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ON CERTAIN STRUCTURES DEFINED ON THE TANGENT BUNDLE

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ABSTRACT. The differential geometry of tangent bundles was studied by several authors, for example: Davies [**4**], Yano and Davies [**5**], Dombrowski [**6**], Ledger and Yano [**9**] and Blair [**1**], among others. It is well known that an almost complex structure defined on a differentiable manifold M of class C^∞ can be lifted to the same type of structure on its tangent bundle $T(M)$. However, when we consider an almost contact structure, we do not get the same type of structure on $T(M)$. In this case we consider an odd dimensional base manifold while our tangent bundle remains to be even dimensional. The purpose of this paper is to examine certain structures on the base manifold M in relation to that of the tangent bundle $T(M)$.

1. Introduction. Let M be an n-dimensional differentiable manifold, and let $T(M)$ be its tangent bundle. Then $T(M)$ is also a differentiable manifold of dimension $2n$ [**11**]. Let $X = \sum_{i=1}^n x^i(\partial/\partial x^i)$ and $\omega = \sum_{i=1}^n \omega^i \, dx^i$ be the expressions in local coordinates for the vector field X and the 1-form ω in M. Let (x^i, y^i) be local coordinates of a point in $T(M)$ induced naturally from the coordinate chart (U, x^i) in M. If f is a function in M, then its vertical lift f^V is a function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$ so that

$$f^V = f \circ \pi. \tag{1.1}$$

The complete lift f^C of f is also a function in $T(M)$ given by

$$f^C = y^i \, \partial_i f = \partial f. \tag{1.2}$$

We can introduce a system of local coordinates $\left(x^h, y^h\right)$ in an open set $\pi^{-1}(U) \subset T(M)$. Here we call $\left(x^h, y^h\right)$ the coordinates in $\pi^{-1}(U)$

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induced from $\left(x^h\right)$. Suppose that we are given in M a tensor field [**11**].

$$S = S^{i\dots,h}_{\ell k\dots,j}\,\frac{\partial}{\partial x^i}\otimes\cdots\otimes\frac{\partial}{\partial x^h}\otimes dx^\ell\otimes\cdots\otimes dx^j.$$

We then define a tensor field $\Upsilon_x S$ in $\pi^{-1}(U)$ by

$$\Upsilon_x S = \left(X^\ell S^{i\dots,h}_{\ell k\dots,j}\right)\frac{\partial}{\partial y^i}\otimes\cdots\otimes\frac{\partial}{\partial y^h}\otimes\cdots dx^j$$

and a tensor field ΥS in $\pi^{-1}(U)$ by

$$\Upsilon_x S = \left(y^\ell S^{i\dots,h}_{\ell k\dots,j}\right)\frac{\partial}{\partial y^i}\otimes\cdots\otimes\frac{\partial}{\partial y^h}\otimes dx^k\cdots\otimes dx^j \tag{1.3a}$$

with respect to the induced coordinates $\left(x^h, y^h\right)$, U being an arbitrary coordinate neighborhood in M. The tensor field $\Upsilon_x S$ and ΥS defined in each $\pi^{-1}(U)$ determine respectively global tensor fields in TM.

Let us assume that there is given an affine connection ∇ in a differentiable manifold M. If f is a function in M, we write ∇f for the gradient of f in M. We apply the operation Υ given by (1.3a) to the gradient ∇f and $\Upsilon(\nabla f)$. We put

$$\nabla_\Upsilon f = \Upsilon(\nabla f). \tag{1.3b}$$

We now define the horizontal lift $f^H$of f in M to the tangent bundle $T(M)$ by

$$f^H = f^c - \nabla_\Upsilon f. \tag{1.4}$$

Definition 1.1. For any arbitrary tensor fields on $T(M)$, we have:

$$(P\otimes Q)^C = P^C\otimes Q^V + P^V\otimes Q^C \tag{1.5a}$$

$$(P\otimes Q)^V = P^V\otimes Q^V \tag{1.5b}$$

and

$$(P\otimes Q)^H = P^V\otimes Q^H + P^H\otimes Q^V. \tag{1.5c}$$

GF- structure. The base space M is said to possess a GF-structure if there exists on M a tensor field F of type $(1,1)$ such that

$$F^2 = a^2 I_n, \tag{1.6}$$

where I_n is the unit tensor field and a is a nonzero complex number.

