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OSCILLATION OF SECOND ORDER DAMPED ELLIPTIC EQUATIONS VIA WEIGHTED AVERAGES TECHNIQUE

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ABSTRACT. By using weighted averages technique, some oscillation criteria for second order damped elliptic differential equation

$$\text{(E)} \qquad \sum_{i,j=1}^{N} D_i\Big[a_{ij}(x)D_j y\Big] + \sum_{i=1}^{N} b_i(x)D_i y + p(x)f(y) = 0$$

are obtained. These criteria are extensions of the results due to Coles for second order linear ordinary differential equation to equation (E).

1. Introduction. Consider the second order damped elliptic differential equation

$$\text{(1.1)} \qquad \sum_{i,\,j=1}^{N} D_i\big[a_{ij}(x)D_j y\big] + \sum_{i=1}^{N} b_i(x)D_i y + p(x)f(y) = 0$$

in $\Omega(a) \subseteq \mathbf{R}^N$, where $x = (x_1, \dots, x_N) \in \mathbf{R}^N$, $N \geq 2$, $D_i y = \partial y/\partial x_i$ for all i, $|x| = [\sum_{i=1}^{N} x_i^2]^{1/2}$, $\Omega(a) = \{x \in \mathbf{R}^N : |x| \geq a\}$ for some $a > 0$.

Throughout this paper, we shall assume that the following conditions hold without further mention.

(A_1) $f \in C(\mathbf{R},\mathbf{R}) \cup C^1(\mathbf{R} - \{0\},\mathbf{R})$, $yf(y) > 0$ and $f'(y) \geq k > 0$ whenever $y \neq 0$;

(A_2) $p \in C^{\mu}_{loc}(\Omega(a),\mathbf{R})$, $b_i \in C^{1+\mu}_{\text{loc}}(\Omega(a),\mathbf{R})$ for all i, and $\mu \in (0,1)$;

(A_3) $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix function with $a_{ij} \in C^{1+\mu}_{\text{loc}}(\Omega(a),\mathbf{R})$ for all i, j, and $\mu \in (0,1)$.

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Denote by $\lambda_{\max}(x)$ the largest eigenvalue of the matrix A. There exists a function $\lambda \in C([a,\infty),\mathbf{R}^+)$ such that

$$\lambda(r) \geq \max_{|x|=r} \lambda_{\max}(x) \quad \text{for} \quad r > a.$$

In what follows, the solution (classical solution) of equation (1.1) is every function of the class $C^{2+\mu}_{loc}(\Omega(a),\mathbf{R})$, $\mu \in (0,1)$, which satisfies equation (1.1) almost everywhere on $\Omega(a)$. We consider only the nontrivial solution of equation (1.1) which is defined for all large $|x|$, cf. [**2**].

The oscillation is considered in the usual sense, i.e., a solution $y(x)$ of equation (1.1) is said to be oscillatory if it has zero on $\Omega(b)$ for every $b \geq a$. Equation (1.1) is said to be oscillatory if every solution (if any exists) is oscillatory. Conversely, equation (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

Here we are concerned with extending oscillation criteria for second order linear ordinary differential equation

$$y''(t) + p(t)y(t) = 0, \quad p \in C([t_0,\infty),\mathbf{R}), \tag{1.2}$$

to that of the second order damped elliptic differential equation of form (1.1). For equation (1.2), the first important simple oscillation criterion is the well-known Fite-Wintner theorem [**3, 8**] which states that if the function $p(t)$ satisfies

$$\lim_{t\to\infty} \int_{t_0}^{t} p(s)\,ds = \infty, \tag{1.3}$$

then equation (1.2) is oscillatory. In fact, Fite [**3**] assumed in addition that $p(t)$ is nonnegative, while Wintner [**8**] proved a stronger result which required a weaker condition involving the integral average, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{t} p(s)\,ds\,dt = \infty. \tag{1.4}$$

Clearly, (1.3) implies (1.4).

In a different direction, Coles [**1**] extended the Wintner theorem by considering weighted averages of the integral of the function $p(t)$ with the form

$$A_\phi(t,t_0) = \frac{\int_{t_0}^t \phi(s) \int_{t_0}^s p(u)\,du\,ds}{\int_{t_0}^t \phi(s)\,ds},$$

where $\phi(s)$ is positive and locally integrable but not an integrable function on $[t_0,\infty)$. He proved that the condition

$$\lim_{t\to\infty} A_\phi(t,t_0) = \infty \tag{1.5}$$

is sufficient for the oscillation of equation (1.2) and he also gave another result for when a similar condition to that of (1.5) fails.

