t) \tag{1.5}$$

holds for every $s \in [0, 1]$ and $t \geq 0$. Whenever there is a $p > 0$ satisfying (1.5) we shall say that ϕ is of positive lower type. We will also say that a nonnegative function η is Δ_2^∞ if there exists a constant C and a positive number t_0 such that

$$\eta(2t) \leq C\eta(t), \tag{1.6}$$

for any $t \geq t_0$.

Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$. For a nonnegative and nondecreasing function ϕ defined on $[0, \infty)$ with $\lim_{t\to 0^+} \phi(t) = 0$, we denote by $L^\phi(\Omega)$ the class of all measurable functions on Ω for which $\int_\Omega \phi(C\,|f|) < \infty$ for some positive constant C. It is clear that for Ω of finite measure, the space $L^\phi(\Omega)$ will remain the same if we change the values of ϕ in a neighborhood of the origin since for any $\lambda > 0$

$$\int_{\{x\in\Omega:|f(x)|<\lambda\}} \phi(C\,|f|) \leq \phi(C\lambda)\,|\Omega| < \infty.$$

The "Luxemburg norm" is introduced as the quantity

$$\|f\|_{L^\phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \phi(|f|/\lambda) \leq 1\right\}$$

That this quantity is finite for $f \in L^\phi(\Omega)$ is a consequence of the Lebesgue dominated convergence theorem. Moreover, when ϕ is a convex growth function, it gives a norm on $L^\phi(\Omega)$ which makes $L^\phi(\Omega)$ a Banach space. If we just know that ϕ is of positive lower type, the quantity $\| \quad \|_{L^\phi}$ defines a translation invariant quasi-metric, turning $L^\phi(\Omega)$ into a metrizable topological vector space. Moreover the metric can be chosen to be invariant under translations. We notice that when $\phi(t) = t^p,\ p \geq 1$, we obtain the Lebesgue spaces. We will keep the classical notation $L^p(\Omega)$ when we need to refer to these specific cases.

The paper is organized as follows. The next section contains the statements of our main results. In Section 3, proofs of the general theorems are given, whereas in the fourth section we derive results relative to $M_S f$ and $\tilde{f}^n$.

2. Main theorems. In the sequel we will consider the following type of growth functions:

$$\phi(t) = \int_0^t a(s)\,ds \quad \text{and} \quad \psi(t) = \int_0^t b(s)\,ds \tag{2.1}$$

for $t \geq 0$, where a and b are positive continuous functions defined on $[0, \infty)$. In addition we suppose that $b(s)$ is nondecreasing.

For $n \geq 1$ let τ^n be the n-dimensional torus. We denote by $\mathcal{M}^n$ the class of measurable real functions defined on τ^n.

Let T be an operator from a subspace D of $\mathcal{M} = \mathcal{M}^1$ into $\mathcal{M}$. Related to T, we consider the iterated operator T^n acting on functions f in a subset of $\mathcal{M}^n$ and given by

$$T^n f(x) = T_1 \circ T_2 \circ \cdots \circ T_n f(x),$$

where by T_j we mean the operator T acting on the x_j-variable, that is,

$$T_j f(x) = T(f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n))(x_j) \quad \text{for} \quad j = 1, \ldots, n.$$

It is clear that, if T is linear and well defined on D, then T^n is linear and well defined on linear combinations of products of n functions in D, each acting on a different variable, that is functions that can be written as $\prod_{i=1}^n f_i(x_i)$ with $f_i \in D$. Also, T^n is well defined over such class of functions if we start with any sublinear operator that "depends on $|f|$," in the sense $|T(f)| \leq T(|f|)$, and its domain D satisfies the property: $|f| \leq g$, $g \in D$ implies $f \in D$. Certainly the operators we are interested in, that is those defined by (1.1) and (1.2), fall in one of the above categories.

