ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 6, 2006

ALMOST PERIODIC FUNCTIONALS ON SOME CLASS OF BANACH ALGEBRAS

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ABSTRACT. In this paper we study the question of characterizing almost periodic functionals on some class of Banach algebras. Some related problems are also discussed.

1. Introduction. Let A be a Banach algebra, A^* its dual and A_1 its closed unit ball. For $\varphi \in A^*$ and $a \in A$, the functionals $\varphi \cdot a$ and $a \cdot \varphi$ are defined by $\langle \varphi \cdot a, b\rangle = \langle \varphi, ab\rangle$ and $\langle a \cdot \varphi, b\rangle = \langle \varphi, ba\rangle$. These operations invert A^* into a Banach A-bimodule. Kitchen [**11**] calls a functional φ almost periodic on A, if the linear operator $L_\varphi : A \to A^*$, defined by $L_\varphi(a) = \varphi \cdot a$, is compact. This is equivalent to the fact that the set $\{\varphi \cdot a : a \in A_1\}$ is relatively norm compact in A^*. For example if $A = L^1(G)$ for a locally compact group G, then this reduces to the classical notion of almost periodicity for $\varphi \in L^\infty(G)$. Since for $a \in A$, $L_\varphi^*(a) = a \cdot \varphi$, it follows that φ is almost periodic if and only if the set $\{a \cdot \varphi : a \in A_1\}$ is relatively norm compact in A^*. By $\mathrm{ap}\,(A)$ we denote the set of all almost periodic functionals on A. Clearly, $\mathrm{ap}\,(A)$ is a norm closed A-subbimodule of A^*. In the theory of representations of Banach algebras and in the Arens regularity theory it is important to have a convenient description of $\mathrm{ap}\,(A)$ for concrete classes of Banach algebras, see [**6, 7, 11, 13, 15, 17, 18**] for a discussion of this problem. The following problem was posed in [**6**, Problem 2]: Characterize those Banach algebras A for which each $\varphi \in A^*$ is almost periodic. It has been proved by Quigg [**18**, Theorem 3.2], that for a C^*-algebra A, $\mathrm{ap}\,(A) = A^*$ if and only if A is scattered and its irreducible representations are finite dimensional. Recall that A is called scattered if every positive functional on A is a sum of pure positive functionals. By Lau and Ülger [**13**, Theorem 3.6], other characterizations of the C^*-algebra A satisfying the equality $\mathrm{ap}\,(A) = A^*$ are given. Among

2000 AMS *Mathematics Subject Classification.* Primary 43A20, 43A60, 46H15, 46L05.
Key words and phrases. Almost periodic functional, almost periodic representation, group algebra, M-ideal, Schur property, C*-algebra.
Received by the editors on April 28, 2004.

them we note the following: $\operatorname{ap}(A) = A^*$ if and only if the space A^* is an ℓ^1-sum of finite dimensional Banach spaces if and only if A^* has the Schur property, i.e., every weakly convergent sequence in A^* is norm convergent. In [**15**, Theorem 4.8], the characterization of almost periodic functionals on Herz algebras is given.

In this paper we study the question of characterizing almost periodic functionals on some well-known classes of Banach algebras. These results generalize and unify the above-mentioned results or parts of them. Some other applications to related problems are also given.

2. Preliminaries. We recall the main definitions and results that we require later. If X is a Banach space, we denote by X^* its dual. The natural duality between X^* and X is denoted by $\langle \varphi, x\rangle$. If $E \subset X$, then $\overline{E}$ will denote the norm closure of E in X. Now let A be an arbitrary Banach algebra. A weak* closed subspace $E \subset A^*$ is said to be an invariant subspace of A^*, if $\varphi \cdot a \in E$ and $a \cdot \varphi \in E$, for all $\varphi \in E$ and $a \in A$. Note that if A has an approximate identity, then a weak* closed subspace $E \subset A^*$ is an invariant subspace of A^* if and only if $a \cdot \varphi \cdot b \in E$ for all $\varphi \in E$ and $a, b \in A$. An invariant subspace E of A^* is said to be minimal if E does not contain other nonzero invariant subspaces of A^*. By $\widehat{A}_c$ we will denote the set of all equivalence classes of irreducible finite-dimensional representations of A. We can see that if $\sigma \in \widehat{A}_c$, then the algebra $A/\ker\sigma$ is isomorphic to the full matrix algebra and consequently, $\ker\sigma$ is a maximal bi-ideal in A of finite codimension. Hence $(\ker\sigma)^{\perp}$ is a minimal finite dimensional invariant subspace of A^*.

Lemma 1. *If A has a bounded approximate identity, then every minimal finite-dimensional invariant subspace E of A^* has the form $(\ker\sigma)^{\perp}$, for some $\sigma \in \widehat{A}_c$.*

Proof. Let E be a nonzero minimal finite-dimensional invariant subspace of A^*. Then $E = J^{\perp}$ for some maximal nontrivial bi-ideal J in A of finite codimension. Note that the quotient algebra A/J also has a bounded approximate identity. Since A/J is finite dimensional, it has a unit element. Hence J is a maximal modular bi-ideal. Then J is contained in a maximal modular nontrivial left ideal I. Let σ be the regular representation of A on A/I. Then σ is a nonzero irreducible finite-dimensional representation of A and $J \subset \ker\sigma = \{a \in A : aA \subset I\}$. By maximality of J we have $J = \ker\sigma$, so that $E = (\ker\sigma)^{\perp}$. $\square$

Let A be an arbitrary Banach algebra. As is known [**6**], the second dual A^{**} of A can be equipped with two Banach algebra multiplications $\circ$ and $*$ (the first and the second Arens multiplication) which extend the multiplication in A (canonically embedded into A^{**}). Namely for $a \in A$, $\varphi \in A^*$ and $F, H \in A^{**}$, we set $\langle F \circ H, \varphi\rangle = \langle F, H \cdot \varphi\rangle$ and $\langle F * H, \varphi\rangle = \langle H, \varphi \cdot F\rangle$, where $H \cdot \varphi$ and $\varphi \cdot F$ are functionals on A defined by $\langle H \cdot \varphi, a\rangle = \langle H, \varphi \cdot a\rangle$ and $\langle \varphi \cdot F, a\rangle = \langle F, a \cdot \varphi\rangle$. If $F \circ H = F * H$ for every $F, H \in A^{**}$, then A is said to be Arens regular. Recall that $\varphi \in A^*$ is said to be weakly almost periodic on A, if the set $\{\varphi \cdot a : a \in A_1\}$ is relatively weakly compact in A^*. As in the introduction we can see that φ is weakly almost periodic if and only if the set $\{a \cdot \varphi : a \in A_1\}$ is relatively weakly compact in A^*. Let $\mathrm{wap}\,(A)$ denote the set of all weakly almost periodic functionals on A. Then, Arens regularity of A is equivalent to the condition that $\mathrm{wap}\,(A) = A^*$ [**6**, Theorem 1]. Throughout the paper by A^{**} we will denote the Banach algebra A^{**} equipped with the first Arens multiplication. It can be seen that for F fixed in A^{**}, the mapping $H \to H \circ F$ is weak*-weak* continuous on A^{**}. The weak*-weak* continuity of the mapping $H \to F \circ H$ is equivalent to the Arens regularity of the algebra A [**6**, Theorem 1]. Note also that $\mathrm{ap}\,(A)$ is a Banach A^{**}-bimodule. Moreover, $\varphi \in \mathrm{ap}\,(A)$ if and only if either $\{F \cdot \varphi : F \in A_1^{**}\}$ or $\{\varphi \cdot F : F \in A_1^{**}\}$ is relatively norm compact in A^{**}.

