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BRANCH CURVES FOR CAMPEDELLI DOUBLE PLANES

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ABSTRACT. Following an idea of Stagnaro, we find new examples of surfaces of general type with $p_g = q = 0$ and $K^2 = 1$ and 2, constructed as branched double covers of the plane, and we determine all possible configurations of branch loci that are invariant under an involution.

1. Introduction. A numerical Godeaux surface is a minimal surface of general type with $p_g = 0$, $K^2 = 1$. A numerical Campedelli surface is a minimal surface of general type with $p_g = 0$, $K^2 = 2$. One method for constructing each of these surfaces is as the minimal resolution of a double cover of the plane branched over a (possibly reducible) curve of degree 10. When the curve has one ordinary quadruple point and five infinitely near triple points, not all on a conic, the surface obtained has $K^2 = 1$; when the curve has six infinitely near triple points, not on a conic, the surface has $K^2 = 2$. This double plane construction was first suggested by Campedelli [**7**] 70 years ago; while isolated examples of these branch curves have been found, there has been little systematic work on their construction.

Recently Stagnaro [**13**] constructed several examples by considering curves invariant under an involution of the plane. In this note we extend this idea to find all possible configurations of branch curves invariant under plane involution; we then compute the torsion of the resulting surfaces.

The subgroup of torsion divisors in the Picard group gives a classification of numerical Godeaux surfaces. The torsion subgroup can be one of $\{1\}$, $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$, or $\mathbf{Z}/5\mathbf{Z}$. The surfaces with torsion $\mathbf{Z}/3\mathbf{Z}, \mathbf{Z}/4\mathbf{Z}$, and $\mathbf{Z}/5\mathbf{Z}$ have been completely classified, see [**12**], and have smooth moduli spaces of dimension 8. Examples of surfaces with trivial torsion, see [**1**], and order two torsion, see [**2, 14**], have been constructed, but little is known about their moduli.

Less is known of numerical Campedelli surfaces. There are several examples of these surfaces constructed as quotients of group actions on

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complete intersections, as well as constructions as double planes, see for example [**8, 11**].

Recently Calabri, Ciliberto and Mendes Lopes [**5**] have classified all numerical Godeaux surfaces with involution. A similar study of numerical Campedelli surfaces with involution has been done by Calabri, Mendes Lopes and Pardini in [**6**].

The paper is organized as follows. In Section 2, we review the double plane surfaces determined by these curves, and in Section 3, we analyze the construction of the singular branch curves that are invariant under an involution. In Section 4 we compute the torsion of the numerical Godeaux surfaces to be of order four. In the case of the numerical Campedelli surfaces, we show that the torsion group must contain $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

2. Double covers of the plane. Let C denote a possibly reducible degree ten curve in the projective plane with one ordinary quadruple point P_0 and five infinitely near triple points $P_1, \dots, P_5$. (An infinitely near triple point refers to a triple point where all three tangent directions coincide, so that after blowing up the plane at the point the proper transform of the curve has a triple point.) We require that the six points $P_0, P_1, \dots, P_5$ do not lie on a conic.

Write $\rho_1 \colon Y_1 \to \mathbf{P}^2$ for the blowup of the plane at $P_0, P_1, \dots, P_5$, with $E_i = \rho_1^{-1}(P_i)$ the exceptional curve above each point P_i. The strict transform of C has an ordinary triple point $P_i{}'$ on each curve E_i for $i = 1, \dots, 5$. Blowing up each of the five $P_i{}'$ resolves the singularities of C, and composing with ρ_1, we obtain a rational map, $\rho \colon Y \to \mathbf{P}^2$. Setting $\overline{C}$ to be the strict transform of C on Y, we have

$$\overline{C} \equiv 10H - 4E_0 - 3\sum_1^5 E_i - 6\sum_1^5 F_i$$

where H denotes the pullback to Y of the class of a line in the plane, F_i denotes the second set of exceptional curves obtained by blowing up Y_1 at $P_1{}', \dots, P_5{}'$ (we abuse notation by letting $E_0, \dots, E_5$ denote the proper transform of the exceptional curves on Y). Then $\overline{C} + \sum_1^5 E_i = 2L$ is an even divisor, and we can form the double cover $\pi \colon X \to Y$ of the surface Y branched along $2L$ (locally $z^2 = f(x,y)$, where $f(x,y)$ is a local equation for the branch curve $\overline{C} + \sum_1^5 E_i$).

If C has no other singularites than those prescribed, then X is a non-singular surface. The curves E_i, $i = 1, \dots, 5$ have self-intersection -2 on Y. Since they are components of the branch locus, $\pi^{-1}(E_i)$ are (-1)-rational curves on X. Contract these five curves to obtain the surface $\overline{X}$.