Generalized almost r-contact structure. The manifold M will be said to possess the generalized almost r-contact structure if there exists on M a tensor field F of type $(1,1)$, r (C^∞) contravariant vector fields U_p and $r(C^\infty)$ 1-forms ω_p, $p = 1, 2, \dots, r$, satisfying [**11**]:

$$F^2 = a^2 I_n + \varepsilon \sum_{p=1}^{r} \omega_p \otimes U_p \tag{1.7a}$$

$$F\, U_p = 0 \tag{1.7b}$$

$$\omega_p \circ F = 0 \tag{1.7c}$$

and

$$\omega_p(U_q) = -\,\frac{a^2}{\varepsilon}\,\delta_{pq} \tag{1.7d}$$

where $p, q = 1, 2, \dots, r$ and δ_{pq} denotes the Kronecker delta while a and ε are nonzero complex numbers.

Nijenhuis tensor. The Nijenhuis tensor $N(X,Y)$ of F is given by [**11**]

$$N\,(X,Y) = [FX, FY] - F\,[FX,Y] - F\,[X,FY] + F^2\,[X,Y] \tag{1.8}$$

2. Induced structure on the tangent bundle. Let us suppose that the base space M admits the generalized almost r-contact structure. Then there exists a tensor field F of type $(1,1)$, $r\,(C^\infty)$ vector fields $U_1, U_2, \dots, U_p$ and $r\,(C^\infty)$ 1-forms $\omega_1, \omega_2, \dots, \omega_p$ such that equations (1.7) are satisfied. Taking complete lifts of equations (1.7) we obtain the following:

$$(F^C)^2 = a^2 I_{2n} + \varepsilon \sum_{p=1}^{r} \{\omega_p^V \otimes U_p^C + \omega_p^C \otimes U_p^V\} \tag{2.1a}$$

$$F^C\, U_p^C = 0 \tag{2.1b}$$

and

$$\omega_p^C(U_q^C) = -\frac{a^2}{\varepsilon}\,\delta_{pq} \tag{2.1d}$$

where p, q take values $1, 2, \ldots, r$ and δ_{pq} denotes the Kronecker delta. Hence we have:

Theorem 2.1. *If the base space M admits the generalized almost r-contact structure it induces the structure in $T(M)$ given by equations* (2.1). *Observe that the vertical and complete lifts satisfy the following conditions* [**11**]:

$$F^C\,U_p^V = 0, \quad F^C\,U_p^C = 0 \tag{2.2a}$$

$$\omega_p^V \circ F^C = 0, \quad \omega_p^C \circ F^V = 0, \quad \omega_p^V \circ F^V = 0 \tag{2.2b}$$

$$\omega_p^V(U_q^V) = 0, \quad \omega_p^V(U_q^C) = -\frac{a^2}{\varepsilon}\,\delta_{pq}. \tag{2.2c}$$

If we let

$$J = F^C + \frac{\varepsilon}{a}\sum_{p=1}^{r}\{\omega_p^V \otimes U_p^V + \omega_p^C \otimes U_p^C\}, \tag{2.3}$$

then in the view of equations (2.2a), (2.2b) *and* (2.2c), *it is easily shown that*

$$J^2 = a^2 I_{2n}.$$

Hence, we have

Theorem 2.2. *If the base space M admits the generalized almost r-contact structure, then the tensor field J, defined by* (2.3) *gives a GF-structure on $T(M)$.*

Now, taking the horizontal lift of equations (1.7) we obtain the following

$$(F^H)^2 = a^2 I_n + \varepsilon \sum_{p=1}^{r} \{\omega_p^V \otimes U_p^H + \omega_p^H \otimes U_p^V\} \tag{2.4a}$$

$$F^H U_p^H = 0, \quad F^H U_p^V = 0 \tag{2.4b}$$

$$\omega_p^H \circ F^H = 0, \quad \omega_p^V \circ F^H = 0 \tag{2.4c}$$

$$\omega_p^H(U_q^H) = 0, \quad \omega_p^H(U_q^V) = -\frac{a^2}{\varepsilon}\,\delta_{pq} \tag{2.4d}$$

$$\omega_p^V(U_q^V) = 0, \quad \omega_p^V(U_q^H) = -\frac{a^2}{\varepsilon}\,\delta_{pq}. \tag{2.4e}$$

Now let us define a (1,1) tensor field $\overset{*}{J}$ on $T(M)$ as follows:

$$\overset{*}{J} = F^H + \frac{\varepsilon}{a} \sum_{p=1}^{r} \{\omega_p^V \otimes U_p^V + \omega_p^H \otimes U_p^H\}. \tag{2.5}$$

In view of the equations (2.4), it can be easily shown that

$$\left(\overset{*}{J}\right)^2 = a^2 I_{2n}.$$

Hence $\overset{*}{J}$ gives a GF-structure on $T(M)$. Consequently, we have the following theorem:

