ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

THE COMPLETE CONTINUITY PROPERTY IN BANACH SPACES

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ABSTRACT. Let X be a complex Banach space. We show that the following are equivalent: (i) X has the complete continuity property, (ii) for every, or equivalently for some, $1 < p < \infty$, for $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy, where f_{r_n} is defined by $f_{r_n}(t) = f(r_n e^{it})$ for $t \in [0, 2\pi]$, (iii) for every, or equivalently for some, $1 < p < \infty$, for every $\mu \in V^p(X)$, the bounded linear operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi \, d\mu$ is compact, where $1/q + 1/p = 1$, (iv) for every, or equivalently for some, $1 < p < \infty$, each $\mu \in V^p(X)$ has a relatively compact range.

Before stating our results we overview the involved concepts and notations of vector-valued harmonic analysis. Throughout this note $(X, \| \cdot \|)$ denotes a complex Banach space, $\mathbf{D}$ denotes the open unit disc in the complex plane, and λ is the normalized Lebesgue measure on $[0, 2\pi]$. For a Banach space Y, we denote by B_Y the closed unit ball of Y. Given $1 < p < \infty$ the space $h^p(\mathbf{D}, X)$ consists of all X-valued harmonic functions f on $\mathbf{D}$ such that

$$\|f\|_p = \sup_{0<r<1} \left(\int_0^{2\pi} \|f(re^{it})\|^p \, d\lambda(t) \right)^{1/p} < \infty.$$

Accordingly, $h^\infty(\mathbf{D}, X)$ is the space of all X-valued bounded harmonic functions on $\mathbf{D}$ equipped with the norm $\|f\|_\infty = \sup_{z \in \mathbf{D}} \|f(z)\|$. For $f \in h^p(\mathbf{D}, X)$ and $n \in \mathbf{Z}$, the Fourier coefficient $\hat{f}(n)$ is computed as

$$\hat{f}(n) = r^{-|n|} \int_0^{2\pi} f(re^{it}) e^{-int} \, d\lambda(t).$$

2000 AMS *Mathematics Subject Classification.* Primary 46B20, 46B22, 46E40 46G10.
Key words and phrases. The complete continuity property, vector-valued Hardy spaces, vector measures, Pettis integrability.
This research is supported by the NSF of China and by the Academy of Finland.
Received by the editors on October 8, 2003, and in revised form on April 7, 2004.

Above $r \in (0,1)$ is arbitrary, since it is clear that $\hat{f}(n)$ is independent from the choice of $0 < r < 1$. We define for $\Lambda \subset \mathbf{Z}$ and $1 < p \le \infty$

$$h^p_\Lambda(\mathbf{D}, X) = \{f \in h^p(\mathbf{D}, X) : \hat{f}(n) = 0 \text{ for } n \notin \Lambda\}.$$

Let $\mathcal{B}$ be the collection of all Borel subsets of $[0, 2\pi]$. If μ is a countably additive X-valued measure on $[0, 2\pi]$, $1 < p < \infty$, the p-variation of μ is defined as

$$\|\mu\|_p = \sup \left(\sum_{E \in \pi} \frac{\|\mu(E)\|^p}{\lambda(E)^{p-1}} \right)^{1/p},$$

where the supremum is taken over all finite partitions of $[0, 2\pi]$, and one applies the usual convention: $\lambda/0$ is 0 or ∞ provided $\lambda = 0$ or $\lambda > 0$, respectively. For $p = \infty$, we set

$$\|\mu\|_\infty = \inf\{C \ge 0 : \|\mu(E)\| \le C\lambda(E) \text{ for all } E \in \mathcal{B}\}.$$

We denote by $V^p(X)$ the space of all countably additive X-valued measures μ on $[0, 2\pi]$ such that $\|\mu\|_p < \infty$. For $\mu \in V^p(X)$, the range of μ is defined as the set $\{\mu(E) : E \in \mathcal{B}\}$. Given $\mu \in V^p(X)$ and let $n \in \mathbf{Z}$, its Fourier coefficients $\hat{\mu}(n)$ are defined through