Proposition 2.1. *The surface $\overline{X}$ is a numerical Godeaux surface.*

Proof. The proof is standard, see, for example, [**3**], and we just sketch it here. We have branch curve $\overline{C}$ on the rational surface Y, where $\rho\colon Y \to \mathbf{P}^2$ is the eleven-fold blowup of the plane. The canonical divisor on the surface Y, K_Y, is given by

$$K_Y \equiv \rho^*\left(K_{\mathbf{P}^2}\right) + E_0 + \sum_1^5 E_i + 2\sum_1^5 F_i.$$

Since Y is obtained from the plane by 11 blow-ups, ${K_Y}^2 = {K_{\mathbf{P}^2}}^2 - 11 = -2$. By the projection formula for the double cover $\pi\colon X \to Y$, $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$, where $2L$ is linearly equivalent to the branch curve. Since this branch curve is

$$\overline{C} + \sum_1^5 E_i \equiv 10H - 4E_0 - 2\sum_1^5 E_i - 6\sum_1^5 F_i,$$

we have $L \equiv 5H - 2E_0 - \sum_1^5 E_i - 3\sum_1^5 F_i$ and $L^2 = -4$.

By Riemann-Roch, $\chi(\mathcal{O}_X) = 1$, where $\chi(\mathcal{O}_X) = 1 - q + p_g$, with irregularity $q = \dim H^1(X, \mathcal{O}_X)$ and $p_g = \dim H^2(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X(K_X))$ the geometric genus of X.

The canonical divisor on the double cover X is $K_X = \pi^*(K_Y + L)$, so that $K_X = \pi^*\big(2H - E - \sum_1^5 F_i\big)$. Therefore, the canonical system $|K_X|$ corresponds to the system of plane conics which pass through the six singular points of the branch curve C. The condition that these six points do not lie on a conic implies that this system is empty; therefore, $p_g = 0$, which implies $q = 0$.

We also have

$$2K_X \equiv \pi^*\left(4H - 2E - 2\sum_1^5 F_i\right),$$

so the bicanonical system is the pullback to X of the system of plane quartics with a double point at P_0, through $P_1, \dots, P_5$ with the same tangent direction as C. This corresponds to a pencil of plane quartics, therefore $\dim\bigl(H^0(X, 2K_X)\bigr) = 2$, thus X is of general type.

Next we compute the self-intersection of K_X on the non-minimal surface X. Since $K_X = \pi^*(K_Y + L)$, ${K_X}^2 = 2(K_Y + L)^2 = -4$. After blowing down the five exceptional curves on X to obtain $\overline{X}$, we have ${K_{\overline{X}}}^2 = 1$. Since $\dim\bigl(H^0(X, 2K_X)\bigr) = 1 + {K_{\overline{X}}}^2$, $\overline{X}$ is minimal. Therefore $\overline{X}$ is a numerical Godeaux surface. □

The case of numerical Campedelli surfaces is similar. We let C be a degree ten plane curve with six infinitely near triple points $P_0, \dots, P_5$. As above we let $\rho\colon Y \to \mathbf{P}^2$ be the blowup of the plane resolving the singularities of the curve C; in this case Y is the twelve-fold blowup of $\mathbf{P}^2$. We require that the six points $P_0, \dots, P_5$ do not lie on a conic, and that C has no additional singularities. Using the same notation as above, we have as proper transform of the branch curve

$$\overline{C} \equiv 10H - 3\sum_0^5 E_i - 6\sum_0^5 F_i$$

and $\overline{C} + \sum_0^5 E_i = 2L$ is an even divisor, so we can form the double cover $\pi\colon X \to Y$ of the surface Y branched along $2L$.

If C has no other singularites than those prescribed, then X is a non-singular surface. The six curves $\pi^{-1}(E_i)$ are -1-exceptional curves on X; contracting these we obtain the surface $\overline{X}$.

Proposition 2.2. *The surface $\overline{X}$ is a numerical Campedelli surface.*

Proof. As above we use Riemann-Roch to see that $\chi(\mathcal{O}_X) = 1$. In this case the canonical divisor on X is $K_X \equiv \pi^*(K_Y + L) = \pi^*\bigl(2H - \sum_0^5 F_i\bigr)$, so again the canonical system corresponds to the system of plane conics through the six singular points of the branch curve C. By our assumption this system is empty, and therefore $p_g = 0$ and $q = 0$.

Also $2K_X = \pi^*\bigl(4H - 2\sum_0^5 F_i\bigr)$, thus the bicanonical system corresponds to plane quartics through all six points P_i with the same tangent direction as C. This is again the pullback of a system of plane quartics; in this case $\dim\left(H^0\left(X, 2K_X\right)\right) = 3$, and X is of general type.