Later on we shall need the following results with which we now proceed. Let G be a locally compact group, and let T be a continuous representation of G on a Banach space X. T is said to be almost periodic if the set $\{T_g x : g \in G\}$ is relatively compact in X for all $x \in X$. The following result is an immediate consequence of the Peter-Weyl theory.

Theorem 1 [**16**, pp. 152–153]. *If T is an almost periodic representation of G on X, then X is a closed linear span of the irreducible finite-dimensional T-invariant subspaces.*

Now define the adjoint representation T^* of G on X^* by $T_g^*\varphi = (T_{g-1})^*\varphi$, $\varphi \in X^*$. Of course, T^* is not (strongly) continuous in general.

Lemma 2. *Assume that the set $\{T_g^*\varphi : g \in G\}$ is relatively norm compact in X^*, for every $\varphi \in X^*$. Then T^* is a continuous representation* (*consequently, T^* is an almost periodic representation*).

Proof. Let $\varphi \in X^*$ be given. We have to show that the function $g \to T_g^*\varphi$ is continuous at $g = 1$, where 1 is the unit element of G. Let $(g_\lambda)_{\lambda\in\lambda}$ be a net in G that converges to 1. Since $T_{g_\lambda}^*\varphi \to \varphi$ in the weak* topology, it follows that φ is the unique norm cluster point of the net $(T_{g_\lambda}^*\varphi)_{\lambda\in\lambda}$. On the other hand, the net $(T_{g_\lambda}^*\varphi)_{\lambda\in\lambda}$ is contained in a relatively norm compact set. This clearly implies that $T_{g_\lambda}^*\varphi \to \varphi$ for the norm topology. $\square$

3. Almost periodic functionals on the image of group algebras.

Before stating the main results of this section we shall need some notation. Let G be a locally compact group equipped with a left Haar measure dg and $L^1(G)$, the group algebra of G. For $g \in G$, the left and the right translations of $f \in L^1(G)$ are defined by ${}_gf(t) = f(g^{-1}t)$ and $f_g(t) = \Delta(g)f(tg)$, respectively, where $\Delta(g)$ is the modular function of G. Recall that ${}_g(f*h) = {}_gf*h$, $(f*h)_g = f*h_g$ and $f*_gh = f_{g-1}*h$ for all $f,h \in L^1(G)$ and all $g \in G$. One of the main results of this section is the following theorem.

Theorem 2. *Let A be a Banach algebra. If there exists a continuous homomorphism $\theta : L^1(G) \to A$, with dense range, then*

$$\mathrm{ap}\,(A) = \overline{\mathrm{span}}\,\{(\ker\sigma)^\perp : \sigma \in \widehat{A}_c\}.$$

Proof. The inclusion $\overline{\mathrm{span}}\,\{(\ker\sigma)^\perp : \sigma \in \widehat{A}_c\} \subset \mathrm{ap}\,(A)$ being clear, we prove reverse inclusion only. Let us define the left and the right regular representations on A as follows: Put $L_g\theta(f) = \theta({}_gf)$ and $R_g\theta(f) = \theta(f_g)$. Let $(e_i)_{i\in I}$ be a bounded (by one) approximate identity for $L^1(G)$. Since $\theta({}_ge_i)\theta(f) \to L_g\theta(f)$, we have $\|L_g\theta(f)\| \leq$

$\|\theta\|\,\|\theta(f)\|$. Thus, since $\{\theta(f) : f \in L^1(G)\}$ is dense in A, L_g (similarly R_g) can be extended (in a unique way) to A as a bounded continuous representation. We will denote this extension again by L_g (respectively by R_g). We can see that $L_g(ab) = (L_g a)b$, $R_g(ab) = a(R_g b)$ and $a(L_g b) = (R_{g^{-1}} a)b$ for all $a, b \in A$ and all $g \in G$. Let T denote the Cartesian product of the representations L and R : $T_{g,s}a = L_g R_s a$, where $a \in A$ and $(g, s) \in G \times G$.

Clearly, $(\theta(e_i))_{i\in I}$ is a bounded approximate identity for A. As in the proof of the Lemma 2, we can see that if $\varphi \in \mathrm{ap}\,(A)$, then $\varphi\cdot\theta\,(e_i) \to \varphi$ and $\theta\,(e_i)\cdot\varphi \to \varphi$ in the norm topology. Since $\mathrm{ap}\,(A)$ is a Banach A-bimodule, by the Cohen-Hewitt factorization theorem [**10**, 32.22], $\mathrm{ap}\,(A) = A\cdot\mathrm{ap}\,(A) = \mathrm{ap}\,(A)\cdot A$. Consequently, every $\varphi \in \mathrm{ap}\,(A)$ can be represented as $\varphi = a\cdot\psi\cdot b$, for some $a, b \in A$ and $\psi \in \mathrm{ap}\,(A)$. We claim that

$$T^*_{g,s}\varphi = (L_s a)\cdot\psi\cdot(R_g b)\,. \tag{3.1}$$

To see this, let $c \in A$ be given. Since

$$\begin{aligned}(R_g b)\,c(L_s a) &= b(L_{g^{-1}}c)(L_s a) = b(L_{g^{-1}}(c(L_s a)))\\ &= b(L_{g^{-1}}(R_{s^{-1}}c))a = b(T_{g^{-1},s^{-1}}c)a,\end{aligned}$$

we have

$$\begin{aligned}\langle (L_s a)\cdot\psi\cdot(R_g b), c\rangle &= \langle \psi, (R_g b)c(L_s a)\rangle = \langle \psi, b(T_{g^{-1},s^{-1}}c)a\rangle\\ &= \langle a\cdot\psi\cdot b, T_{g^{-1},s^{-1}}c\rangle = \langle T^*_{g,s}\varphi, c\rangle.\end{aligned}$$