In order to state our first result we introduce further classes of growth functions. By $\mathcal{C}_n$ we shall mean the class of pairs (ψ, ϕ) with ψ and ϕ as in (2.1) and for which there exists a constant C such that the inequality

$$\int_1^t \frac{a(\lambda)}{\lambda} \log^{n-1}\left(\frac{t}{\lambda}\right) d\lambda \leq Cb(Ct) \tag{2.2}$$

holds for every $t \geq 1$. On the other hand, we set

$$\mathcal{C}'_n = \{(\psi, \varphi) \in \mathcal{C}_n / a \text{ is } \Delta_2^\infty\}.$$

Theorem 2.3. *If for each pair (η, ρ) in $\mathcal{C}_1$, respectively $\mathcal{C}'_1$, there exists a constant C such that*

$$\int_\tau \rho(|Tf(t)|)\, dt \leq C + C \int_\tau \eta(C|f(t)|)\, dt \tag{2.4}$$

for every f in L^η, then, for any given (ψ, ϕ) in $\mathcal{C}_n$, respectively $\mathcal{C}'_n$, there exist a constant C such that

$$\int_{\tau^n} \phi(|T^n f(x)|)\, dx \leq C + C \int_{\tau^n} \psi(C|f(x)|)\, dx \tag{2.5}$$

for every f in L^ψ such that $T^n f$ is well defined.

Remark 2.6. We would like to remark that if (ρ, η) is a pair in $\mathcal{C}^1$, respectively $\mathcal{C}'_1$, then the one-dimensional Hardy-Littlewood maximal function, respectively the one-dimensional Hilbert transform, satisfies an inequality of the type (1.4). This follows by Theorem (2.1) in [**7**], respectively Theorem (2.18) in [**6**]. In fact, a careful look at the proofs there shows that the extra hypothesis $b(s) \to \infty$ is not needed for (i) $\Rightarrow$ (ii), respectively (2.19) $\Rightarrow$ (2.20).

The next theorems establish that, under some extra conditions, Theorem 2.3 is sharp in the sense that for some specific operators the requirements $(\psi, \varphi) \in \mathcal{C}_n$ and $(\psi, \varphi) \in \mathcal{C}'_n$ are needed.

Theorem 2.7. *Suppose that the function $ta(e^t)$ is of positive lower type and that there exist positive constants $\delta_0 < \pi$, ξ, c_1 and c_2, such that for $\delta \in (0, \delta_0)$ the estimate*

$$\frac{c_1}{\theta} \leq \left| T\left(\frac{1}{\delta}\,\mathcal{X}_{[-\delta,\delta]}\right)(\theta)\right| \leq \frac{c_2}{\theta} \tag{2.8}$$

holds for every θ in $[\xi\delta, (\pi/2)]$. Then, if the modular inequality

$$\int_{\tau^n} \phi(|T^n f(x)|)\, dx \leq C_0 + C_0 \int_{\tau^n} \psi(C_0|f(x)|)\, dx \tag{2.9}$$

holds for every f of separate variables in $L^\psi(\tau^n)$, there exists a constant C such that

$$\int_1^t \frac{a(\lambda)}{\lambda} \log^{n-1}\left(\frac{t}{\lambda}\right) d\lambda \leq Cb(Ct) \tag{2.10}$$

for every $t \geq 1$.

The above theorems can be applied to obtain sharp boundedness results for M_S and $\tilde{f}^n$, defined in (1.2) and (1.1) respectively.

Theorem 2.11. *Assume that the function $ta(e^t)$ is of positive lower type and that a is Δ_2^∞. Then the following statements are equivalent.*

(2.12) *The pair (ψ, ϕ) belongs to $\mathcal{C}_n$.*

(2.13) *There exists C such that*

$$\int_{\tau^n} \phi(M_S f(x))\, dx \leq C + C \int_{\tau^n} \psi(C|f(x)|)\, dx$$

for every f in $L^\psi(\tau^n)$.

Theorem 2.14. *Assume that the function $ta(e^t)$ is of positive lower type and that a is Δ_2^∞. Then the following statements are equivalent.*

(2.15) *The pair (ψ,ϕ) belongs to $\mathcal{C}'_n$.*

$$\int_{\tau^n} \phi(|\tilde{f}^n(x)|)\,dx \le C + C\int_{\tau^n} \psi(C|f(x)|)\,dx \tag{2.16}$$

holds for every f such that $\tilde{f}^n$ is well defined.