In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether or not solutions of the equation under consideration are oscillatory. For the semi-linear elliptic differential equation

$$\sum_{i,j=1}^N D_i\big[a_{ij}(x)D_j y\big] + p(x)f(y) = 0, \tag{1.6}$$

the oscillation theory has been widely discussed in the literature, see, for example, [**5, 7, 9–11, 13**] and other references contained therein. In particular, Noussair and Swanson [**5**] first employed an N-dimensional vector Riccati transformation and established Fite-Wintner type oscillation criteria for equation (1.6), see [**5**, Theorem 4]. The survey paper by Swanson [**7**] contains a complete bibliography up to 1979. Very recently, a classical theorem due to Kamenev [**4**] (as extended and improved by Phiols [**6**] and Yan [**12**]) was extended to equation (1.6), cf. [**10, 11**]. Unfortunately, their results cannot be applied to the second order damped elliptic differential equation (1.1). Motivated by this fact, in this paper, we use the N-dimensional vector Riccati transformation which has been developed further here and weighted averages technique similar to that exploited by Coles [**1**] and establish oscillation criteria for equation (1.1). These criteria are extensions of the results due to Coles for second order linear ordinary differential equation (1.2) to equation (1.1), thereby improving the main results in [**5**]. To the best of our knowledge, very little is known about the oscillation of equation (1.1) in general form, especially, when the coefficient functions $b_i(x)$ for all i, and $p(x)$ are allowed to change sign on $\Omega(a)$.

2. Main results. First of all, we introduce the following principle notations without further mention. For arbitrary functions $\rho \in C^1([a,\infty),\mathbf{R}^+)$ and $\lambda\eta \in C^1([a,\infty),\mathbf{R})$, we define for all $r \geq a$

$$h(r) = \frac{k}{\omega_N}\,\frac{r^{1-N}}{\lambda(r)\rho(r)}, \qquad g(r) = \frac{\rho'(r)}{\rho(r)} + \frac{2k}{\omega_N}\,\eta(r)r^{1-N},$$
$$\theta(r) = \rho(r)\Bigg\{\int_{S_r}\Bigg[p(x) - \frac{1}{4k}\,B^T A^{-1}B - \frac{1}{2k}\sum_{i=1}^{N} D_i b_i\Bigg]\,d\sigma + \frac{k}{\omega_N}\,\lambda(r)\eta^2(r)r^{1-N} - [\lambda(r)\eta(r)]'\Bigg\},$$

and

$$\theta_1(r) = \theta(r) - \frac{g^2(r)}{4h(r)}, \qquad \theta_2(r) = \theta_1(r) + \frac{1}{2}\left[\frac{g(r)}{h(r)}\right]',$$

where $S_r = \{x \in \mathbf{R}^N : |x| = r\}$ for all $r > 0$, $B^T = (b_1(x),\dots,b_N(x))$, σ denotes the measure on S_r and ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$, i.e., $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$.

Let $\Phi(r,a)$ denote the class of all nonnegative and local integrable functions $\phi(s)$ on $[a,\infty)$ with $\int_a^\infty \phi(s)\,ds \not\equiv 0$. For arbitrary functions $\phi \in \Phi(r,a)$ and $\psi \in C([a,\infty),\mathbf{R})$, we define for all $r \geq a$

$$\alpha(r,a) = \int_a^r \phi(s)\,ds, \quad \beta(r,a) = \int_a^r \frac{\phi^2(s)}{h(s)}\,ds,$$

and

$$X(\phi,\psi;r,a) = \frac{1}{\alpha(r,a)}\int_a^r \phi(s)\int_a^r \psi(u)\,du\,ds.$$

Members of the function class Φ will be called weight functions.