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-int}\, d\mu(t).$$

We let

$$V^p_\Lambda(X) = \{\mu \in V^p(X) : \hat{\mu}(n) = 0 \text{ for } n \notin \Lambda\}.$$

If $\mu \in V^p(X)$, $1 < p < \infty$, one can give sense to $\int_0^{2\pi} \phi(t)\, d\mu(t)$ for any $\phi \in L^q(0, 2\pi)$ (first for simple functions, then by extending on $L^q(0, 2\pi)$ using density), where $1/p + 1/q = 1$. Furthermore we have $\|\int_0^{2\pi} \phi\, d\mu\| \le \|\mu\|_p \|\phi\|_q$, see [**1**]. In particular, for $z \in \mathbf{D}$, $z = re^{i\theta}$, it is possible to define

$$P(\mu)(z) = \int_0^{2\pi} P_r(t - \theta)\, d\mu(t),$$

where $P_r(t) = (1 - r^2)/(1 - 2r\cos(t) + r^2)$ is the Poisson kernel. It is known that $P(\mu) \in h^p(\mathbf{D}, X)$ and, moreover, the correspondence

$\mu \mapsto P(\mu)$ yields an isomorphism between $V^p_\Lambda(X)$ and $h^p_\Lambda(\mathbf{D}, X)$ for $\Lambda \subset \mathbf{Z}$, see [**1**, Theorem 1.1] and [**4**].

If $1 \le p < \infty$, $f \in L^p(0, 2\pi; X)$, the p-Pettis-norm of f is defined by

$$|||f|||_p = \sup_{\eta \in B_{X'}} \left(\int_0^{2\pi} |\langle \eta, f(t) \rangle|^p \, d\lambda(t) \right)^{1/p},$$

A sequence f_n in $L^p((0, 2\pi), X)$ is said to be p-Pettis-Cauchy, if f_n is a Cauchy sequence for the norm $||| \cdot |||_p$.

We arrive at a central notion of this work. If Y is another Banach space, a bounded linear operator $T : X \to Y$ is said to be completely continuous, or Dunford-Pettis, if it maps weakly convergent sequences in X into norm convergent sequences in Y. Recall that X is said to have the complete continuity property, CCP in short, if each bounded linear operator $T : L^1(0, 2\pi) \to X$ is completely continuous. The CCP was introduced in [**8**], we refer to [**6, 9, 10**] for more information about this property. It is, e.g., known [**8**] that every space with the weak Radon-Nikodym property, see [**7**] for this notion, has the CCP. The simplest examples of Banach spaces with the CCP are separable dual spaces, since it is well known that they have the RNP.

Let $\Lambda \subset \mathbf{Z}$, X is said to have the type I-Λ-complete continuity property, I-Λ-CCP in short, if every $\mu \in V^\infty_\Lambda(X)$ has a relatively compact range [**10**]. X is said to have the type II-Λ-complete continuity property, II-Λ-CCP in short, if every $\mu \in V^1_\Lambda(X)$ which is λ-continuous, has a relatively compact range [**10**]. It is clear from the definitions that type II-Λ-CCP implies the type I-Λ-CCP, the type I-Λ-RNP, respectively II-Λ-RNP, implies the type I-Λ-CCP, II-Λ-CCP, [**3, 5**]. For $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, we denote by f_{r_n} the function in $L^p(0, 2\pi)$ defined by $f_{r_n}(t) = f(r_n e^{it})$ for $t \in [0, 2\pi]$. The following characterization of the type I-Λ-CCP has been given by Robdera and Saab [**10**, Theorem 3.3].

Theorem 1. *Let $\Lambda \subset \mathbf{Z}$. Then X has the type I-Λ-CCP if and only if for every $f \in h^\infty_\Lambda(\mathbf{D}, X)$, $r_n \uparrow 1$, the sequence f_{r_n} in $L^\infty(0, 2\pi; X)$ is 1-Pettis-Cauchy.*

The following result is key to all other results of this note.