Corollary 2.17. *If (ψ,ϕ) is in $\mathcal{C}'_n$ and if in addition, ψ is Δ_2 and ϕ of positive lower type, then $\tilde{f}^n$ is well defined for every f in $L^\psi(\tau^n)$ and the inequality*

$$||\tilde{f}^n||_{L^\phi(\tau^n)} \le C||f||_{L^\psi(\tau^n)}$$

holds, with C independent on f.

Remark 2.18. In [**10**], Zygmund proved that $\tilde{f}^n$ is well defined for every f in $L\ \log^{n-1} L(\tau^n)$. Then, applying Theorem 2.14, we can get inequality (2.16) for every f in $L^\psi(\tau^n)$, whenever $L^\psi(\tau^n) \subset L\log^{n-1} L(\tau^n)$. This inequality improves estimate (1.4) appearing in that paper. In fact, it is easy to check that for any $\varepsilon > 1$ the pair (ψ,ϕ_ε) belongs to $\mathcal{C}'_n$ for $\psi(t) \simeq t(\log^+ t)^{n-1}$ and $\phi_\varepsilon(t) \simeq t(\log^+ t)^{-\varepsilon}$ and satisfies all the general requirements of that theorem.

3. Proof of Theorems 2.3 and 2.7.

Proof of Theorem 2.3. We proceed by induction on the dimension. The case $n = 1$ is assumed by hypothesis. Now, suppose that 2.5 holds for $n = m-1$ and let us check the case $n = m$. Let $(\psi,\phi) \in \mathcal{C}_m$. Since

$$\begin{aligned} Cb(Ct) &\ge \int_1^t \frac{a(s)}{s}\,\log^{m-1}\left(\frac{t}{s}\right)ds \\ &= C\int_1^t \frac{a(s)}{s}\left(\int_s^t \frac{\log^{m-2}(t/r)}{r}\,dr\right)ds \\ &= C\int_1^t \frac{\log^{m-2}(t/r)}{r}\left(\int_1^r \frac{a(s)}{s}\,ds\right)dr \\ &= C\int_1^t \frac{b_1(r)}{r}\log^{m-2}\left(\frac{t}{r}\right)dr \end{aligned}$$

where

$$(3.1) \qquad b_1(r) = \int_1^r \frac{a(s)}{s}\, ds, \quad r > 1 \quad \text{and} \quad b(r) = 0, \quad 0 < r \le 1,$$

it is immediate that the pair (ψ, ψ_1), with $\psi'[_1(t) = b_1(t)$, belongs to $\mathcal{C}_{m-1}$. In addition, from (3.1) , clearly we have that $(\psi_1, \phi) \in \mathcal{C}_1$. So, using the inductive hypothesis and denoting $(x_1, x_2, \dots, x_m) = (x_1, x')$, we get

$$\begin{aligned}
\int_{\tau^m} \phi\left(|T^m f(x_1, \dots, x_m)|\right)\, dx_1 \dots dx_m \\
&= \int_{\tau^{m-1}} \left(\int_\tau \phi(|T(T_2 \circ \cdots \circ T_m f)(\cdot, x')(x_1)|)\, dx_1 \right) dx' \\
&\le C + C \int_{\tau^{m-1}} \left(\int_\tau \psi_1(|(T_2 \circ \cdots \circ T_m f)(x_1, x')|)\, dx_1 \right) dx' \\
&= C + C \int_\tau \left(\int_{\tau^{m-1}} \psi_1(|(T^{m-1} f(x_1, \cdot))(x')|)\, dx' \right) dx_1 \\
&\le C + C \int_\tau \left(\int_{\tau^{m-1}} \psi(|f(x_1, x')|)\, dx' \right) dx_1 \\
&= C + C \int_{\tau^m} \psi(|f(x_1, \dots, x_m)|)\, dx_1, \dots, dx_m,
\end{aligned}$$

proving the theorem for the class $\mathcal{C}^n$. For the case $(\psi, \phi) \in \mathcal{C}'_n$, we only need to check that the function given by (3.1) is Δ_2^∞. We know that $a(t)$ satisfies (1.6) for $t \ge t_0$, which we may assume greater than one. Then, taking $t \ge 2t_0$ and using the continuity of a, it is straightforward to check that b_1 satisfies (1.6) for such values of t. $\square$

In order to prove Theorem 2.7 we shall need two technical lemmas.