Finally, we have ${K_X}^2 = 2\left(K_Y + L\right)^2 = -4$; blowing down the six exceptional curves E_i we have ${K_{\overline{X}}}^2 = 2$ on the minimal surface $\overline{X}$. Therefore $\overline{X}$ is a numerical Campedelli surface. $\square$

We check in Section 3 that for our examples of branch curves, each resulting Campedelli surface has $\left|2K_{\overline{X}}\right|$ free from base points, thus the bicanonical map is a morphism.

3. The branch curves. Consider the space of degree ten plane curves defined by homogeneous polynomial $F(X, Y, Z) = 0$, where we use homogeneous coordinates $[X\colon Y\colon Z]$ on $\mathbf{P}^2$. The polynomial $F(X, Y, Z)$ depends on 66 coefficients, so we have a 65-dimensional projective space of degree ten plane curves. Following Stagnaro [**13**], we consider the polynomials invariant under the involution $[X\colon Y\colon Z] \to [X\colon -Y\colon Z]$. The subspace of degree ten plane curves invariant under this action is 35-dimensional. As we require that the six singular points of the branch curve do not lie on a conic, two of these points must be invariant with respect to this involution. We use a computer algebra system to generate the polynomials for the branch curves.

3.1 Godeaux branch curves. We first find branch curves for numerical Godeaux surfaces, that is, curves C with one order four point and five infinitely near triple points. We place the order four point P_0 at $[1\colon 0\colon 0]$ and a triple point P_1 at $[0\colon 0\colon 1]$ with infinitely near tangent direction $X = 0$ (so after blowing up P_1, the transform of the curve passes through ${P_1}'$). These points are invariant under the involution, and each of these singularities imposes six conditions on invariant polynomials $F(X, Y, Z)$, so the space of these curves is 23-dimensional. To impose the remaining four infinitely near triple points, we choose two pairs of points, P_2, P_3 and P_4, P_5 occurring as orbits of the involution. Each orbit will impose twelve linear conditions on the coefficients of the polynomial F. Allowing the points to vary, we can find curves C with the required singularities.

There is a 16-dimensional projective space of sextics invariant under this involution. It is two conditions for these curves to have a double point at P_0 and one condition to pass through P_1 with tangent $X = 0$. A tacnode imposes six conditions on the space of plane curves; therefore, there exists a sextic S with a double point at P_0, through P_1 with the tangent $X = 0$, and with tacnodes at $P_2, \dots, P_5$ with the same tangent direction as C. The curve S intersects C with multiplicity 8 at P_0, 6 at P_1, and 12 at P_2, P_3, P_4, P_5. This give a total intersection of $8 + 6 + 48 = 62$; therefore, S must be a component of C and C is reducible. By considering this sextic S, we analyze the possible configurations of branch curves.

Theorem 3.1. *Let C be a degree ten plane curve, invariant under the involution, with one order four point and five infinitely near triple points, so that the double plane ramified over C is a numerical Godeaux surface. Let C_i, D_i denote irreducible plane curves of degree i. Then C must be one of the following*:

(1) $C_4 + C_6$	(6) $C_2 + C_8$
(2) $C_2 + D_2 + C_6$	(7) $C_2 + C_4 + D_4$
(3) $C_1 + C_4 + C_5$	(8) $C_3 + D_3 + C_4$
(4) $C_1 + D_1 + C_8$	(9) $C_2 + D_2 + C_3 + D_3$.
(5) $C_1 + D_1 + C_4 + D_4$	

The configuration of C as in case (1) is given by Stagnaro [**13**], and case (6) can be obtained from Stagnaro's example by a birational transformation of the plane. Examples of C as in cases (5) and (9) were given by Oort and Peters [**10**]; the other cases for C yield new examples.

Proof. To enumerate the possible branch curves C, we consider the sextic S, which is invariant under the involution, has a double point at P_0, passes through $P_1, {P_1}'$, and has tacnodes at P_2, P_3, P_4, P_5. Since S is a component of C, the other components of C must form a degree four curve, with a double point at P_0, a tacnode at P_1, through P_2, P_3, P_4, P_5 with the same tangent direction as S.