A standard "finite ε-mesh technique" shows that if $\varphi \in \mathrm{ap}\,(A)$, then the set $\{a\cdot\varphi\cdot b : a, b \in A_1\}$ is relatively norm compact in A^*. From this and from the equality (3.1), we deduce that $\mathrm{ap}\,(A)$ is an invariant subspace for T^* and moreover, the set $\{T^*_{g,s}\varphi : (g, s) \in G \times G\}$ is relatively norm compact in A^* for every $\varphi \in \mathrm{ap}\,(A)$. By Lemma 2, the restriction of T^* to $\mathrm{ap}\,(A)$ is an almost periodic representation. Hence by Theorem 1, $\mathrm{ap}\,(A)$ is a closed linear span of the irreducible finite-dimensional T^*-invariant subspaces. Now, by Lemma 1, it remains to show that irreducible finite-dimensional T^*-invariant subspaces are exactly minimal finite-dimensional invariant subspaces of A^*. For this it is enough to show that finite-dimensional T^*-invariant subspaces are exactly finite-dimensional invariant subspaces of A^*. Let E be a finite-dimensional T^*-invariant subspace, and let $\varphi \in E$ be given. By the definition of the vector-valued integral

$$\int_{G\times G} \tilde{h}(g)k(s)T^*_{g,s}\,\varphi\,dg\,ds$$

is in E, where $\tilde{h}(g) = \Delta(g^{-1})h(g^{-1})$. A simple calculation shows that the last integral is equal to $\theta(h)\cdot\varphi\cdot\theta(k)$. Since $\{\theta(f) : f \in L^1(G)\}$ is dense in A, we have $a\cdot\varphi\cdot b \in E$ for all $a, b \in A$. Now let E be a finite-dimensional invariant subspace of A^* and $\varphi \in E$. Using the equality (3.1) again, we get that

$$\begin{aligned} T^*_{g,s}\varphi &= \text{weak}^* - \lim_i T^*_{g,s}\left(\theta(e_i)\cdot\varphi\cdot\theta(e_i)\right) \\ &= \text{weak}^* - \lim_i \left(\theta({}_s e_i)\cdot\varphi\cdot\theta(e_i)_g\right). \end{aligned}$$

Hence $T^*_{g,s}\varphi \in E$ for all $(g, s) \in G\times G$. This completes the proof. □

The second main result of this section is the following theorem.

Theorem 3. *Let A be an Arens regular Banach algebra. If there exists a continuous homomorphism $\theta : L^1(G) \to A^{**}$, with dense range, then*

$$\mathrm{ap}\,(A) = \overline{\mathrm{span}}\left\{(\ker\sigma)^\perp : \sigma \in \widehat{A}_c\right\}.$$

Proof. Let $(e_i)_{i\in I}$ be a bounded approximate identity for $L^1(G)$, and let $E_i = \theta(e_i)$. Then $(E_i)_{i\in I}$ is a bounded approximate identity for A^{**}. We may assume without loss of generality that the net $(E_i)_{i\in I}$ converges weak* to some $E \in A^{**}$. Let us see that E is the identity for A^{**}. Let $F \in A^{**}$ be given. Weak*-continuity of $H \to H\circ F$ implies that $E_i\circ F \to E\circ F$ in the weak*-topology. Since $E_i\circ F \to F$ in the norm, we obtain $E\circ F = F$. On the other hand, since A is Arens regular, the mapping $H \to F\circ H$ is weak*-weak* continuous. It follows that $F\circ E_i \to F\circ E$ in the weak*-topology and hence $F\circ E = F$.

Let L and R be the left and the right regular representations on A^{**}, respectively (see the proof of the Theorem 2). We recall that $L_g(F\circ H) = (L_gF)\circ H$, $R_g(F\circ H) = F\circ(R_gH)$ and $F\circ(L_gH) =$

$(R_{g^{-1}}F) \circ H$ for all $F, H \in A^{**}$ and all $g \in G$. Denote by T, the Cartesian product of the representations L and R: $T_{g,s}F = L_g R_s F$, where $F \in A^{**}$ and $(g,s) \in G \times G$. $\varphi \in A^{*\cdot}$ and $F \in A^{**}$ are given. We can see that $\varphi = E \cdot \varphi$. On the other hand, since A is Arens regular, we have $\langle F, \varphi \cdot E\rangle = \langle E * F, \varphi\rangle = \langle E \circ F, \varphi\rangle = \langle F, \varphi\rangle$, so that $\varphi = \varphi \cdot E$. Hence φ can be represented as $\varphi = E \cdot \varphi \cdot E$. As in the proof of Theorem 2, we can see that $T^*_{g,s}\varphi = (L_s E) \cdot \varphi \cdot (R_g E)$. Now let $\varphi \in \mathrm{ap}\,(A)$ be given. Since $\mathrm{ap}\,(A)$ is a Banach A^{**}-bimodule, it follows that $\mathrm{ap}\,(A)$ is an invariant subspace for T^*. Further, since $\{L_s E : s \in G\}$ and $\{R_g E : g \in G\}$ are bounded sets in A^{**}, from the last equality we can deduce that the set $\left\{T^*_{g,s}\varphi : (g,s) \in G \times G\right\}$ is relatively norm compact in A^*, for every $\varphi \in \mathrm{ap}\,(A)$. By Lemma 2, $T^*|_{\mathrm{ap}\,(A)}$, the restriction of T^* to $\mathrm{ap}\,(A)$, is an almost periodic representation. Hence, by Theorem 1, $\mathrm{ap}\,(A)$ is a closed linear span of the irreducible finite-dimensional $T^*|_{\mathrm{ap}\,(A)}$-invariant subspaces. As in the proof of Theorem 2, we can see that irreducible finite-dimensional $T^*|_{\mathrm{ap}\,(A)}$-invariant subspaces are exactly minimal finite-dimensional invariant subspaces of A^*. Further, since A is Arens regular and A^{**} has a unit element, A has a bounded approximate identity [**3**, p. 147, Corollary 8]. Consequently, by Lemma 1, every finite-dimensional invariant subspace of A^* has the form $(\ker \sigma)^{\perp}$, for some $\sigma \in \widehat{A}_c$. This completes the proof. $\square$

For the next result we recall [**1**] that a closed subspace E of a Banach space X is said to be an L-ideal if there exists a projection $P : X \to E$ such that $\|x\| = \|Px\| + \|x - Px\|$, for every $x \in X$. E is said to be an M-ideal if $E^{\perp}$ is an L-ideal in X^*. Below we shall need the following result [**1**, Proposition 1.7]. Let E and F be two L-ideals, and let P_E and P_F be the associated projections, respectively. If $E \cap F = \{0\}$, then $P_E P_F = P_F P_E = 0$.