Lemma 3.2. *Let $a(t)$ be a nonnegative continuous function defined for $t > 1$. Then the following statements are equivalent*

(i) *The function $ta(e^t)$ is of positive lower type*;

(ii) *There exists a constant C such that $\int_1^\delta (a(\lambda)/\lambda)\, d\lambda \le C a(\delta) \log\, \delta$*;

(iii) *There exists $\varepsilon > 0$ such that $(1/\log t)^\varepsilon \int_1^t (a(\lambda)/\lambda)\, d\lambda$ is nondecreasing.*

(iv) *There exists $\beta < 1$ such that $a(t) \log^\beta t$ is quasi-increasing.*

Proof. First, we prove that (i) is equivalent to (ii). By an obvious change of variable,

$$\int_1^\delta \frac{a(\lambda)}{\lambda}\, d\lambda = \int_0^{\log \delta} ta(e^t)\, \frac{dt}{t}$$

Then, denoting $f(t) = ta(e^t)$, we have to prove that the inequality

$$\int_0^s f(t)\, \frac{dt}{t} \le Cf(s)$$

is equivalent to the fact that f is of positive lower type. But this result follows easily from Lemma 2.3 in [**9**, p. 515].

Next, to see that (ii) is equivalent to (iii), we just need to note that the derivative of the function $g(t) = (1/\log^\varepsilon t) \int_1^t (a(\lambda)/\lambda)\, d\lambda$ is nonnegative for some $\varepsilon > 0$ if and only if (ii) holds. A straightforward calculation of the derivative gives the desired equivalence.

Now suppose that (i) is true. Using that a function $\eta(t)$ is of positive lower type if and only if $\eta(t)/t^\alpha$ is quasi increasing for some $\alpha > 0$, (iv) follows easily. Finally assuming (iv) we have

$$\begin{aligned}\int_1^\delta \frac{a(t)}{t}\, dt &= \int_1^\delta \frac{a(t)(\log t)^\beta}{t(\log t)^\beta}\, dt \\ &\le Ca(\delta)(\log \delta)^\beta \int_1^\delta \frac{dt}{t(\log t)^\beta} \\ &= ca(\delta)(\log\delta)^\beta (\log \delta)^{1-\beta} \\ &= ca(\delta)\log\delta,\end{aligned}$$

which proves (ii). $\square$

Lemma 3.3. *If the function $ta(e^t)$ is of positive lower type, then for any $n \in \mathbf{N}$, $n \ge 2$, there exists a constant C such that*

$$(\log\delta)^{n-1} \int_1^\delta \frac{a(\lambda)}{\lambda}\, d\lambda \le C \int_1^\delta \frac{a(\lambda)}{\lambda}\, (\log\lambda)^{n-1}\, d\lambda, \quad \delta > 1. \tag{3.4}$$

Proof. By integrating by parts, we have

$$\int_1^\delta \frac{a(\lambda)}{\lambda} (\log \lambda)^{n-1} d\lambda = (\log \delta)^{n-1} \int_1^\delta \frac{a(\lambda)}{\lambda}\, d\lambda \\ - (n-1) \int_1^\delta \left(\int_1^\lambda \frac{a(s)}{s}\, ds \right) \frac{(\log \lambda)^{n-2}}{\lambda}\, d\lambda.$$