If S is irreducible, then C cannot have a line as component. Therefore the possibilities are cases (1) and (2). The case of irreducible sextic and quartic composing C is given in Stagnaro [**13**]. In case (2) we obtain new examples of branch curves, one where each conic is invariant under the involution, and another where the conics are mapped to each other by the involution. In the case where $C = C_2 + D_2 + C_6$, where the conics C_2 and D_2 are each invariant, a set of polynomials defining the curves is $xz + 2y^2$, $xz + y^2$, and

$$x^4y^2 + 8x^2y^4 - 2y^6 + 12x^3y^2z + 21xy^4z + 23x^2y^2z^2 + 20y^4z^2 - 6x^3z^3 \\ + 6xy^2z^3 - 18x^2z^4 - 14y^2z^4 - 15xz^5;$$

here $P_2 = [-1\colon 1\colon 1]$, $P_3 = [-1\colon -1\colon 1]$, $P_4 = [-2\colon 1\colon 1]$, $P_5 = [-2\colon -1\colon 1]$. The conics have the same tangent at P_0, which results in an additional double point on E_0; this double point can be resolved without changing the invariants of the resulting double cover.

In the case where the two conics are exchanged by the involution, an example is given by $xy - 2y^2 + xz$, $-xy - 2y^2 + xz$, and

$$y^2(25600x^4 + 189024x^3z + 411326x^2z^2 - 122868xz^3 - 6075z^4) \\ + y^4(-201439x^2 - 742742xz + 101430z^2) + 396165y^6 \\ - 23296x^4z^2 - 72800x^3z^3 + 42273x^2z^4 + 3402xz^5.$$

The singular points in this example are $P_2 = [1\colon 1\colon 1]$, $P_4 = [9/2\colon 3\colon 1]$, and P_3, P_5 their images under the involution $Y \to -Y$.

Next suppose S is reducible. If S contains a line as a component, the line must be tangent to P_i for $i = 2, 3, 4$, or 5. In this case S splits as a line and a quintic, or as two lines and a quartic. If S has only one line as component, the other component of S must be an irreducible quintic, and we obtain case (3). An example of such a curve is given by the line $x = 2z$, the quartic $49x^2y^2 + 16y^4 - 114xy^2z + 4x^2z^2 = 0$, and

$$1225x^3y^2 + 396xy^4 - 5044x^2y^2z - 720y^4z - 3996x^3z^2 + 3821xy^2z^2 \\ + 17208x^2z^3 + 2590y^2z^3 - 18396xz^4.$$

Here $P_2 = [2\colon 1\colon 1]$, $P_3 = [2\colon -1\colon 1]$, $P_4 = [1\colon 2\colon 1]$, $P_5 = [1\colon -2\colon 1]$. In this case C has an additional double point away from $P_0, \dots, P_5$, as the line is tangent to the quintic at P_2 and P_3, so there is one additional point of intersection. This ordinary double point does not affect the invariants of the resulting surface.

In the case where the tangents to both pairs P_2, P_3 and P_4, P_5 are collinear (and necessarily components of C), we obtain either case (4), where the other component of C is an irreducible octic, and case (5), where the octic reduces into two quartics. Examples of both of these cases can be obtained by applying a Cremona transformation to previous examples. We have the general fact:

Remark 3.2. Consider a plane curve of degree d with order n point at P_0, order m point at P_1 and at $P_1{}'$. Then the image of this curve under the Cremona transformation centered at $P_0, P_1, P_1{}'$ is a plane curve of degree $2d - 2m - n$, with an order $d - 2m$ point at P_0 and order $d - n - m$ points at $P_1, P_1{}'$.

Examples of case (4), C composed of two lines and an irreducible octic, can be obtained by applying this Cremona transformation to the curves in case (2). Similarly an example of case (5) can be obtained by transforming the curves in the last case (4).

If the sextic S does not have a linear component, then it either contains a conic, or it consists of two cubics. When S does contain a conic as component, then the conic must be tangent to C at $P_2, \dots, P_5$. In this case this conic is a component of the branch curve C, and the remaining octic is either irreducible, as in case (6), composed of two quartics, as in case (7), or composed of a sextic and conic, so that again the branch curve C consists of a sextic and two conics. The case (6) can be obtained from case (1), and case (7) from case (8), by a Cremona transformation. Another example of C composed of a sextic and two conics is obtained by the Cremona transformation of the curves in case (3). Note that this example of a sextic and two conics making up C is distinct from that previously given for case (2), as this sextic has a triple point at P_0, and is different from S.