We shall also need the following notation. Let λ be a nonempty index set, and suppose $\{X_\lambda : \lambda \in \lambda\}$ is a collection of Banach spaces. By $X = \left[\sum_{\lambda\in\lambda} \oplus X_\lambda\right]_\alpha$ we mean the Banach space of all $\{x_\lambda\}$ such that $\{\|x_\lambda\|\}$ is in $c_0\,(\lambda)$ or $\ell_1\,(\lambda)$ or $\ell_\infty\,(\lambda)$ for the case $\alpha = c_0$ or $\alpha = \ell_1$ or $\alpha = \ell_\infty$, respectively, and $x_\lambda \in X_\lambda$ for each $\lambda \in \lambda$. The norm on X is to be the norm of $\{\|x_\lambda\|\}$.

Theorem 4. *Assume that the hypotheses of Theorem* 2 (*or the hypotheses of Theorem* 3) *are satisfied. Moreover, assume that for each* $\sigma \in \widehat{A}_c$, $\ker \sigma$ *is an* M*-ideal. Then,*

$$\mathrm{ap}\,(A) = \Big(\sum_{\sigma \in A_c} \oplus \, (\ker \sigma)^{\perp} \Big)_{l^1}. \tag{3.2}$$

Proof. For an arbitrary $\sigma \in \widehat{A}_c$, we put $E_\sigma = (\ker \sigma)^{\perp}$. Then E_σ is an L-ideal in A^*. Let P_σ be the associated projection. Let two distinct $\sigma', \sigma'' \in \widehat{A}_c$ be given. Since $E_{\sigma'}$ and $E_{\sigma''}$ are minimal invariant subspaces of A^*, we have $E_{\sigma'} \cap E_{\sigma''} = \{0\}$, and therefore $P_{\sigma'} P_{\sigma''} = P_{\sigma''} P_{\sigma'} = 0$. It follows that, if $\varphi_{\sigma_i} \in E_{\sigma_i}$, $i = 1, \dots, n$, where $\sigma_i \neq \sigma_j (i \neq j)$, then

(3.3)

$$\begin{aligned} \|\varphi_{\sigma_1} + \cdots + \varphi_{\sigma_n}\| &= \|P_{\sigma_1}(\varphi_{\sigma_1} + \cdots + \varphi_{\sigma_n})\| \\ &\quad + \|\varphi_{\sigma_1} + \cdots + \varphi_{\sigma_n} - P_{\sigma_1}(\varphi_{\sigma_1} + \cdots + \varphi_{\sigma_n})\| \\ &= \|\varphi_{\sigma_1}\| + \|\varphi_{\sigma_2} + \cdots + \varphi_{\sigma_n}\| \\ &= \cdots = \|\varphi_{\sigma_1}\| + \cdots + \|\varphi_{\sigma_n}\|. \end{aligned}$$

Let Y denote the right-hand side of (3.2), Y_c the linear subspace of Y consisting of all functions with finite support. By Theorem 2 (or Theorem 3), we have

$$\mathrm{ap}\,(A) = \overline{\mathrm{span}} \left\{ (\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c \right\}.$$

From this and from the identity (3.3), we see that the linear operator $S : Y_c \to ap(A)$, defined by

$$S : (\varphi_j)_{j \in J} \longrightarrow \sum_{j \in J} \varphi_j,$$

is an isometry with dense range. Since Y_c is dense in Y, S extends by continuity to an isometric isomorphism of Y onto the $\mathrm{ap}\,(A)$. This completes the proof. $\square$

Let A be an arbitrary Banach algebra. We put $A^* \cdot A = \{\varphi \cdot a : \varphi \in A^*, a \in A\}$. If A has a bounded approximate identity, then by the Cohen-Hewitt factorization theorem [**10**, 32.22], $A^* \cdot A$ is a norm closed linear subspace of A^*. The following result is related to a result of Duncan and Ülger [**7**, Theorem 2.3].

Theorem 5. *Let A be a semi-simple Banach algebra with a bounded approximate identity. Moreover, assume that the following conditions are satisfied.*

a) *Every irreducible representation of A is finite-dimensional.*

b) *For each $a \in A$, the left multiplication operator $L_a : A \to A$, defined by $L_a : b \to ab$ is weakly compact. Then,*

$$\mathrm{ap}\,(A) = A^* \cdot A = \overline{\mathrm{span}}\,\big\{(\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c\big\}.$$

Proof. Obviously, we have

$$\overline{\mathrm{span}}\,\big\{\,(\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c\big\} \subset \mathrm{ap}\,(A).$$

As in the proof of Theorem 2, we can see that

$$\mathrm{ap}\,(A) = \{a \cdot \varphi : \varphi \in \mathrm{ap}\,(A), a \in A\}\,.$$

Hence $\mathrm{ap}\,(A) \subset A^* \cdot A$. Now it remains to show that

$$A^* \cdot A \subseteq \overline{\mathrm{span}}\,\big\{(\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c\big\}.$$

Assume that for some $\varphi \in A^*$ and $a \in A$,

$$\varphi \cdot a \notin \overline{\mathrm{span}}\,\big\{(\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c\big\}.$$

Then there is an F in A^{**} such that $\langle F, \varphi \cdot a\rangle \neq 0$ but $\langle F, \psi\rangle = 0$, for all $\psi \in (\ker \sigma)^{\perp}$ and all $\sigma \in \widehat{A}_c$. In particular, we have $\langle a \circ F, \psi\rangle = \langle F, \psi \cdot a\rangle = 0$, for all $\psi \in (\ker \sigma)^{\perp}$ and all $\sigma \in \widehat{A}_c$. We recall [**6**, Lemma 3] that A is a right ideal in A^{**} if and only if L_a is a weakly compact operator for each $a \in A$. Hence $a \circ F \in A$. From the last equality we see that $a \circ F \in \ker \sigma$, for all $\sigma \in \widehat{A}_c$. Since A is a semi-simple algebra, we have $a \circ F = 0$. This contradicts the fact that $\langle a \circ F, \varphi\rangle = \langle F, \varphi \cdot a\rangle \neq 0$. The proof is complete. $\square$