Theorem 2.1. *Suppose that there exist functions $\phi \in \Phi(r,a)$, $\rho \in C^1([a,\infty),\mathbf{R}^+)$ and $\lambda\eta \in C^1([a,\infty),\mathbf{R})$ such that*

$$g(r) \geq 0 \quad \textit{for} \quad r \geq a \tag{2.1}$$

and

$$\lim_{r\to\infty}\int_a^r \frac{\phi(s)[\alpha(s,a)]^\delta}{\beta(s,a)}\,ds=\infty \quad \textit{for some} \quad \delta, \quad 0\le\delta<1. \tag{2.2}$$

If

$$\lim_{r\to\infty} X(\phi,\theta_1;r,a)=\infty, \tag{2.3}$$

then equation (1.1) *is oscillatory.*

Proof. Let $y=y(x)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we assume that $y(x)\neq 0$ for $x\in\Omega(a)$. Furthermore, we suppose that $y(x)>0$ for all $x\in\Omega(a)$, since the substitution $u=-y$ transforms equation (1.1) into an equation of the same form subject to the assumption of theorem. Hence the N-dimensional vector Riccati operator

$$W(x)=\frac{1}{f(y)}\,A(x)Dy+\frac{1}{2k}\,B \tag{2.4}$$

exists on $\Omega(a)$, where $Dy=(D_1y,\dots,D_Ny)^T$. Differentiation of the ith component of (2.4) with respect to x_i gives

$$D_iW_i(x)=-\frac{f'(y)}{f^2(y)}\,D_iy\left[\sum_{i=1}^N a_{ij}D_jy\right]+\frac{1}{f(y)}\,D_i\left[\sum_{j=1}^N a_{ij}D_jy\right]+\frac{1}{2k}\,D_ib_i,$$

for all i. Summation over i and use of equations (1.1) and (2.4) lead to

$$\begin{aligned}\operatorname{div}W(x)&=-\frac{f'(y)}{f^2(y)}\,(Dy)^TADy-\frac{1}{f(y)}\big[p(x)f(y)+B^TDy\big]+\frac{1}{2k}\sum_{i=1}^N D_ib_i\\ &\le -k\left[W-\frac{1}{2k}\,B\right]^TA^{-1}\left[W-\frac{1}{2k}\,B\right]-p(x)\\ &\quad -B^TA^{-1}\left[W-\frac{1}{2k}\,B\right]+\frac{1}{2k}\sum_{i=1}^N D_ib_i\\ &=-kW^TA^{-1}W-p(x)+\frac{1}{4k}\,B^TA^{-1}B+\frac{1}{2k}\sum_{i=1}^N D_ib_i.\end{aligned} \tag{2.5}$$

Now, we introduce the generalized Riccati-type substitution and let

$$Z(r) = \rho(r)\left[\int_{S_r} W(x)\cdot\nu(x)\,d\sigma + \lambda(r)\eta(r)\right] \quad \text{for} \quad r \ge a, \tag{2.6}$$

where $\nu(x) = x/r$, $r = |x| \neq 0$, denotes the outward unit normal to S_r. By means of the Green formula in (2.6) and noting (2.5), we have

(2.7)

$$\begin{aligned} Z'(r) &= \frac{\rho'(r)}{\rho(r)}\,Z(r) + \rho(r)\left\{\int_{S_r} \operatorname{div} W(x)\,d\sigma + [\lambda(r)\eta(r)]'\right\} \\ &\le \frac{\rho'(r)}{\rho(r)}\,Z(r) - \rho(r)\left\{k\int_{S_r} (W^T A^{-1} W)(x)\,d\sigma \right. \\ &\quad \left. + \int_{S_r}\left[p(x) - \frac{1}{4k}\,B^T A^{-1} B - \frac{1}{2k}\sum_{i=1}^{N} D_i b_i\right] d\sigma - [\lambda(r)\eta(r)]'\right\}. \end{aligned}$$

In view of (A_3),

$$(W^T A^{-1} W)(x) \ge \lambda_{\max}^{-1}(x)\,|W(x)|^2.$$

By the Schwartz inequality,

$$\int_{S_r} |W(x)|^2\,d\sigma \ge \frac{r^{1-N}}{\omega_N}\left[\int_{S_r} W(r)\cdot\nu(x)\,d\sigma\right]^2.$$