Lastly, when S is composed of two cubics, we obtain either case (8) or (9); an example of case (8) is given by polynomials $36x^3 - 9xy^2 - 48x^2z + 14y^2z + 7xz^2$, $15x^2y^2 - 4y^4 - 2xy^2z - 9x^2z^2$ and $xy^2 + 4y^2z + 9xz^2 - 14z^3$, with singularities at $P_2 = [1{:}\,1{:}\,1]$, $P_3 = [1{:}\,-1{:}\,1]$, $P_4 = [-7/3{:}\,\sqrt{21}{:}\,1]$, $P_5 = [-7/3{:}\,-\sqrt{21}{:}\,1]$. An example of case (9) is given in [**10**]. $\square$

3.2 Campedelli branch curves. We proceed as above to find branch curves for the numerical Campedelli construction. Two of the triple points are necessarily invariant under the involution, and we choose two orbits of points for the remaining four singularities. Again each of the invariant triple points will impose six conditions on the 35-dimensional space of invariant degree 10 curves. The remaining triple points will each impose 12 conditions, and allowing the points to vary we may find such curves.

In this case there exist (possibly reducible) degree four curves Q_0 and Q_1, invariant under the involution, which pass through each of the six singular points with the same tangent direction as C, and have a tacnode at P_0 or P_1, respectively. These quartics must intersect C with multiplicity 42, and therefore must be components of C. Thus, C is necessarily reducible.

Choose invariant points $P_0 = [1\colon 0\colon 0]$ with tangent direction $Z = 0$ and $P_1 = [0\colon 0\colon 1]$ with tangent $X = 0$.

Theorem 3.3. *The possible configurations for degree ten curves C invariant under the involution,with six infinitely near triple points and no other singularities, are as follows, where C_i, D_i and E_i denote irreducible curves of degree i*:

(1) $C_2 + C_4 + D_4$

(2) $C_1 + D_1 + C_4 + D_4$

(3) $C_2 + D_2 + C_6$

(4) $C_2 + D_2 + C_3 + D_3$

(5) $C_2 + D_2 + E_2 + C_4$.

Proof. The first three cases are the configurations of Campedelli curves given by Stagnaro [**13**]; case (5) is the classical Campedelli construction, while the configuration in case (4) was given by Campedelli-Kulikov-Oort, see [**8**]. Here we give examples of each class, and complete the classification to prove that any such Campedelli branch curve, invariant under involution, must have one of these forms.

We use the quartics Q_0 and Q_1. The general plane quartic polynomial, invariant under involution, has nine coefficients. A tacnode at P_0, with tangent $Z = 0$, imposes three linear conditions; it is one condition to require the curve to pass through P_1 with tangent $X = 0$, and requiring this curve to pass through two pairs of points P_2, P_3 and P_4, P_5, with designated tangent directions places an additional four linear conditions on the coefficients of the polynomial. Thus, for any choice of points P_2 and P_4 and corresponding tangent directions, we can find a possibly reducible quartic Q_0 (and similarly Q_1).

If both quartics are irreducible, then we obtain the first and second cases, where Q_0 and Q_1 are components of C. An example of polynomials defining the components of C as in case (1) are quartics $123y^4 - 205xy^2z + 85x^2z^2 - 32y^2z^2 + 29xz^3$ and $-160x^2y^2 - 1587y^4 + 145x^3z + 3289xy^2z - 1687x^2z^2$, together with the conic $5x^2 - 108y^2 + 62xz + 41z^2$, where the singularities are at the points $P_2 = [1\colon 1\colon 1]$, $P_3 = [1\colon -1\colon 1]$, $P_4 = [(23/5)\colon 2\colon 1]$, $P_5 = [(23/5)\colon -2\colon 1]$.

For case (2), the curve C has bitangents through P_2, P_4 and P_3, P_5 and these two lines are components of C, for example, the lines $x + y + z$ and $x - y + z$ which are tangent to the quartic curve $56x^3z - 35x^2y^2 + 72x^2z^2 - 132xy^2z + 36y^4$ and the quartic curve $40x^2z^2 + 6xy^2z + 56xz^3 - y^4 - 35y^2z^2$ at $[-2\colon 1\colon 1]$, $[3\colon -4\colon 1]$ and $[-2\colon -1\colon 1]$, $[3\colon 4\colon 1]$. In this case there is one additional singularity of C, namely at the intersection point of the two lines. This is an ordinary double point, whose resolution does not affect the invariants of the resulting double cover.

More generally, given any conic, invariant under our involution, and a choice of points P_2 and P_4 on the conic (and their images under the involution, P_3 and P_5), we can find Q_0 and Q_1. For the general choice of these points, these curves will not have additional singularities, the six points $P_0, \dots, P_5$ will not lie on a conic, and we will obtain a Campedelli branch curve configuration.

When the quartic Q_0 is reducible, it must decompose as two irreducible conics, and coincide with Q_1; thus, we obtain the third case. We have two conics, C_1 and C_2, both through P_0 and P_1 with the designated tangent directions, C_1 through P_2, P_3 and C_2 through P_4, P_5. A dimension count shows that for general points $P_2, \dots, P_5$, there is a pencil of sextics, invariant under the involution, tangent to C_1 and C_2 at P_0 and P_1, with tacnodes at P_2, P_3, P_4, P_5 with the tangents designated by C_1 and C_2. By Bertini's theorem the general member of this pencil will be an irreducible sextic with no additional singularities, as in case (3).