Theorem 2.3. *If the base space M admits the generalized almost r-contact structure, and the induced structure on $T(M)$ is given by the equations* (2.4) *and if $\overset{*}{J}$ is a $(1,1)$ tensor field defined on $T(M)$ by equation* (2.5)*, then $\overset{*}{J}$ gives a GF-structure on $T(M)$.*

Example of generalized almost r-contact structure. Here we shall give an example of the generalized almost r-contact structure. Let us consider a manifold X with an almost complex structure $\mathfrak{I}$. Let (M, T, X) be a fibered manifold over X with fibers of dimension r. Let

$U_1, U_2 \ldots, U_r$ be the vector fields on M, tangent to the fibers and linearly independent at every point of M. Let D denote the n-dimensional distribution on M spanned by these vector fields. Moreover, let E be a horizontal distribution on M such that at every point $x \in M$ we have $T_x(M) = D_x \oplus E_x$.

Let us define 1-forms ω_p, $p = 1, 2, 3 \ldots, r$ on M as follows

$$\omega_p \circ E = 0, \quad \omega_p(U_q) = -\frac{a^2}{E}\,\delta_{pq}. \tag{2.6}$$

Let us also define $(1,1)$ tensor field F at every $X \in M$ such that

$$F\,U_p = 0, \quad p = 1, 2, 3 \ldots, r \tag{2.7}$$

$$F\,X = T_*^{-1}\,\Im\,T_*\,X \quad \text{for} \quad X \in E, \tag{2.8}$$

where T_* denotes the differential of T.

Thus the following proposition is trivial:

Proposition 2.1. *Let (M, T, X) be a fibered manifold over an almost complex manifold X with fibers of dimension r. Let $U_1, U_2 \ldots, U_r$ be the vector fields on M, tangent to the fibers and linearly independent at every point. Let E be the horizontal distribution on M. Then the vector fields $U_1, U_2 \ldots, U_r$ together with tensors ω_p and F define on M a foliated generalized almost r-contact structure.*

3. Prolongations of F-structure to the tangent bundle of order "r." In this section we will study prolongations of F-structure to the tangent bundle of order "r."

F-structure. Let F be a nonzero tensor field of type $(1,1)$ and of class $C^{\propto}$ on an n-dimensional manifold M such that [**8**]

$$F^k + (-)^{k+1\,F} = 0 \quad \text{and} \quad F^w + (-)^{w+1}\,F \neq 0 \quad \text{for} \quad 1 < \mathrm{w} < \mathrm{k} \tag{3.1}$$

where k is a fixed positive integer greater than 2. Such a structure on M is called an F-structure of rank 'r' and degree k. If the rank of F is constant and $r = r(F)$, then M is called an F-structure manifold of degree k (≥ 3). The case when k is odd has been considered in this paper.

Let the operators on M be defined as follows [**8**]

$$l = (-)^{k} F^{k-1} \quad \text{and} \quad m = I + (-)^{k+1} F^{k-1} \tag{3.2}$$

where I denotes the identity operator on M.

For the operators defined by (3.2) we have

$$l + m = I \quad \text{and} \quad l^2 = l \quad \text{and} \quad m^2 = m. \tag{3.3}$$

For F satisfying (3.1), there exist complementary distributions L and M corresponding to the projection operators l and m, respectively. If rank (F) =constant on M, then $\dim L = r$ and $\dim M = n - r$. We have the following results [**8**]

$$Fl = lF = F \quad \text{and} \quad Fm = mF = 0 \tag{3.4a}$$

$$F^{k-1} l = -l \quad \text{and} \quad F^{k-1} m = 0. \tag{3.4b}$$

Let M be an n-dimensional manifold of class $C^{\propto}$, and let $T_p(M) = \cup_{p\in M} T_p(M)$ be the tangent bundle over the manifold M. Let us denote $T_s^r(M)$, the set of all tensor fields of class $C^{\propto}$ and of type (r, s) in M, and let $T(M)$ be the tangent bundle over M.