These theorems have several corollaries. We first remark that a class of Banach algebras satisfying the hypotheses of Theorem 2 are sufficiently large. For example, let T be a continuous representation of G on a Banach space X. For $f \in L^1(G)$, the operator $T_f : X \to X$, defined by

$$T_f x = \int_G f(g) T_g x \, dg,$$

is a bounded linear operator on X. Let $L_T(G)$ denote the closure of $\{T_f : f \in L^1(G)\}$ with respect to the uniform operator topology. Hence, Theorem 2 can be applied to the algebras $L_T(G)$. Let $X = L^p(G)$, $1 < p < \infty$, and let T be the left regular representation on $L^p(G)$. In this case, as in [**9**], $L_T(G)$ will be denoted by $PF_p(G)$. We can see that $PF_p(G)$ is the completion of $L^1(G)$ relative to the convolution operator norm:

$$|||f|||_p = \sup\left\{\|f * h\|_p : h \in L^p(G),\ \|h\|_p \le 1\right\}, \quad f \in L^1(G).$$

Clearly, $|||f|||_p \le \|f\|_1$. We shall also need the following notation. By $C_0(G)$ we denote the space of continuous functions on G which are null at infinity. $\mathrm{AP}\,(G)$ denotes the space of almost periodic functions on G.

Corollary 1. $\mathrm{ap}\,(PF_p(G)) = PF_p(G)^*$ *if and only if* G *is compact.*

Proof. For the compact group G the equality $\mathrm{ap}\,(PF_p(G)) = PF_p(G)^*$ has been proved in [**13**, Corollary 8.7]. Now assume that $\mathrm{ap}\,(PF_p(G)) = PF_p(G)^*$ and G is not compact. We see that $id : L^1(G) \to PF_p(G)$ is a continuous homomorphism with dense range. If $\varphi \in \mathrm{ap}\,(PF_p(G))$, then by Theorem 2, $id^*\varphi \in \mathrm{AP}\,(G)$. For $h \in L^p(G) - \{0\}$ and $k \in L^q(G) - \{0\}$, $(1/p + 1/q = 1)$, let $[h \otimes k]$ denote the functional on $PF_p(G)$, defined by $\langle [h \otimes k], f\rangle = \langle f * h, k\rangle$. It is easy to verify that $id^*[h \otimes k] = h * k^\nu$, where $k^v(g) = k(g^{-1})$. Hence we have $h * k^\nu \in \mathrm{AP}\,(G)$. Because $h * k^\nu \in C_0(G)$, we obtain $h * k^\nu = 0$, since no nontrivial almost periodic function on a noncompact locally compact group can vanish at infinity [**4**, p. 41, Corollary 3.8]. Let (V) be a net of compact symmetric neighborhoods of 1 contracting to $\{1\}$, e_V, the nonnegative function on G supported by V and satisfying $e_V = e_V^\nu$ and $\|e_V\|_1 = 1$. Then $0 = h * e_V \to h$ in L^p-norm as $V \to \{1\}$. Hence, we have $h = 0$. This contradicts $h \neq 0$. $\square$

Below, we will assume that G is an abelian group. In this case, under the hypotheses of Theorem 2, A becomes a commutative Banach algebra. By Σ_A we will denote the structure space of A. A standard Banach algebra technique shows that θ^* is homeomorphic identified Σ_A with the hull $(\ker\theta)$. We see that hull $(\ker\theta)$ is a closed subset of $\widehat{G}$, the dual group of G.

Corollary 2. *Under the hypotheses of Theorem* 2*,*

$$\mathrm{ap}\,(A) = \overline{\mathrm{span}}\ \Sigma_A.$$

Corollary 3. *Assume that the hypotheses of Theorem* 2 *are satisfied and J is a closed ideal of A. Then*

$$\mathrm{ap}\,(A/J) = \overline{\mathrm{span}}\,\mathrm{hull}\,(J).$$

A subset K of $\widehat{G}$ is strongly independent if, for every choice distinct points $\chi_1, \dots, \chi_k$ of K and integers $n_1, \dots, n_k$, the equality $\chi_1^{n_1} \cdot \cdots \cdot \chi_k^{n_k} = 1$ implies $n_1 = \cdots = n_k = 0$ [**2**, p. 69]. Recall also that [**20**, p. 98], a subset K of $\widehat{G}$ is a Kronecker set if K has the following property: to every continuous function f on K of absolute value one, and to every $\varepsilon > 0$, there is $g \in G$, such that $\sup_{\chi\in K} |f(\chi) - \chi(g)| < \varepsilon$.

Corollary 4. *Assume that the hypotheses of Theorem* 2 *are satisfied and* hull $(\ker\theta)$ *is a strongly independent subset of* $\widehat{G}$. *Then* ap (A) *is isomorphic to the space* $\ell^1\,(\Sigma_A)$.

Proof. By Corollary 2, $\mathrm{ap}\,(A) = \overline{\mathrm{span}}\,\Sigma_A$. Therefore, it is enough to show that, for every choice $\phi_1, \dots, \phi_n \in \Sigma_A$ and $\lambda_1, \dots, \lambda_n \in C - \{0\}$,

$$\left\|\sum_{i=1}^n \lambda_i \phi_i\right\|_{A^*} \geq \frac{1}{\|\theta\|} \sum_{i=1}^n |\lambda_i|. \tag{3.4}$$

Let $\theta^*\phi_i = \chi_i$, $i = 1, \dots, n$, where $\chi_1, \dots, \chi_n \in \mathrm{hull}\,(\ker\theta)$. Then, we have

$$\left\|\sum_{i=1}^n \lambda_i \phi_i\right\|_{A^*} \geq \frac{1}{\|\theta^*\|} \left\|\theta^*\left(\sum_{i=1}^n \lambda_i \phi_i\right)\right\|_\infty = \frac{1}{\|\theta\|} \left\|\sum_{i=1}^n \lambda_i \chi_i\right\|_\infty.$$

Let f be the function on $\{\chi_1, \dots, \chi_n\}$, defined by $f(\chi_i) = |\lambda_i|/\lambda_i$, $i = 1, \dots, n$. Let an arbitrary ε be such that $0 < \varepsilon < 1$ be given. Since a finite strongly independent set is a Kronecker set [**20**, Theorem 5.1.3], there exists $g_0 \in G$, such that $\sup_i |f(\chi_i) - \chi_i(g_0)| < \varepsilon$.

Consequently, we have

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i\chi_i\right\|_\infty &\geq \left|\sum_{i=1}^n \lambda_i\chi_i(g_0)\right|\\ &\geq \left|\sum_{i=1}^n \lambda_i f(\chi_i)\right| - \sum_{i=1}^n |\lambda_i|\,|f(\chi_i) - \chi_i(g_0)|\\ &\geq (1-\varepsilon)\sum_{i=1}^n |\lambda_i|.\end{aligned}$$

This completes the proof. □

Corollary 5. *We have* $\operatorname{ap}(PF_p(G)) = \overline{\operatorname{span}}\,\widehat{G}$.