Then, in order to obtain (3.4) we only need to show that there exists $\alpha \in (0,1)$ such that

$$\int_1^\delta \frac{(\log \lambda)^{n-2}}{\lambda} \left(\int_1^\lambda \frac{a(s)}{s}\, ds \right) d\lambda \leq \frac{\alpha}{n-1} (\log \delta)^{n-1} \int_1^\delta \frac{a(\lambda)}{\lambda}\, d\lambda.$$

But this inequality follows from the hypothesis and Lemma 3.2. In fact, by that lemma, there exists $\varepsilon > 0$ such that the function

$$h(\lambda) = \frac{1}{(\log \lambda)^\varepsilon} \int_1^\lambda \frac{a(s)}{s}\, ds$$

is nondecreasing. Therefore

$$\begin{aligned} \int_1^\delta \frac{(\log \lambda)^{n-2}}{\lambda} \left(\int_1^\lambda \frac{a(s)}{s}\, ds \right) d\lambda &= \int_1^\delta \frac{(\log \lambda)^{n-2+\varepsilon}}{\lambda}\, h(\lambda)\, d\lambda \\ &\leq h(\delta) \int_1^\delta \frac{(\log \lambda)^{n-2+\varepsilon}}{\lambda}\, d\lambda \\ &= \frac{(\log \delta)^{n-1}}{n-1+\varepsilon} \int_1^\delta \frac{a(s)}{s}\, ds \\ &= \frac{\alpha}{n-1} (\log \delta)^{n-1} \int_1^\delta \frac{a(s)}{s}\, ds, \end{aligned}$$

where $\alpha = (n-1)/(n-1+\varepsilon)$. This concludes the proof of the lemma. □

Now, we are in position to proceed with the proof of Theorem 2.7.

Proof of Theorem 2.7. From 2.8, we have

$$\left|\left\{\theta \in \tau : \left|T\Big(\frac{1}{\delta}\,\mathcal{X}_{[-\delta,\delta]}\Big)(\theta)\right| > \gamma\right\}\right| \geq \left|\left\{\theta \in \Big[\xi\delta, \frac{\pi}{2}\Big] \text{ and } \theta < \frac{c_1}{\gamma}\right\}\right| = \frac{c_1}{\gamma} - \xi\delta > \frac{c_1}{2\gamma} \tag{3.5}$$

whenever $\gamma \in (2c_1/\pi,\, c_1/(2\xi\delta))$.

Let $F_\delta(\theta_1, \dots, \theta_n) = \prod_{i=1}^n f_\delta(\theta_i)$ with $f_\delta(s) = (1/\delta)\mathcal{X}_{[-\delta,\delta]}(s)$. Notice that

$$T^n(F_\delta)(\bar\theta) = \prod_{i=1}^n T(f_\delta)(\theta_i) = T(f_\delta)(\theta_1) G_\delta(\theta'),$$

where $\bar\theta = (\theta_1, \dots, \theta_n)$ and $G_\delta(\theta') = \prod_{i=2}^n T(f_\delta)(\theta_i)$ with $\theta' = (\theta_2, \dots, \theta_n)$. Then, from (3.5) and for C_0 a positive constant to be determined later, we get

$$\begin{aligned}
&\int_{\tau^n} \phi\left(C_0\left|T^n(F_\delta)(\bar\theta)\right|\right) d\bar\theta \\
&= C_0 \int_0^\infty a(C_0\lambda) |\{\bar\theta \in \tau^n : |T^n(F_\delta)(\bar\theta)| > \lambda\}|\, d\lambda \\
&= C_0 \int_0^\infty a(C_0\lambda) \left|\left\{\bar\theta \in \tau^n : |Tf_\delta(\theta_1)| > \frac{\lambda}{G_j(\theta')}\right\}\right| d\lambda \\
&= C_0 \int_0^\infty a(C_0\lambda) \int_{\tau^{n-1}} \left|\left\{\theta \in \tau : |Tf_\delta(\theta)| > \frac{\lambda}{|G_\delta(\theta')|}\right\}\right| d\theta'\, d\lambda \\
&\geq C \int_0^\infty \frac{a(C_0\lambda)}{\lambda} \left(\int_{\tau^{n-1}} \mathcal{X}_{A_\delta}(\lambda, \theta') |G_\delta(\theta')|\, d\theta'\right) d\lambda,
\end{aligned} \tag{3.6}$$

where

$$A_\delta = \left\{(\lambda, \theta') \in (0, \infty) \times \tau^{n-1} : \frac{2c_1}{\pi} < \frac{\lambda}{|G_\delta(\theta')|} < \frac{c_1}{2\xi\delta}\right\}.$$