Theorem 2. *Let $\Lambda \subset \mathbf{Z}$ and assume that X has the type II-Λ-CCP. Then, for $1 < p < \infty$, $f \in h^p_\Lambda(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy.*

It is well known that when $\Lambda = \mathbf{Z}$, the type I-Λ-CCP and type II-Λ-CCP coincide with the CCP [**10**]. This fact, in combination with Theorems 1 and 2, gives the following characterization of the CCP which, together with Theorem 5 below, is our main result.

Theorem 3. *X has the CCP if and only if for every, or equivalently for some, $1 < p < \infty$, $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy.*

Proof. The condition is clearly necessary by Theorem 2 as the type II-$\mathbf{Z}$-CCP and the CCP are equivalent. Assume next that for some $1 < p < \infty$, for every $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy. Then in particular for every $f \in h^\infty(\mathbf{D}, X)$, $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy. Hence f_{r_n} is 1-Pettis-Cauchy. By Theorem 1 this implies that X has the type I-$\mathbf{Z}$-CCP, i.e., the CCP. This finishes the proof. $\square$

One should compare Theorem 3 with the following well-known characterization of the RNP: a complex Banach space X has the RNP if and only if for every, equivalently for some, $1 < p < \infty$, for every $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is convergent in $L^p(0, 2\pi; X)$.

In the proof of Theorem 2 we will use the following lemma, which is essentially known, but we include a proof for the sake of completeness.

Lemma 4. *Let $1 < p < \infty$ and $\mu \in V^p(X)$. Then the range of μ is relatively compact if and only if the operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi(t)\, d\mu(t)$ is compact, where $1/q + 1/p = 1$.*

Proof. Assume first that the range of $\mu \in V^p(X)$ is relatively compact. It is clear that the operator T is well defined and bounded on $L^q(0, 2\pi)$ [**1**, p. 349]. We claim that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $\phi \in L^q(0, 2\pi)$ satisfying $\lambda(\operatorname{supp}(\phi)) \le \delta$ with

$\|\phi\|_q \leq 1$ we have $\|T\phi\| \leq \varepsilon$. Here $\operatorname{supp}(\phi) = \{x : \phi(x) \neq 0\}$ is the support of ϕ.

By [**1**, Proposition 1.1], there exists a positive function $g \in L^p(0, 2\pi)$ such that, for all $\phi \in L^q(0, 2\pi)$, one has

$$\left\| \int_0^{2\pi} \phi(t)\, d\mu(t) \right\| \leq \int_0^{2\pi} g(t)|\phi(t)|\, d\lambda(t).$$

Therefore

$$\|T\phi\| \leq \int_0^{2\pi} g(t)|\phi(t)|\, d\lambda(t) \leq \|\phi\|_q \bigg(\int_{\operatorname{supp}(\phi)} |g(t)|^p\, d\lambda(t) \bigg)^{1/p}.$$

Then the claim follows easily from the absolute continuity of the Lebesgue integrals.

Now since the range of μ is relatively compact, the set $T(B_{L^\infty(0,2\pi)})$ is also relatively compact as $B_{L^\infty(0,2\pi)}$ is the closed absolute convex hull of $\{\chi_A : A \in \mathcal{B}\}$, where we denote by χ_A the characteristic function of A.

Let $\varepsilon > 0$ be fixed, and let $0 < \delta < 1$ be the positive number according to the claim. Let $\phi \in L^q(0, 2\pi)$ be such that $\|\phi\|_q \leq 1$. We let $\phi = \phi_1 + \phi_2$, where $\phi_1(t) = \phi(t)$ if $|\phi(t)| \leq 1/\delta$ and $\phi_1(t) = 0$ otherwise. Then

$$\lambda(\operatorname{supp}(\phi_2))/\delta \leq \int_{\operatorname{supp}(\phi_2)} \frac{d\lambda(t)}{\delta^q} \leq \int_{\operatorname{supp}(\phi_2)} |\phi_2(t)|^q\, d\lambda(t) \leq 1.$$