Thus, by (2.7), we obtain

$$\begin{aligned} Z'(r) &\le \frac{\rho'(r)}{\rho(r)}\,Z(r) - \rho(r)\left\{\frac{kr^{1-N}}{\omega_N\lambda(r)}\left[\int_{S_r} W(x)\cdot\nu(x)\,d\sigma\right]^2 \right. \\ &\quad \left. + \int_{S_r}\left[p(x) - \frac{1}{4k}\,B^T A^{-1} B - \frac{1}{2k}\sum_{i=1}^{N} D_i b_i\right] d\sigma - [\lambda(r)\eta(r)]'\right\} \\ &= \frac{\rho'(r)}{\rho(r)}\,Z(r) - \rho(r)\left\{\frac{kr^{1-N}}{\omega_N\lambda(r)}\left[\frac{Z(r)}{\rho(r)} - \lambda(r)\eta(r)\right]^2 \right. \\ &\quad \left. + \int_{S_r}\left[p(x) - \frac{1}{4k}\,B^T A^{-1} B - \frac{1}{2k}\sum_{i=1}^{N} D_i b_i\right] d\sigma - [\lambda(r)\eta(r)]'\right\} \\ &= -\theta(r) + g(r)Z(r) - h(r)Z^2(r), \end{aligned}$$

that is, for $r \geq a$,

$$Z'(r) \leq -\theta(r) + g(r)Z(r) - h(r)Z^2(r). \tag{2.8}$$

Completing squares of $Z(r)$ in (2.8) yields

$$Z'(r) \leq -\theta_1(r) - h(r)\left[Z(r) - \frac{g(r)}{2h(r)}\right]^2. \tag{2.9}$$

Now integrating from a to r on both sides of (2.9), we have

$$Z(r) + \int_a^r h(s)\left[Z(s) - \frac{g(s)}{2h(s)}\right]^2 ds \leq Z(a) - \int_a^r \theta_1(s)\, ds. \tag{2.10}$$

Multiplying (2.10) by $\phi(r)$ and integrating it from a to r, we get

$$\begin{aligned}\int_a^r \phi(s)Z(s)\, ds + \int_a^r \phi(s) \int_a^r h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du\, ds \\ \leq \alpha(r,a)[Z(a) - X(\phi, \theta_1; r, a)].\end{aligned} \tag{2.11}$$

By (2.3), there exists a $b > a$ such that

$$Z(a) - X(\phi, \theta_1; r, a) < 0 \quad \text{for all} \quad r \geq b.$$

Then, by (2.11), for all $r \geq b$,

$$H(r) := \int_a^r \phi(s) \int_a^s h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du\, ds \leq -\int_a^r \phi(s)Z(s)\, ds.$$

By (2.1), we obtain

$$H(r) \leq H(r) + \frac{1}{2}\int_a^r \frac{\phi(s)g(s)}{h(s)}\, ds \leq -\int_a^r \phi(s)\left[Z(s) - \frac{g(s)}{2h(s)}\right] ds.$$

Noting that $H(r)$ is nonnegative, and using the Schwartz inequality, we obtain

$$\begin{aligned} H^2(r) &\leq \left(\int_a^r \phi(s)\left|Z(s) - \frac{g(s)}{2h(s)}\right| ds\right)^2 \\ &\leq \left[\int_a^r \frac{\phi^2(s)}{h(s)}\, ds\right]\left[\int_a^r h(s)\left(Z(s) - \frac{g(s)}{2h(s)}\right)^2 ds\right] \\ &= \frac{\beta(r,a)}{\phi(r)}\, H'(r). \end{aligned} \tag{2.12}$$

On the other hand,

(2.13)

$$H(r) \geq \int_b^r \phi(s)\left(\int_a^b h(u)\left(Z(u) - \frac{g(u)}{2h(u)}\right)^2 du\right) ds = M_1\alpha(r,b),$$

where

$$M_1 = \int_a^b h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du.$$

From (2.12) and (2.13), we get

$$\frac{M_1^\delta \phi(r)[\alpha(r,a)]^\delta}{\beta(r,a)} \leq H^{\delta-2}(r)H'(r) \quad \text{for all} \quad r \geq b. \tag{2.14}$$