Setting

$$A_\delta^\lambda = \left\{\theta' \in \tau^n : \frac{2\xi\delta}{c_1}\,\lambda < |G_\delta(\theta')| < \frac{\pi}{2c_1}\,\lambda\right\}$$

we have $(I_\delta^\lambda)^{n-1} \subset A_\delta^\lambda$, where

$$I_\delta^\lambda = \left\{\theta \in \tau : \left(\frac{2\xi\delta\lambda}{c_1}\right)^{1/(n-1)} < |Tf_\delta(\theta)| < \left(\frac{\pi\lambda}{2c_1}\right)^{1/(n-1)}\right\}.$$

Therefore, from (3.6), we obtain

(3.7)

$$\int_{\tau^n} \phi\left(C_0 \left|T^n(F_\delta)(\bar\theta)\right|\right)\, d\bar\theta \geq C \int_0^\infty \frac{a(C_0\lambda)}{\lambda}\left(\int_{I_\delta^\lambda} |Tf_\delta(\theta)|\, d\theta\right)^{n-1} d\lambda.$$

Now, since (2.8) holds, we have

$$\begin{aligned} I_\delta^\lambda &\supset \left\{\theta \in \left[\xi\delta, \frac{\pi}{2}\right] : \left(\frac{2\xi\delta\lambda}{c_1}\right)^{1/n-1} \frac{1}{c_1} \leq \frac{1}{\theta} \leq \frac{1}{c_2}\left(\frac{\pi}{2c_1}\,\lambda\right)^{1/(n-1)}\right\} \\ &\supset \left[\xi\delta, \frac{\pi}{2}\right] \cap \left[\frac{a}{\lambda^{1/(n-1)}}, \frac{b}{(\lambda\delta)^{1/(n-1)}}\right] \end{aligned}$$

with $\delta < (b/a)^{n-1}$, where $a = c_2(2c_1/\pi)^{1/n-1}$ and $b = c_1(c_1/(2\xi))^{1/(n-1)}$.

It is not difficult to see that the above intersection will give only three nonempty results:

(i) $[a/(\lambda^{1/(n-1)}), (\pi/2)]$ when $\lambda \in [b_0, (b_1/\delta)]$

(ii) $[a/(\lambda^{1/(n-1)}), (b/(\delta^{1/(n-1)}\lambda^{1/(n-1)})]$ when $\lambda \in [(b_1/\delta), (b_2/\delta^{n-1})]$

(iii) $[\xi\delta, (b/\lambda^{1/(n-1)}\delta^{1/(n-1)})]$ when $\lambda \in [(b_2)/(\delta^{n-1}), (b_3)/(\delta^n)]$,

where b_0, b_1, b_2 and b_3 are constants depending on c_1, c_2, n and ξ. Therefore, splitting the integral over λ, we have

(3.8)

$$\begin{aligned} &\int_{\tau^n} \phi\left(C_0 \left|T^n(F_\delta)(\bar\theta)\right|\right)\, d\bar\theta \\ &\quad \geq C\Bigg(\int_{b_0}^{b_1/\delta} \frac{a(C_0\lambda)}{\lambda} \log^{n-1}(\alpha_1\lambda)\, d\lambda \\ &\qquad + \int_{b_1/\delta}^{b_2/\delta^{n-1}} \frac{a(C_0\lambda)}{\lambda} \log^{n-1}\left(\frac{\alpha_2}{\delta}\right) d\lambda \\ &\qquad + \int_{b_2/\delta^{n-1}}^{b_3/\delta^n} \frac{a(C_0\lambda)}{\lambda} \log^{n-1}\left(\frac{\alpha_3}{\lambda\delta^n}\right) d\lambda\Bigg). \end{aligned}$$