For example, in this case we have a family of branch curves given by the conics $y^2 + xz$ and $y^2 - 4xz$, with $P_2 = [-4\colon 2\colon 1]$, $P_3 = [-4\colon -2\colon 1]$, $P_4 = [1\colon 2\colon 1]$, $P_5 = [1\colon -2\colon 1]$, and the pencil of sextics generated by

$$\left(y^2 + xz\right)\left(4x^2 - 5y^2 + 12xz + 4z^2\right)^2$$

and

$$\left(y^2 - 4xz\right)\left(x^2 - 5y^2 + 3xz + 16z^2\right)^2 .$$

One reducible member of the pencil of sextics is the curve composed of two cubics, each tangent to C_1, C_2 at $P_2, \dots, P_5$, one cubic tangent at P_0, the other at P_1. This gives case (4); in our example, this reducible member is composed of the cubic curves defined by $x^3 - 2xy^2 + 6x^2z - 3y^2z + 13xz^2$ and $3xy^2 + 4x^2z - 8y^2z + 16z^3$.

Lastly, given the conics C_1 and C_2 as above, suppose there exists an invariant conic Q tangent to C at four points P_2, P_3, P_4, and P_5. In this case C_1, C_2, and Q must be components of C, and the remaining component must be of degree four, through each of the six points $P_0, \dots, P_5$ with coinciding tangent direction. These requirements impose six conditions on the eight-dimensional space of invariant quartics, and there is a two-dimensional family of such curves. Again Bertini's theorem guarantees that the general member have no additional singularities. An example of this final case is given by C_1, C_2 as above, and the conic Q defined by $5x^2 - 4y^2 + 6xz + 5z^2$ is tangent to C_1 at $P_2 = [-1\colon 1\colon 1]$, $P_3 = [-1\colon -1\colon 1]$ and tangent to C_2 at $P_4 = [1\colon 2\colon 1]$, $P_5 = [1\colon -2\colon 1]$. Then the net of quartics is given by

$$\begin{aligned}\alpha\left(12y^4 + 20x^3z - 52xy^2z - 24x^2z^2 + 20xz^3\right)&\\ + \beta\left(-5y^4 + 15xy^2z + 20x^2z^2\right)&\\ + \gamma\left(20x^2y^2 - 16y^4 + 24xy^2z + 20y^2z^2\right).&\end{aligned} \qquad \square$$

It is not known whether all minimal surfaces of general type with $p_g = 0$, $K^2 = 2$, must have bicanonical map a morphism; for all known examples the bicanonical system is base point free. We prove this is the case for our examples.

Proposition 3.4. *For each of the above configurations of Campedelli branch curves, the minimal model of the double cover, $\overline{X}$, has bicanonical system free from base points.*

Proof. Consider the first configuration of branch curve for a Campedelli surface, with C_2 representing the pullback to Y of the conic component of the branch curve, through the points $P_2, \dots, P_5$, Q_0 the quartic with a tacnode at P_0, through each of the remaining P_i with the same tangent as C_2, and Q_1 the quartic with a tacnode at P_1, through the remaining five P_i with the tangent as C_2. Then we have three members of $|2K|$,

$$\begin{aligned} M_1 &= 2\left(\pi^{-1}(Q_0) + \pi^*(F_0)\right) \\ M_2 &= 2\left(\pi^{-1}(Q_1) + \pi^*(F_1)\right) \\ M_3 &= 2\left(\pi^{-1}(C_2) + \pi^*(L_{01})\right) \end{aligned}$$

where L_{01} represents the pullback to Y of the line through P_0 and P_1. Since the proper transforms of Q_0 and Q_1 are disjoint, any base point of $|2K|$ must lie on $F_0 \cap Q_1$ or $F_1 \cap Q_0$. But C_2 does not pass through F_0 or F_1 (since the conic does not contain the points P_0 or P_1), and the line L_{01} will meet F_0 and F_1 at the points contracting E_0 and E_1. Thus there are no base points. The other cases are proved similarly. □

4. Torsion.

Proposition 4.1. *The numerical Godeaux surfaces constructed above have torsion* $\mathbf{Z}/4\mathbf{Z}$.

As the torsion of the surface is unchanged by blowing down curves, we compute the torsion working on the double cover X. To prove the proposition, we first note that the surface X has order two torsion, using the following lemma.