This implies that

$$M_1^\delta \int_b^r \frac{\phi(s)[\alpha(s,a)]^\delta}{\beta(s,a)}\, ds \leq \frac{1}{1-\delta}\,\frac{1}{H^{1-\delta}(b)} < \infty,$$

which contradicts condition (2.2). □

Corollary 2.1. *Let Condition* (2.3) *in Theorem* 2.1 *be replaced by*

$$\lim_{r\to\infty} \int_a^r \frac{g^2(s)}{h(s)}\, ds < \infty$$

and

$$\lim_{r\to\infty} X(\phi,\theta;r,a) = \infty;$$

then the conclusion of Theorem 2.1 *holds.*

Theorem 2.2. *Let the functions ϕ, ρ, η be as in Theorem* 2.1 *such that* (2.2) *holds. If*

$$\lim_{r\to\infty} X(\phi,\theta_2;r,a) = \infty, \tag{2.15}$$

then equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we have that (2.8) holds for all $r \geq a$. Define

$$V(r) = Z(r) - \frac{g(r)}{2h(r)};$$

then (2.8) can be rewritten as

$$V'(r) \leq -\theta_2(r) - h(r)V^2(r). \tag{2.16}$$

Inequality (2.16) is of the same type as inequality (2.9). Hence we can use a similar procedure to complete the proof of Theorem 2.2. □

Remark 2.1. For equation (1.6), let $\delta = 0$ and $\phi(r) = h(r)$; then Theorem 2.2 improves Theorem 4 in [**5**].

The following two oscillation criteria (Theorem 2.3 and Theorem 2.4) treat the cases when it is not possible to verify easily conditions (2.3) or (2.15).

Lemma 2.1 (cf. [**1**]). *Suppose that* $\varrho(r) \in C([a,\infty),[0,\infty))$ *is nondecreasing with* $\int_a^\infty \phi(s)\,ds = \infty$. *Then*

(1) $\displaystyle\frac{1}{\alpha(r,a)}\int_a^r \phi(s)\varrho(s)\,ds$ *is nondecreasing in* r;

(2) *If* $\displaystyle\frac{1}{\alpha(r,a)}\int_a^r \phi(s)\varrho(s)\,ds$ *is bounded on* $[a,\infty)$, *so is* $\varrho(s)$.

Theorem 2.3. *Let the functions* ϕ, ρ, η *be as in Theorem* 2.1 *such that* (2.1) *and* (2.2) *hold. If*

$$\lim_{r\to\infty} X(\phi,\theta_1;r,a) > -\infty \tag{2.17}$$

and

$$\lim_{r\to\infty} \frac{1}{\int_a^r h(s)\,ds}\int_a^r h(s)\int_a^r \theta_1(u)\,du\,ds = \infty, \tag{2.18}$$

then equation (1.1) *is oscillatory.*

Proof. Proceeding as the proof of Theorem 2.1, we have that (2.10) and (2.11) hold for all $r \geq a$. Thus, by (2.17), there exist constants M_2 and $b_1 > a$ such that, for all $r > b_1$,

$$\frac{1}{\alpha(r,a)}\left\{\int_a^r \phi(s)Z(s)\,ds + \int_a^r \phi(s)\int_a^s h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]du\,ds\right\} \leq Z(a) - X(\phi,\theta_1;r,a) \leq M_2.$$

Now, we claim that

$$\frac{1}{\alpha(r,a)}\int_a^r \phi(s)\int_a^s h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]du\,ds \quad \text{is bounded on} \quad [b_1,\infty).$$

If not, by Lemma 2.1 (1), it tends to ∞ and so, for large r,

$$\int_a^r \phi(s)Z(s)\,ds + \frac{1}{2}\int_a^r \phi(s)\int_a^s h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du\,ds$$
$$\leq \alpha(r,a)\left[M_2 - \frac{1}{2\,\alpha(r,a)}\int_a^r \phi(s)\int_a^s h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du\,ds\right] < 0.$$

Next, one proceeds as in proof of Theorem 2.1 to contradict (2.2). So, by Lemma 2.1 (2), we get, for $r \geq a$

$$\int_a^r h(u)\left[Z(u) - \frac{g(u)}{2h(u)}\right]^2 du < \infty. \tag{2.19}$$

Thus, by (2.1), (2.10) and (2.19), there exist constants $M_3 > 0$ and $b_2 > a$ such that, for $r \geq b_2$,

$$\int_a^r \theta_1(s)\,ds \leq M_3 - Z(r) \leq M_3 - \left[Z(r) - \frac{g(r)}{2h(r)}\right] \leq M_3 + \left|Z(r) - \frac{g(r)}{2h(r)}\right|.$$

Hence

$$\int_a^r h(s)\int_a^s \theta_1(u)\,du\,ds \leq M_3\int_a^r h(s)\,ds + \int_a^r h(s)\left|Z(s) - \frac{g(s)}{2h(s)}\right|ds. \tag{2.20}$$