Now, taking $C_0 = \alpha_1$ and changing $\alpha_1\lambda = t$ in the first integral, the sum of the three integrals can be written as

$$\begin{aligned}(3.9)\qquad &\int_{\alpha_1 b_0}^{\alpha_1 b_1/\delta} \frac{a(t)}{t}\,\log^{n-1}(t)\,dt + \int_{\alpha_1 b_1/\delta}^{\alpha_1 b_2/\delta^{n-1}} \frac{a(t)}{t}\,\log^{n-1}\left(\frac{\alpha_2}{\delta}\right) dt \\ &\qquad + \int_{\alpha_1 b_2/\delta^{n-1}}^{\alpha_1 b_3/\delta^{n}} \frac{a(t)}{t}\,\log^{n-1}\left(\frac{\alpha_3\alpha_1}{t\delta^n}\right) dt.\end{aligned}$$

Without loss of generality, we may assume that $\alpha_1 b_0 = 1$. Then, applying Lemma 3.3 to the first integral, we get

$$\int_{1}^{\alpha_1 b_1/\delta} \frac{a(t)}{t}\,\log^{n-1}(t)\,dt \geq \log^{n-1}\left(\frac{\alpha_1 b_1}{\delta}\right)\int_{1}^{\alpha_1 b_1/\delta} \frac{a(t)}{t}\,dt.$$

Therefore (3.9) is bounded from below by

$$\begin{aligned}&\log^{n-1}\frac{(\alpha_1 b_1)}{\delta}\int_{1}^{\alpha_1 b_1/\delta} \frac{a(t)}{t}\,dt + \log^{n-1}\left(\frac{\alpha_2}{\delta}\right)\int_{\alpha_1 b_1/\delta}^{\alpha_1 b_2/\delta^{n-1}} \frac{a(t)}{t}\,dt \\ &\qquad + \int_{\alpha_1 b_2/\delta^{n-1}}^{\alpha_1 b_3/\delta^{n}} \frac{a(t)}{t}\,\log^{n-1}\left(\frac{\alpha_3\alpha_1}{t\delta^n}\right) dt.\end{aligned}$$

Since all the integrands are positive, from (3.8), taking $\alpha_0 = \min\{\alpha_1 b_1, \alpha_2, \alpha_1\alpha_3, \alpha_1 b_3\}$, we obtain

$$\begin{aligned}\int_{\tau^n} \phi(\alpha_1 |T^n(F_\delta(\bar\theta))|)\,d\bar\theta &\geq C\Bigg(\log^{+(n-1)}\left(\frac{\alpha_0}{\delta}\right)\int_{1}^{\alpha_1 b_2/\delta^{n-1}} \frac{a(t)}{t}\,dt \\ &\qquad + \int_{\alpha_1 b_2/\delta^{n-1}}^{\alpha_1 b_3/\delta^{n}} \frac{a(t)}{t}\,\log^{+(n-1)}\left(\frac{\alpha_0}{t\delta^n}\right) dt\Bigg) \\ &\geq C\int_{1}^{\alpha_1 b_3/\delta^{n}} \frac{a(t)}{t}\,\log^{+(n-1)}\left(\frac{\alpha_0}{t\delta^n}\right) dt \\ &= C\int_{1}^{\alpha_0/\delta^{n}} \frac{a(t)}{t}\,\log^{+^{n-1}}\left(\frac{\alpha_0}{t\delta^n}\right) dt.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\tau^n} \psi(\alpha_1 F_\delta(\bar\theta))\,d\bar\theta &= \psi\left(\frac{\alpha_1}{\delta^n}\right)\int_{[-\delta,\delta]^n} d\bar\theta \\ &= c\delta^n \psi\left(\frac{\alpha_1}{\delta^n}\right) \\ &\leq cb\left(\frac{\alpha_1}{\delta^n}\right)\end{aligned}$$

Finally, this inequality together with (2.9) and (3.10) give

$$\begin{aligned}\int_1^{\alpha_0/\delta^n} \frac{a(t)}{t} \log^{n-1}\left(\frac{\alpha_0}{t\delta^n}\right) dt &\le c + cb\left(\frac{\alpha_1}{\delta^n}\right)\\ &\le cb\left(\frac{c}{\delta^n}\right),\end{aligned}$$

for δ small enough, since b is nondecreasing. Clearly, the latter inequality implies (2.10) by taking $s = \alpha_0/\delta^n$.