4. Herz algebras and the Schur property. Let G be a locally compact abelian group. For $1 < p < \infty$, let $A_p(G)$ denote the linear subspace of $C_0(G)$ consisting of all functions of the form

$$f = \sum_{n=1}^\infty h_n * k_n^v$$

where $h_n' s$ are in $L^p(G)$, $k_n' s$ in $L^q(G)$, $(1/p+1/q=1)$, $k_n^v(x) = k_n(x^{-1})$ and

$$\sum_{n=1}^\infty \|h_n\|_p \|k_n\|_q < \infty.$$

The norm of f is the infimum of these sums over all such representations of f. Obviously we have $\|f\|_\infty \leq \|f\|_{A_p}$ for all $f \in A_p(G)$. $A_p(G)$ is often called a Herz algebra. As is known [**9**], $A_p(G)$ is a commutative semi-simple regular Banach algebra. The structure space of $A_p(G)$ can be identified with a group G via Dirac measures $\delta_g (g \in G)$. Recall also that $L^1(\widehat{G})$ is isometric isomorphic to $A_2(G)$ via Fourier transform F. As functions which are continuous on G with a compact support are dense in $L^p(G)$, $A_2(G)$ is dense in $A_p(G)$. It follows that $F : L^1(\widehat{G}) \to A_p(G)$ is a continuous homomorphism with dense range. Now applying Corollary 2, we have the following.

Corollary 6 [**15**, Theorem 4.8]. $\mathrm{ap}\,(A_p(G)) = \overline{\mathrm{span}}\,\{\delta_g : g \in G\}$.

Let $PM_p(G)$ denote the algebra of all operators on $L^p(G)$ which commute with translation. As is known [**9**] $PM_p(G)$ can be identified with the dual space of $A_p(G)$. The elements of $PM_p(G)$ are called p-pseudomeasures. There is an obvious notion "support" for p-pseudomeasures. Let K be a compact subset of G and $PM_p(K)$, the space of p-pseudomeasures which are supported on K. Note that $PM_p(K)$ is the dual space of $A_p(G)/J_K$, where J_K is the smallest closed ideal in $A_p(G)$ whose hull is K.

Recall that a Banach space X has the Schur property, if every weakly convergent sequence in X is norm convergent. Lust-Piquard [**15**, Theorem 2.14] has shown that, if K is compact and dispersed (i.e. K has no nonempty perfect subset), then $PM_p(K)$ has the Schur property. Under a metrizability hypothesis on K, the converse of this fact was proved in [**15**, Theorem 2.8] and [**14**, Proposition 3]. However, we have the following.

Theorem 6. *Let K be an arbitrary compact subset of G. If $PM_p(K)$ has the Schur property, then K is dispersed.*

For the proof of this theorem we need the following lemma which is proved in [**17**]. We include a proof for completeness.

Lemma 3. *Let A be an arbitrary Banach algebra. If the space A^* has the Schur property, then* $\mathrm{ap}\,(A) = A^*$.

Proof. Suppose that A^* has the Schur property. Let φ be an element in A^*. We have to show that the operator $L_\varphi : A \to A^*$ defined by $L_\varphi(a) = \varphi \cdot a$ is compact. To prove this, let $(a_n)_{n\in N}$ be a sequence in A_1. Since A^* has the Schur property, A does not contain an isomorphic copy of ℓ^1 [5, Theorem 3]. Hence, by Rosenthal's ℓ^1-theorem [**19**], the sequence $(a_n)_{n\in N}$ has a subsequence, denoted again $(a_n)_{n\in N}$, which is weakly Cauchy. It follows that the sequence $(L_\varphi(a_n))_{n\in N}$ is weakly Cauchy in A^*. As A^* has the Schur property, it is weakly sequentially complete. Consequently, the sequence $(L_\varphi(a_n))_{n\in N}$ converges weakly, so in norm in A^*. This proves that φ is almost periodic on A, so that $A^* = \mathrm{ap}\,(A)$. $\square$

Proof of Theorem 6. Assume that $PM_p(K)$ has the Schur property. Since $PM_p(K)$ is the dual space of $A_p(G)/J_K$, by Lemma 3,

$$\mathrm{ap}\;(A_p(G)/J_K) = PM_p(K).$$

We have already noted that the Fourier transform F maps $L^1(\widehat{G})$ onto a dense subalgebra of $A_p(G)$. Now, applying Corollary 3, we get that

$$\mathrm{ap}\;(A_p(G)/J_K) = \overline{\mathrm{span}}\left\{\delta_g : g \in K\right\}.$$

Hence, we have

$$PM_p(K) = \overline{\mathrm{span}}\left\{\delta_g : g \in K\right\}. \tag{4.1}$$

Let μ be an arbitrary finite regular Borel measure on G. μ can be considered as an element of $PM_p(G)$ for the pairing

$$\langle \mu, f\rangle = \int_G f(g)\,d\mu(g), \quad f \in A_p(G).$$

We remark also that $\mathrm{supp}\,\mu$ in terms of the support of $PM_p(G)$ and $\mathrm{supp}\,\mu$ in usual terms are the same. Now we claim that $F^*\mu = \hat{\mu}(\chi)$, where $\hat{\mu}(\chi)$ is the Fourier-Stieltjes transform of μ. To see this let $f \in L^1(\widehat{G})$ be given. We can write

$$\begin{aligned}\int_{\widehat{G}} (F^*\mu) f(\chi)\,d\chi = \langle \mu, F(f)\rangle &= \int_G \hat{f}(g)\,d\mu(g)\\ &= \int_G \left(\int_{\widehat{G}} f(\chi)\overline{\chi(g)}\,d\chi\right) d\mu(g)\\ &= \int_{\widehat{G}} \left(\int_G \overline{\chi(g)}\,d\mu(g)\right) f(\chi)\,d\chi = \int_{\widehat{G}} \hat{\mu}(\chi) f(\chi)\,d\chi.\end{aligned}$$

This being true for all $f \in L^1(\widehat{G})$, we get that $F^*\mu = \hat{\mu}(\chi)$.

Now let μ be an arbitrary continuous regular Borel measure supported on K. To prove that K is dispersed, it is enough to show that μ is identically zero [**12**, p. 52, Theorem 10]. Since $\mu \in PM_p(K)$, it follows from the equality (4.1) that $F^*\mu$ can be approximated in the $\|\cdot\|_\infty$-norm by the linear combinations of the characters $\{\overline{g(\chi)} : g \in K\}$. Hence, $\hat{\mu}(\chi^{-1})$ is an almost periodic function on $\widehat{G}$. Let Φ be the invariant mean on AP$\,(\widehat{G})$.