The Schwartz inequality yields that

$$\int_a^r h(s)\left|Z(s)-\frac{g(s)}{2h(s)}\right| ds \le \left[\int_a^r h(s)\,ds\right]^{1/2}\left[\int_a^r h(s)\left[Z(s)-\frac{g(s)}{2h(s)}\right]^2 ds\right]^{1/2}.$$

This and (2.20) imply that

$$\frac{1}{\int_a^r h(s)\,ds}\int_a^r h(s)\int_a^s \theta_1(u)\,du\,ds \le M_2 + \left[\frac{\int_a^r h(s)\big[Z(s)-(g(s))/(2h(s))\big]^2 ds}{\int_a^r h(s)\,ds}\right]^{1/2}. \tag{2.21}$$

Observing (2.19) and $h(r) > 0$ for $r > a$, we get that the right side of (2.21) is bounded; this contradicts (2.18). □

Procedure of the proof of Theorem 2.3, we can also prove the following theorem.

Theorem 2.4. *Let the functions* ϕ, ρ, η *be as in Theorem* 2.1 *such that* (2.2) *holds. If*

$$\lim_{r\to\infty} X(\phi,\theta_2;r,a) > -\infty \tag{2.22}$$

and

$$\lim_{r\to\infty}\frac{1}{\int_a^r h(s)\,ds}\int_a^r h(s)\int_a^r \theta_2(u)\,du\,ds = \infty, \tag{2.23}$$

then equation (1.1) *is oscillatory.*

Remark 2.2. In order that (2.2) can be satisfied by a nonnegative local integrable function ϕ, it is necessary that $\int_a^\infty \phi(s)\,ds = \infty$.

Remark 2.3. It should be pointed out here that the term $1/(2k)B$ appearing in (2.4) is very important. Without this term, our method does not apply to equation (1.1), cf. [**5, 7, 9–11, 13**].

Remark 2.4. The above results hold true if we replace condition (A_1) with the following one:

(A_1') $f \in C(\mathbf{R}, \mathbf{R})$, $yf(y) > 0$ and $f(y)/y \geq k > 0$ whenever $y \neq 0$.

But in this case, the function $p(x)$ should be nonnegative on $\Omega(a)$.

Finally, we present examples that illustrate the results of this paper. These examples are new and not covered by any of the known criteria in [**5, 9–11, 13**].

Example 2.1. Consider equation (1.1) with $N = 2$, where

$$\begin{aligned} A(x) &= \operatorname{diag}\left(\frac{1}{|x|}, \frac{1}{|x|}\right), \quad b_i(x) = \frac{1}{|x|}, \quad i = 1, 2, \\ p(x) &= e^{|x|}\left\{\frac{2 + \cos|x| - 2|x|\sin|x|}{4|x|^{3/2}} + \frac{1}{4|x|}\right\}, \\ f(y) &= y + y^3, \end{aligned} \tag{2.24}$$

for $x \in \Omega(\pi/2)$. Let

$$\eta(r) = \pi r \quad \text{and} \quad \rho(r) = e^{-r},$$

then

$$g(r) = 0 \quad \text{and} \quad h(r) = \frac{e^r}{2\pi}.$$

A direct computation implies that

$$\theta(r) = \pi\left[-\, r^{-1/2}\sin r + \frac{1}{2}\, r^{-1/2}(2 + \cos r)\right],$$

$$\begin{aligned} \int_{\pi/2}^{r} \theta(s)\, ds &= \pi\left[r^{1/2}(2 + \cos r) - 2\left(\frac{\pi}{2}\right)^{1/2}\right] \\ &\geq \pi\left[r^{1/2} - 2\left(\frac{\pi}{2}\right)^{1/2}\right]. \end{aligned}$$

Let $\phi(r) = r$, $\delta = 0$, then

$$\begin{aligned} \int_{\pi/2}^{r} \frac{\phi(s)}{\beta(s, (\pi/2))}\, ds &= \frac{1}{2\pi}\int_{\pi/2}^{r} s\left[\int_{\pi/2}^{s} \frac{u^2}{e^u}\, du\right]^{-1} ds \\ &\geq \frac{1}{4\pi}\left[\int_{\pi/2}^{r} \frac{u^2}{e^u}\, du\right]^{-1} \int_{\pi/2}^{r} s\, ds, \end{aligned}$$

$$\frac{1}{\int_{\pi/2}^{r}\phi(s)\,ds}\int_{\pi/2}^{r}\phi(s)\int_{\pi/2}^{s}\theta(u)\,du\,ds \geq \frac{\pi}{r^2-(\pi/2)^2}\int_{\pi/2}^{r}\left[s^{3/2}-2\left(\frac{\pi}{2}\right)^{1/2}s\right]ds.$$

So, all the hypotheses of Theorem 2.2 are satisfied, and hence equation (2.24) is oscillatory.