Therefore $\lambda(\operatorname{supp}(\phi_2)) \leq \delta$. One obtains that $\|T\phi_2\| \leq \varepsilon$ by the claim. Moreover $T\phi_1 \in M_\delta := T(\delta^{-1} B_{L^\infty(0,2\pi)})$. Hence $\operatorname{dist}(T\phi, M_\delta) \leq \varepsilon$ for all $\phi \in L^q(0, 2\pi)$ with $\|\phi\|_q \leq 1$. This implies that the set $\{T\phi : \phi \in L^q(0, 2\pi), \|\phi\|_q \leq 1\}$ is relatively compact as M_δ is relatively compact and $\varepsilon > 0$ is arbitrary.

Conversely, assume that the operator T is compact. Then $T(B_{L^\infty(0,2\pi)})$ is relatively compact as we have $B_{L^\infty(0,2\pi)} \subset B_{L^q(0,2\pi)}$. We deduce that the range of μ being a subset of $T(B_{L^\infty(0,2\pi)})$, is also relatively compact, which ends the proof. $\square$

Proof of Theorem 2. Assume that X has the type II-Λ-CCP, $1 < p < \infty$, $f \in h^p_\Lambda(\mathbf{D}, X)$ and $r_n \uparrow 1$. For any $\eta \in X'$, the function $\langle \eta, f \rangle$ belongs to $h^p_\Lambda(\mathbf{D}, \mathbf{C})$. Therefore, by the classical result, there exists $f_\eta \in L^p(0, 2\pi)$ with $\hat{f}_\eta(n) = 0$ for $n \notin \Lambda$ satisfying

$$\langle \eta, f(re^{i\theta}) \rangle = \int_0^{2\pi} P_r(\theta - t) f_\eta(t)\, d\lambda(t),$$

for $\theta \in [0, 2\pi]$ and $0 \leq r < 1$. Now, for $E \in \mathcal{B}$ we can define $\mu(E) \in X''$ by

$$\langle \mu(E), \eta \rangle = \int_E f_\eta(t)\, d\lambda(t).$$

Since $f_\eta(t) = \lim_{r\uparrow 1} \langle \eta, f(re^{it}) \rangle$ almost everywhere for $t \in [0, 2\pi]$, we get by Fatou's lemma

$$\begin{aligned} |\langle \mu(E), \eta \rangle| &\leq \int_E \lim_{r\uparrow 1} |\langle \eta, f(re^{it}) \rangle|\, d\lambda(t) \\ &\leq \|\eta\| \liminf_{r\uparrow 1} \int_E \|f(re^{it})\|\, d\lambda(t). \end{aligned}$$

It follows that

$$\|\mu(E)\| \leq \liminf_{r\uparrow 1} \int_E \|f(re^{it})\|\, d\lambda(t).$$

Let π be a finite partition of $[0, 2\pi]$. We may estimate

$$\begin{aligned} \sum_{E\in\pi} \frac{\|\mu(E)\|^p}{\lambda(E)^p}\, \lambda(E) &\leq \sum_{E\in\pi} \liminf_{r\uparrow 1} \left(\int_E \frac{\|f(re^{it})\| d\lambda(t)}{\lambda(E)} \right)^p \lambda(E) \\ &\leq \sum_{E\in\pi} \liminf_{r\uparrow 1} \int_E \|f(re^{it})\|^p\, d\lambda(t) \\ &\leq \liminf_{r\uparrow 1} \sum_{E\in\pi} \int_E \|f(re^{it})\|^p\, d\lambda(t) \\ &= \lim_{r\uparrow 1} \int_0^{2\pi} \|f(re^{it})\|^p\, d\lambda(t) = \|f\|_p^p < \infty \end{aligned}$$

by Jensen's inequality. Consequently, $\mu \in V^p(X'')$. The same proof as [**1**, Theorem 1.1] shows that the range of μ is actually contained in X.

It follows easily from the definition of μ that $\hat{\mu}(n) = 0$ whenever $n \notin \Lambda$, i.e., $\mu \in V^p_\Lambda(X)$.