Let us introduce an equivalence relation $\sim$ in the real line. Let $r \geq 1$ be a fixed integer. If two mappings $F : R \to M$ and $G : R \to M$ satisfy the conditions [**11**]

$$F^h(0) = G^h(0), \ \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, \ \ldots, \ \frac{d^r F^h(0)}{dt^r} = \frac{d^r G^h(0)}{dt^r}$$

the mappings F and G being represented respectively by $x^h = F^h(t)$ and $x^h = G^h(t)$, $t \in R$ with respect to local coordinates x^h in a coordinate neighborhood $\{U, x^h\}$ containing the point $P = F(0) = G(0)$, then we say that the mapping F is equivalent to G. Each equivalence class determined by the equivalence relation $\sim$ is called an r-jet of M and denoted by $J_P^r(F)$. The set of all r-jets of M is called the tangent bundle of order r and denoted by $T_r(M)$. Let $\{U, x^h\}$ be a coordinate neighborhood of M. Then an r-jet $J_P^r(F)$, $P \in U$, is expressed in a unique way by the set $\left(x^h, y^h_{(1)}, \dots, y^h_{(r)}\right)$, x^h being the coordinates of P in U and $y^h_{(1)}, y^h_{(2)}, \dots, y^h_{(r)}$ are defined respectively as follows:

$$y^h_{(1)} = \frac{dF^h(0)}{dt},\ y^h_{(2)} = \frac{1}{2!}\frac{d^2F^h(0)}{dt^2}, \dots,\ y^h_{(r)} = \frac{1}{r!}\frac{d^rF^h(0)}{dt^r},$$

where F has the local expression $x^h = F^h(t)$, $t \in R$ and $P = F(0)$. Thus we can naturally introduce a topology in $T_r(M)$ in such a way that the tangent bundle $T_r(M)$ of order r becomes a differentiable manifold.

The λ-lift of a tensor field F of type $(1,1)$ with local components F^h_i in M to $T_r(M)$ has components of the form [**11**]

$$F^{(\lambda)} : \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & v \\ \left(F^h_1\right)^{(0)} & 0 & 0 & \cdots & 0 & 0 \\ \left(F^h_1\right)^{(1)} & \left(F^h_1\right)^{(0)} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(F^h_1\right)^{(\lambda)} & \left(F^h_1\right)^{(\lambda-1)} & \cdots & \left(F^h_1\right)^{(0)} & 0 & 0 \end{bmatrix} \tag{3.5}$$

for $\lambda = 0, 1, \dots, r$ with respect to the induced coordinates $\left(y^A\right)$ in $T_r(M)$.

Some results. Now we obtain the following results on the λ-lift of F satisfying (3.1). For any $F, G \in T^1_1(M)$ the following holds [**11**]

$$\begin{aligned} \left(G^{(\lambda)}F^{(\mu)}\right)x^{(r)} &= G^{(\lambda)}\left(F^{(\mu)}x^{(r)}\right) \\ &= G^{(\lambda)}(Fx)^{(\mu)} \\ &= (G(Fx))^{(\lambda+\mu-r)} \\ &= (GF)^{(\lambda+\mu-r)}x^r. \end{aligned}$$

We find with $\lambda = \mu = r$, the following

$$\left(G^{(r)}F^{(r)}\right)x^{(r)} = (GF)^{(r)}x^{(r)} \quad \text{for every} \quad x \in T^1_0(M). \tag{3.6}$$

If $P(s)$ denotes a polynomial of variable s, then we have [**11**]

$$P\left(F^{(r)}\right) = (P(F))^{(r)}. \tag{3.7}$$

Let F satisfy (3.7); then F defines an F-structure in M satisfying

$$F^k + (-)^{k+1} F = 0$$

which in view of (3.7) yields

$$\left(F^{(r)}\right)^k + (-)^{k+1} F^{(r)} = 0. \tag{3.8}$$

Therefore, $F^{(r)}$ defines an F-structure in $T_r(M)$. The converse can be proved in a similar manner. Consequently we have the following theorem which generalizes Proposition 5.4 of [**11**].

Theorem 3.1. *The r-lift $F^{(r)}$ defines an F-structure in $T_r(M)$ if and only if F defines an F-structure on M.*

Let us denote $N_{F^{(r)}}$ and N_F, the Nijenhuis tensors of $F^{(r)}$ and F respectively.

Then we have

$$N_{F^{(r)}} = (N_F)^{(r)} \quad \text{for} \quad F \in T_1^1(M) \tag{3.9}$$

in $T_r(M)$.

We know that the F-structure is integrable if and only if

$$N(x, y) = 0$$

which, in view of (3.8), is equivalent to

$$N_{F^{(r)}}(x, y) = 0. \tag{3.10}$$

Thus, $F^{(r)}$ is integrable if and only if F is integrable in M. The converse follows in an obvious manner. Hence we have the following theorem.

Theorem 3.2. *The r-lift $F^{(r)}$ is integrable in $T_r(M)$ if and only if F is integrable in M.*

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