Lemma 4.2 (Beauville [**4**]). *Let Y be a smooth surface with no torsion, $\{C_i\}_{i\in I}$ a collection of smooth disjoint curves on Y and $\pi\colon X \to Y$ a connected double cover branched along $\cup_{i\in I} C_i$. Define a map*

$\varphi\colon \mathbf{Z}/2\mathbf{Z}^I \to \operatorname{Pic} Y \otimes \mathbf{Z}/2\mathbf{Z}$ by sending $\sum n_i C_i$ to its class in $\operatorname{Pic} Y$. If $e = \sum_{i\in I} C_i$, then the group $\operatorname{Pic}_2 X$ of $2-$torsion elements in $\operatorname{Pic} X$ is isomorphic to $\ker(\varphi)\,/\,(\mathbf{Z}/2\mathbf{Z})\,e$.

If $\sum_{i\in J} C_i \equiv 2A$ is an even divisor on Y, where $J \subset I$, then the map from the kernel of φ to $\operatorname{Pic} X$ sends $\sum_{i\in J} C_i$ to $\sum_{i\in J} \pi^{-1}(C_i) - \pi^*(A)$. For components C_i of the branch locus of $\pi\colon X \to Y$, $2\pi^{-1}(C_i) = \pi^*(C_i)$, thus $\sum_{i\in J} \pi^{-1}(C_i) - \pi^*(A)$ is 2-torsion in $\operatorname{Pic} X$.

In the case of the surface constructed using the branch curve composed of a quartic and a sextic, $C = Q_4 + S_6$, both the proper transform of Q_4 and of S_6 on Y are even divisors, so we obtain order two torsion. In particular,

$$\tau_1 = \pi^{-1}(Q_4) - \pi^*(2H - E - 2F_1 - F_2 - F_3 - F_4 - F_5)$$

and

$$\tau_2 = \pi^{-1}(S_6) - \pi^*(3H - E - F_1 - 2F_2 - 2F_3 - 2F_4 - 2F_5)$$

are both order two, with

$$\tau_1 + \tau_2 = \pi^{-1}(C) - \pi^*(L) \equiv 0.$$

Alternately we note that for any torsion divisor τ on X, by Riemann-Roch the divisor $D_\tau = K_X + \tau$ is effective. Thus if τ is of order two, $2D_\tau \equiv 2K_X$. As $K_X = \pi^*(K_Y + L) = \pi^*\left(2H - E - \sum_1^5 F_i\right)$, the bicanonical system $|2K_X|$ corresponds to the pencil of plane quartics which have a double point at the order four point of C, and which pass through each of the five triple points of C with the same tangent direction as C. Therefore $\pi^*(Q_4 + 2F_1)$ is in the bicanonical system, and

$$2\left(\pi^{-1}(Q_4) + \pi^*(F_1)\right) \equiv 2K_X.$$

Thus X has order two torsion. We next determine the base points of the tricanonical system to finish the computation of the torsion, applying the following.

Lemma 4.3 (Miyaoka [**9**]). *For a minimal Godeaux surface, the number of base points of $|3K|$ is equal to*

$$\begin{cases} 0 \text{ if } \operatorname{Tors} = 0 \text{ or } \mathbf{Z}/2\mathbf{Z} \\ 1 \text{ if } \operatorname{Tors} = \mathbf{Z}/3\mathbf{Z} \text{ or } \mathbf{Z}/4\mathbf{Z} \\ 2 \text{ if } \operatorname{Tors} = \mathbf{Z}/5\mathbf{Z}. \end{cases}$$

Thus if $|3K|$ has one base point, since there is order two torsion the torsion group must be $\mathbf{Z}/4\mathbf{Z}$.

We use the projection formula to compute the base points of the tricanonical system. Since $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$ and $K_X = \pi^*(K_Y + L)$, we have

$$\pi_*\mathcal{O}_X(3K_X) = \mathcal{O}_Y(3K_Y + 3L) \oplus \mathcal{O}_Y(3K_Y + 2L).$$

As $3K_Y + 3L \equiv 6H - 3E - 3\sum_1^5 F_i$ and $3K_Y + 2L \equiv H - E$, the tricanonical system on X corresponds to sections of $\mathcal{O}_Y(3K_Y + 3L)$, the pencil of plane sextics with a triple point at P_0, and double points at each of the P_i where one tangent direction coincides with C, together with sections of $\mathcal{O}_Y(3K_Y + 2L)$, the pencil of lines through P_0.

We have

$$3K_X \equiv \pi^{-1}(C) + \pi^*(H - E);$$

since the pencil of lines $|\pi^*(H - E)|$ has no base points, any base point of $|3K_X|$ must lie on the branch curve $\pi^{-1}(C)$.