The measure μ is λ-continuous as $\mu \in V^p(X)$. It follows from the definition of the type II-Λ-CCP that the range of μ is relatively compact. By Lemma 4 the operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi(t)\, d\mu(t)$ is compact. Hence the adjoint operator $T^* : X' \to L^p(0, 2\pi)$ is also compact.

For $\eta \in X'$ and $\phi \in L^q(0, 2\pi)$, one has

$$\langle T^*\eta, \phi\rangle = \langle \eta, T\phi\rangle = \left\langle \eta, \int_0^{2\pi} \phi(t)\, d\mu(t)\right\rangle = \int_0^{2\pi} \phi(t) f_\eta(t)\, d\lambda(t).$$

Therefore $T^*\eta = f_\eta$. For each $\eta \in B_{X'}$, the function f_η belongs to $L^p(0, 2\pi)$, so we can identify f_η with its harmonic extension via the Poisson kernel in $\mathbf{D}$. By the classical result

$$\lim_{n,m\uparrow\infty} \|f_\eta(r_m\cdot) - f_\eta(r_n\cdot)\|_p = 0.$$

We deduce that

$$\lim_{n,m\uparrow\infty} \sup_{\eta\in B_{X'}} \|f_\eta(r_m\cdot) - f_\eta(r_n\cdot)\|_p = 0$$

as the set $\{f_\eta : \eta \in B_{X'}\} = T^*(B_{X'})$ is relatively compact in $L^p(0, 2\pi)$. The proof is complete.

From the proof of Theorem 3 and the isomorphism between $h^p(\mathbf{D}, X)$ and $V^p(X)$, it is clear that we have the following characterizations of the CCP.

Theorem 5. *The following statements are equivalent*:

(i) *X has the CCP.*

(ii) *For every $1 < p < \infty$, every $\mu \in V^p(X)$ has a relatively compact range.*

(iii) *For some $1 < p < \infty$, every $\mu \in V^p(X)$ has a relatively compact range.*

(iv) *For every $1 < p < \infty$, for every $\mu \in V^p(X)$, the corresponding operator T on $L^q(0, 2\pi)$ is compact, where $1/p + 1/q = 1$.*

(v) *For some* $1 < p < \infty$, *for every* $\mu \in V^p(X)$, *the corresponding operator* T *on* $L^q(0, 2\pi)$ *is compact, where* $1/p + 1/q = 1$.

Remarks. (i) It was shown in [**2**] that a complex Banach space X has the type I-$\mathbf{N}$-CCP, or equivalently the type II-$\mathbf{N}$-CCP, called the analytic CCP, if and only if for each $1 \le p < \infty$, $f \in h^p_{\mathbf{N}}(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} in $L^p(0, 2\pi)$ is p-Pettis-Cauchy, see also [**11**]. One can easily use the argument used in the proof of Theorem 3 to give another proof of this result. We should also notice that the method used in [**2**] does not work in the CCP-case. The reason is that in [**2**] one uses the fact that, for every $f \in h^p_{\mathbf{N}}(\mathbf{D}, X)$, there exist $g \in h^\infty_{\mathbf{N}}(\mathbf{D}, X)$ and $h \in h^\infty_{\mathbf{N}}(\mathbf{D}, \mathbf{C})$ such that $f = g/h$. This is no longer true for functions in $h^p(\mathbf{D}, X)$. One should also compare our Theorem 2 with [**10**, Theorem 3.4], which deals only with the case $p = 1$ and assumes that Λ is a Riesz-set.

(ii) We can also formulate a similar result as Theorem 5 for the analytic CCP, but in this case we use $\mu \in V^p_{\mathbf{N}}(X)$ for $1 \le p < \infty$. $p = 1$ is allowed as for $f \in h^1_{\mathbf{N}}(\mathbf{D}, X)$, the corresponding measure μ in the proof of Theorem 2 is in $V^1_{\mathbf{N}}(X)$, hence μ is λ-continuous by the vector-valued Riesz theorem.

Acknowledgments. The authors are most grateful to O. Blasco and J. Diestel for many helpful discussions.

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