4. Proofs of Theorems 2.11, 2.14 and Corollary 2.17.

Proof of Theorem 2.11. To see that (2.12) implies (2.13), we first note that hypothesis (2.4) of Theorem 2.3 is satisfied by the one-dimensional Hardy-Littlewood maximal function as was pointed out in Remark 2.6. So, by that theorem, we get (2.13) for every bounded f. Since $M_S f \le M^n f$, the result for every f in L^ψ follows by the monotone convergence theorem.

Now, assume that (2.13) holds. It is clear that $M_S f = M^n f$ for every f of separate variables. On the other hand, an easy calculation shows that the Hardy-Littlewood maximal function satisfies (2.8). Then from Theorem 2.7 we get (2.12). □

Proof of Theorem 2.14. Again by Remark 2.6, hypothesis (2.4) of Theorem 2.3 is satisfied by the one-dimensional Hilbert transform. So, from that theorem, (2.16) follows. The reciprocal is a consequence of Theorem 2.7, since $\tilde{f}$ satisfies (2.8) as is shown in [**6**, see Lemma (3.8)].

Proof of Corollary 2.17. As it was pointed out in Section 2 for the general operator T^n, being $\tilde{f}$ linear, $\tilde{f}^n$ is well defined for linear combinations of functions that can be written as $\prod_{i=1}^n f_i(x_i)$ with $f_i = \mathcal{X}_{I_i}$, where I_i are intervals in τ. Then, from Theorem 2.14 we get (2.16) for those functions. Since ϕ is of positive lower type, it is immediate that the inequality

$$\|\tilde{f}^n\|_{L^\phi(\tau^n)} \le c\|f\|_{L^\psi(\tau^n)}, \tag{4.1}$$

holds for such class of functions. But, being ψ in the Δ_2 class these functions are dense in $L^\psi(\tau^n)$ by just following the same argument used for Lebesgue spaces. So, the results stated in the corollary follow easily from this fact and (4.1). □

REFERENCES

1. S. Bloom and R. Kerman, *Extrapolation of L_p data from a modular inequality*, Canad. Math. Bull. **45** (2002), 25–35.

2. ———, *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110** (1994), 149–167.

3. A. Cianchi, *An optimal theorem of Marcinkiewicz type in Orlicz spaces*, J. Funct. Anal. **153** (1998), 357–381.

4. ———, *Hardy inequalities in Orlicz spaces*, Trans. Amer. Math. Soc. **351** (1959), 2459–2478.

5. M. Guzmán, *Real variable methods in Fourier analysis*, North-Holland Math. Stud., North-Holland, Amsterdam, 1981.

6. E. Harboure, O. Salinas and B. Viviani, *Orlicz boundedness for certain classical operators*, Colloq. Math. **91** (2002), 263–282.

7. H. Kita, *On maximal functions in Orlicz spaces*, Proc. Amer. Math. Soc. **124** (1996), 3019–3025.

8. V. Kokilashvili and M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, 1991.

9. J. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana University Math. J. **28** (1979), 511–544.

10. A. Zygmund, *On the boundary values of functions of several complex variables*, I, Fund. Math. **36** (1949), 207–235.

Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Güemes 3450, 3000 Santa Fe, República Argentina
E-mail address: harbour@ceride.gov.ar

Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Güemes 3450, 3000 Santa Fe, República Argentina
E-mail address: salinas@ceride.gov.ar

Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Güemes 3450, 3000 Santa Fe, República Argentina
E-mail address: viviani@ceride.gov.ar