Since

$$\langle \Phi, g(\chi)\rangle = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}$$

and since μ is a continuous measure, we have

$$\left\langle \Phi, \overline{g(\chi)}\, \hat{\mu}(\chi^{-1})\right\rangle = \mu\{g\} = 0, \quad g \in K.$$

This shows that the Fourier-Bohr coefficients of $\hat{\mu}(\chi^{-1})$ are zero. By the uniqueness theorem we obtain that $\hat{\mu}(\chi^{-1}) \equiv 0$ and hence $\mu = 0$. This completes the proof. $\square$

5. Almost periodic functionals on C^*-algebras. Recall that the following characterization of almost periodic functionals on C^*-algebras is given by Quigg [**18**, Theorem 3.4]. Let π_{ap} be the direct sum of all the finite dimensional cyclic representations obtained from the application of the Gelfand-Naimark-Segal construction to the states of A, and denote the corresponding central projection by z_{ap}. Then $\mathrm{ap}\,(A) = z_{\mathrm{ap}} \cdot A^*$. However, we have the following.

Corollary 7. *For an arbitrary C^*-algebra A,*

$$\mathrm{ap}\,(A) = \Bigg(\sum_{\sigma \in \widehat{A}_c} \oplus (\ker \sigma)^{\perp} \Bigg)_{\ell^1}.$$

Proof. Let A be a C^*-algebra. It is well known that A is Arens regular and A^{**} can be identified with the enveloping von Neumann algebra of A. Let Γ be the unitary group in A^{**} endowed with the discrete topology. It follows from the Russo-Dye theorem [**3**, p. 210], $id : L^1(\Gamma) \to A^{**}$ is a continuous homomorphism with dense range. On the other hand, the M-ideals in C^*-algebras are exactly the closed two-sided ideals [**21**]. Thus, the result now follows from Theorem 4. $\square$

Let M be a von Neumann algebra and M_* its predual. It is well known that M_* is a Banach M-bimodule. By $\mathrm{ap}_*(M)$ we denote the set of all σ-continuous almost periodic functionals on M. Clearly, $\mathrm{ap}_*(M)$ is an M-subbimodule of M_*. A closed subspace $E \subset M_*$ is called an invariant subspace of M_*, if $a \cdot \varphi \cdot b \in E$ for all $\varphi \in E$ and $a, b \in A$. An invariant subspace E of M_* is said to be minimal if E does not contain other nonzero invariant subspaces of M_*. As usual, we call a projection p in a von Neumann algebra M finite dimensional if the algebra pMp is finite dimensional. By $P_c(M)$ we shall denote the set of all minimal finite dimensional central projections of M.

Quigg [**18**, Theorem 2.8], has shown that $\mathrm{ap}_*(M) = (M_{\mathrm{ap}})_*$, where M_{ap} is the largest ideal of M which is a direct sum of matrix algebras. The following theorem gives another characterization of σ-continuous almost periodic functionals on von Neumann algebras.

Theorem 7. *For an arbitrary von Neumann algebra M,*

$$\mathrm{ap}_*(M) = \left(\sum_{p \in P_c(M)} \oplus\, p \cdot M_* \right)_{\ell^1}.$$

Proof. Let $\{E_i : i \in I\}$ be the family of all minimal finite-dimensional invariant subspaces of M_*. We first show that

$$\mathrm{ap}_*(M) = \overline{\mathrm{span}}\,\{E_i : i \in I\}. \tag{5.1}$$

Since each E_i, $i \in I$, is a finite-dimensional invariant subspace of M_*, we have $E_i \subset \mathrm{ap}_*(M)$ and consequently $\overline{\mathrm{span}}\,\{E_i : i \in I\} \subset \mathrm{ap}_*(M)$. To show the reverse inclusion, let U denote the group of all unitary elements of M. Let us equip U with the discrete topology and define a representation T of $U \times U$ on M_* by $T(\gamma, s)\varphi = \gamma^{-1} \cdot \varphi \cdot s$, where $\gamma, s \in U$ and $\varphi \in M_*$. Clearly, $\mathrm{ap}_*(M)$ is an invariant subspace for T. A standard "finite ε-mesh technique" shows that the set $\{\gamma^{-1}\cdot\varphi\cdot s : (\gamma, s)\in U\times U\}$ is relatively compact in M_* for all $\varphi\in \mathrm{ap}_*(M)$. Hence, the restriction of T to $\mathrm{ap}_*(M)$ is an almost periodic representation. By Theorem 1, $\mathrm{ap}_*(M)$ is a closed linear span of the irreducible finite-dimensional T-invariant subspaces. However, since the linear span of U is dense in M, irreducible finite-dimensional T-invariant subspaces are exactly minimal finite-dimensional invariant subspaces of M_*. This proves (5.1). Further, if E is a minimal finite-dimensional invariant subspace of M_*, then $E^{\perp}$ is a maximal bi-ideal in M of finite codimension. It follows that there exists a minimal finite-dimensional central projection $p\in M$ such that $E^{\perp}=(1-p)M$ and, consequently, $E=p\cdot M_*$. The converse is also true. If p is a minimal finite-dimensional central projection in M, then $p\cdot M_*$ is a minimal finite-dimensional invariant subspace of M_*. Let p_i be the associated projection of E_i, $i\in I$. Since $E_i\cap E_j=\{0\}$, $i\neq j$, it follows that the projections $\{p_i : i\in I\}$ are mutually orthogonal. Furthermore, the natural mapping

$$M \longrightarrow (p_i\, M \oplus (1-p_i)\, M)_{\ell_\infty},$$

is clearly isometric, whence $\|\varphi\| = \|p_i\cdot\varphi\| + \|\varphi - p_i\cdot\varphi\|$, for all $\varphi\in M_*$ and $i\in I$. Consequently, for a given $\varphi_{i_1}\in E_{i_1},\dots,\varphi_{i_n}\in E_{i_n}$, since $p_{i_n}\cdot\varphi_{i_m}=0$, $n\neq m$, we have

$$\begin{aligned}
&\|\varphi_{i_1}+\cdots+\varphi_{i_n}\|\\
&= \|p_{i_1}\cdot(\varphi_{i_1}+\cdots+\varphi_{i_n})\| + \|\varphi_{i_1}+\cdots+\varphi_{i_n} - p_{i_1}\cdot(\varphi_{i_1}+\cdots+\varphi_{i_n})\|\\
&= \|\varphi_{i_1}\| + \|\varphi_{i_2}+\cdots+\varphi_{i_n}\| = \cdots = \|\varphi_{i_1}\| + \|\varphi_{i_2}\| + \cdots + \|\varphi_{i_n}\|.
\end{aligned}$$

Now the proof is completed as in the proof of Theorem 4. $\square$

This theorem also gives a characterization of almost periodic functionals on C^*-algebras.