Example 2.2. Consider equation (1.1) with $N \geq 2$ where

$$\begin{aligned} &A(x) = \operatorname{diag}(|x|^{1-N}, \cdots, |x|^{1-N}),\\ &b_i(x) = 0 \quad \text{for all} \quad i,\\ &f(y) = y + y^{2N+1}, \end{aligned} \tag{2.25}$$

for $x \in \Omega(1)$. Let $\rho(r) = 1$ and $\eta(r) = 0$; then $h(r) = 1/\omega_N$, $g(r) = 0$. Choose $p(x)$ with $\theta(r) = \bar{\theta_n}(r)$ for $r \in [2n-1, 2n+1)$, $n \in \mathbf{N}_0 = \{1, 2, \dots\}$, and

$$\begin{aligned}\bar{\theta_n}(r) &= \int_{S_r} p(x)\,d\sigma\\ &= \begin{cases} 0 & \text{if} \quad 2n-1 \leq r \leq 2n,\\ 2r-4n+1 & \text{if} \quad 2n < r \leq 2n+(1/2),\\ -2r+4(n+1) & \text{if} \quad 2n+(1/2) < r < 2n+1,\end{cases}\end{aligned}$$

then $\int_{2n-1}^{2n+1}\theta_n(s)\,ds = 2$ for $n \in N_0$. We have, for $r \in (2n+1, 2n+3)$,

$$\begin{aligned}\int_1^r h(s)\int_1^s \theta(u)\,du\,ds &= \frac{1}{\omega_N}\left[\sum_{i=1}^{n}\int_{2i-1}^{2i+1}\int_1^s \bar{\theta_n}(u)\,du\,ds + \int_{2n+1}^{r}\int_1^s \bar{\theta_n}(u)\,du\,ds\right]\\ &\geq \frac{1}{\omega_N}\sum_{i=1}^{n}\int_{2i-1}^{2i+1}\left[\int_1^3 + \int_3^5 + \cdots + \int_{2i-3}^{2i-1}\right]ds\\ &= \frac{n(n+1)}{\omega_N}.\end{aligned}$$

Then

$$\lim_{r\to\infty}\frac{1}{\int_1^r h(s)\,ds}\int_1^r h(s)\int_1^s \theta(u)\,du\,ds=\infty.$$

Choose

$$\phi(s)=\begin{cases}1 & \text{if} \quad 2n-1\le s\le 2n,\\ 0 & \text{if} \quad 2n<s\le 2n+1,\end{cases}$$

then

$$\frac{\int_1^r \phi(s)\int_1^s\theta(u)\,du\,ds}{\int_1^r\phi(s)\,ds}=0.$$

Further, for $0<\delta<1$ and $2n+1\le r\le 2n+3$,

$$\begin{aligned}\int_1^r &\phi(s)[\beta(s,1)]^{-1}[\alpha(s,a)]^{\delta}\,ds\\ &=\frac{1}{\omega_N}\int_1^r\phi(s)\bigg[\int_1^s\phi^2(u)\,du\bigg]^{-1}\bigg(\int_1^s\phi(u)\,du\bigg)^{\delta}\,du\,ds\\ &=\frac{1}{\omega_N}\bigg[\int_1^2+\int_3^4+\cdots+\int_{2n-1}^{2n}+\int_{2n+1}^r\bigg]\\ &\ge\frac{1}{\delta\omega_N}\big[\big(1^\delta-0\big)+\big(2^\delta-1^\delta\big)+\cdots+\big(n^\delta-(n-1)^\delta\big)\big]\\ &=\frac{n^\delta}{\delta\omega_N}\to\infty\quad\text{as}\quad r\to\infty.\end{aligned}$$

Thus, all assumptions of Theorem 3.3 are satisfied and equation (2.25) is oscillatory.

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