Also

$$3K_X \equiv M = \pi^*\Big(6H - 3E - 3\sum F_i\Big),$$

thus any base point of $|3K_X|$ must be a base point of the pencil $|M|$.

For each of our two examples of C, computing the pencil of plane sextics, we find that $|M|$ has one base point that lies on $\pi^{-1}(C)$. Thus the surface X, and therefore $\overline{X}$, has order four torsion.

Proposition 4.4. *The numerical Campedelli surfaces constructed above have 2-torsion subgroup* $\mathrm{Tors}_2 = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ *in cases* (1)–(4) *of Theorem* 3.3*, and* $\mathrm{Tors}_2 = (\mathbf{Z}/2\mathbf{Z})^3$ *in case* (5).

Proof. Case (5) of Theorem 3.3, the classical Campedelli configuration, has torsion group equal to $(\mathbf{Z}/2\mathbf{Z})^3$, see [**8**]. In the other cases we consider the bicanonical system on X, for each of the branch curves C of the double cover $\pi: X \to Y$,

(i) C is a conic and two quartics,

(ii) C is two lines and two quartics,

(iii) C is two conics and a sextic,

(iv) C is two conics and two cubics.

In case (i), we have branch divisor $C = C_2 + Q_0 + Q_1$ with

$$\begin{aligned} C_2 &\equiv 2\left(H - \sum_2^5 F_i\right) \\ Q_0 &\equiv 2\left(2H - 2F_0 - \sum_1^5 F_i\right) \\ Q_1 &\equiv 2\left(2H - F_0 - 2F_1 - \sum_2^5 F_i\right); \end{aligned}$$

thus,

$$\begin{aligned} \tau_1 &= \pi^{-1}(C_2) - \pi^*\left(H - \sum_2^5 F_i\right) \\ \tau_2 &= \pi^{-1}(Q_0) - \pi^*\left(2H - 2F_0 - \sum_1^5 F_i\right) \\ \tau_3 &= \pi^{-1}(Q_1) - \pi^*\left(2H - F_0 - 2F_1 - \sum_2^5 F_i\right) \end{aligned}$$

are each order two torsion divisors, with

$$\tau_1 + \tau_2 + \tau_3 = \pi^{-1}(C) - \pi^*(L) \equiv 0.$$

In case (ii), we have $C = L_1 + L_2 + Q_0 + Q_1$ with

$$\begin{aligned} L_1 + L_2 &\equiv 2(H - F_2 - F_3 - F_4 - F_5) \\ Q_0 &\equiv 2\left(2H - 2F_0 - \sum_1^5 F_i\right) \\ Q_1 &\equiv 2\left(2H - F_0 - 2F_1 - \sum_2^5 F_i\right); \end{aligned}$$

thus,

$$\tau_1 = \pi^{-1}(L_1 + L_2) - \pi^*\left(H - \sum_2^5 F_i\right)$$
$$\tau_2 = \pi^{-1}(Q_0) - \pi^*\left(2H - 2F_0 - \sum_1^5 F_i\right)$$
$$\tau_3 = \pi^{-1}(Q_1) - \pi^*\left(2H - F_0 - 2F_1 - \sum_2^5 F_i\right)$$

are each order two torsion divisors, with

$$\tau_1 + \tau_2 + \tau_3 = \pi^{-1}(C) - \pi^*(L) \equiv 0.$$

In cases (iii) and (iv), the branch curve is $C = C_1 + C_2 + S_6$ with

$$\begin{aligned} C_1 &\equiv 2(H - F_0 - F_1 - F_2 - F_3) \\ C_2 &\equiv 2(H - F_0 - F_1 - F_4 - F_5) \\ S_6 &\equiv 2(3H - F_0 - F_1 - 2F_2 - 2F_3 - 2F_4 - 2F_5), \end{aligned}$$

and 2-torsion divisors

$$\begin{aligned} \tau_1 &= \pi^{-1}(C_1) - \pi^*(H - F_0 - F_1 - F_2 - F_3) \\ \tau_2 &= \pi^{-1}(C_2) - \pi^*(H - F_0 - F_1 - F_4 - F_5) \\ \tau_3 &= \pi^{-1}(S_6) - \pi^*(3H - F_0 - F_1 - 2F_2 - 2F_3 - 2F_4 - 2F_5), \end{aligned}$$

$$\tau_1 + \tau_2 + \tau_3 = \pi^{-1}(C) - \pi^*(L) \equiv 0.$$

Therefore in each case, we have $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \subseteq \mathrm{Tors}_2(X)$. By Lemma 4.2, all 2-torsion on the double cover comes from the decomposition of the branch curve, thus

$$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} = \mathrm{Tors}_2(X). \qquad \square$$

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