Corollary 8. *For an arbitrary C^*-algebra A,*

$$\mathrm{ap}\,(A) = \Big(\sum_{p\in P_c(A^{**})} \oplus\, p\cdot A^*\Big)_{\ell^1}.$$

Proof. It is easy to see that $\operatorname{ap}(A) = \operatorname{ap}_*(A^{**})$. Thus, the result follows from Theorem 7. $\square$

Remark 1. Let A be a C^*-algebra. As in the proof of Theorem 7, we can see that $\{p \cdot A^* : p \in P_c(A^{**})\}$ is exactly the set of all minimal finite-dimensional invariant subspaces of A^*. Since A has a bounded approximate identity, by Lemma 1, we have $\{p \cdot A^* : p \in P_c(A^{**})\} = \{(\ker \sigma)^{\perp} : \sigma \in \widehat{A}_c\}$. Hence, Corollary 8 is a reformulation of Corollary 7.

The following simple fact presents certain interest.

Proposition 1. *Let H be a Hilbert space, and let A be a Banach algebra of compact operators on H. Then the following assertions are equivalent.*

a) $\operatorname{ap}(A) = A^*$.

b) A_1 *is relatively compact in the strong operator topology.*

Proof. a) $\Rightarrow$ b). Note that A^* is (linearly isometric) $L^1(H)/A^{\perp}$, where $L^1(H)$ is the space of trace-class operators on H. For x, y in H, by $x \otimes y$ we denote the one-dimensional operator on H defined by $x \otimes y : z \to (z, y)x$. Now let $(T_n)_{n\in N}$ be an arbitrary sequence in A_1. For given x, y in $H\backslash\{0\}$, since $x \otimes y + A^{\perp}$ is in $\operatorname{ap}(A)$, $(T_n)_{n\in N}$ has a subsequence denoted again by $(T_n)_{n\in N}$, such that the sequence $T_n \cdot (x \otimes y + A^{\perp}) = T_n x \otimes y$ converges in A^*. Consequently, we have

$$\|T_n x \otimes y - T_m x \otimes y\| = \|T_n x - T_m x\| \, \|y\| \longrightarrow 0.$$

It follows that the sequence $(T_n x)_{n\in N}$ converges.

b) $\Rightarrow$ a). Since the linear span of $\{x \otimes y + A^{\perp} : x, y \in H\}$ is dense in A^*, it is enough to show that $x \otimes y + A^{\perp}$ is in $\operatorname{ap}(A)$ for every x, y in H. To see this, let $(T_n)_{n\in N}$ be an arbitrary sequence in A_1, and let $x \in H$. Since the set $(T_n x)_{n\in N}$ is relatively compact, it has a subsequence denoted again by $(T_n x)_{n\in N}$ such that $T_n x \to z$, for some $z \in H$. Hence

$$T_n \cdot (x \otimes y + A^{\perp}) = T_n x \otimes y \longrightarrow z \otimes y.$$

The proof is complete. $\square$

Corollary 9. *Let H be a Hilbert space, and let A be a C^*-algebra of compact operators on H. Then the following assertions are equivalent.*

a) A_1 *is relatively compact in the strong operator topology.*

b) A *is a c_0-sum of finite dimensional C^*-algebras.*

Proof. As shown in [**13**, Lemma 4.1], under the hypotheses of the corollary, $\mathrm{ap}\,(A) = A^*$ if and only if A is a c_0-sum of finite dimensional C^*-algebras. Thus the result follows from Proposition 1. $\square$

Now let A and B be two unital C^*-algebras and $A\widehat{\otimes}B$ their projective tensor product. Let U_A and U_B be the group of unitary elements of A and B, respectively. By the Russo-Dye theorem [**3**, p. 210], $\overline{\text{a.c.h.}}U_A = A_1$ and $\overline{\text{a.c.h.}}U_B = B_1$, where a.c.h. is the "absolute convex hull". Then $\Gamma = \{u \otimes v : u \in U_A,\ v \in U_B\}$ is a group in $A\widehat{\otimes}B$ and, by the definition of the projective tensor norm, $\overline{\text{a.c.h.}}\,(\Gamma) = \big(A\widehat{\otimes}B\big)_1$. Let Γ be equipped with the discrete topology. Then the mapping $\mathrm{id} : L^1(\Gamma) \to A\widehat{\otimes}B$ is a continuous homomorphism with dense range (it can be deduced even more: $A\widehat{\otimes}B$ is isomorphic to a quotient algebra of $L^1(\Gamma)$). Now, applying Theorem 2, we have the following.

Corollary 10. *Let A and B be two unital C^*-algebras. Then*

$$\mathrm{ap}\,\big(A\widehat{\otimes}B\big) = \overline{\mathrm{span}}\,\big\{(\ker\sigma)^{\perp} : \sigma \in \big(A\widehat{\otimes}B\big)_c^{\wedge}\big\}.$$

Let K and S be two compact Hausdorff spaces and $K \times S$ their Cartesian product.

Corollary 11. *We have* $\mathrm{ap}\,\big(C(K)\widehat{\otimes}\,C(S)\big) = \ell^1(K \times S)$.

Proof. It is well known that the structure space of $C(K)\widehat{\otimes}C(S)$ can be identified canonically with $K \times S$ via the mapping $(x, y) \to \delta_x \otimes \delta_y (x \in K,\ y \in S)$, where $\delta_x \otimes \delta_y$ is defined by $\langle \delta_x \otimes \delta_y, f \otimes g\rangle = f(x)g(y),\ f \in C(K),\ g \in C(S)$. By Corollary 10, we have

$$\mathrm{ap}\,\big(C(K)\widehat{\otimes}C(S)\big) = \overline{\mathrm{span}}\,\big\{\delta_x \otimes \delta_y : (x, y) \in K \times S\big\}.$$

On the other hand, we can see that, for every choice of $x_1, \dots, x_n \in K$, $y_1, \dots, y_n \in S$ and complex numbers $\lambda_1, \dots, \lambda_n$,

$$\|\lambda_1 \delta_{x_1} \otimes \delta_{y_1} + \cdots + \lambda_n \delta_{x_n} \otimes \delta_{y_n}\| = |\lambda_1| + \cdots + |\lambda_n| .$$

Now the proof is completed as in the proof of Theorem 4. □

Acknowledgment. I am very grateful to the referee for valuable remarks and suggestions.

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