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REMARKS ON SPACES OF REAL RATIONAL FUNCTIONS

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ABSTRACT. Let $\mathrm{RRat}_k(\mathbf{C}P^n)$ denote the space of basepoint-preserving conjugation-equivariant holomorphic maps of degree k from S^2 to $\mathbf{C}P^n$. A map $f : S^2 \to \mathbf{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbf{C}P^n$. Let $\mathrm{RF}_k(\mathbf{C}P^n)$ denote the subspace of $\mathrm{RRat}_k(\mathbf{C}P^n)$ consisting of full maps. We first determine $H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p)$ for all primes p. Then we prove that the inclusion $\mathrm{RF}_k(\mathbf{C}P^n) \hookrightarrow \mathrm{RRat}_k(\mathbf{C}P^n)$ and a natural map $\alpha_{k,n} : \mathrm{RF}_k(\mathbf{C}P^n) \to SO(k)/SO(k-n)$ are homotopy equivalences up to dimensions $k-n$ and $n-1$, respectively.

1. Introduction. Let $\mathrm{Rat}_k(\mathbf{C}P^n)$ denote the space of based holomorphic maps of degree k from the Riemannian sphere $S^2 = \mathbf{C} \cup \infty$ to the complex projective space $\mathbf{C}P^n$. The basepoint condition we assume is that $f(\infty) = [1, \dots, 1]$. Such holomorphic maps are given by rational functions:

(1.1)

$$\mathrm{Rat}_k(\mathbf{C}P^n) = \{(p_0(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbf{C} \text{ of degree } k \text{ and such that there are no roots common to all } p_i(z)\}.$$

There is an inclusion $\mathrm{Rat}_k(\mathbf{C}P^n) \hookrightarrow \Omega^2_k \mathbf{C}P^n \simeq \Omega^2 S^{2n+1}$. Segal [**9**] proved that the inclusion is a homotopy equivalence up to dimension $k(2n-1)$. (Throughout this paper, to say that a map $f : X \to Y$ is a homotopy equivalence up to dimension d is intended to mean that f induces isomorphisms in homotopy groups in dimensions less than d, and an epimorphism in dimension d.) Later, the stable homotopy type of $\mathrm{Rat}_k(\mathbf{C}P^n)$ was described in [**3**] as follows. Let

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$\Omega^2 S^{2n+1} \underset{s}{\simeq} \bigvee_{1\leq q} D_q(S^{2n-1})$ be Snaith's stable splitting of $\Omega^2 S^{2n+1}$. Then

$$\mathrm{Rat}_k(\mathbf{C}P^n) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-1}). \tag{1.2}$$

In particular, the induced homomorphism $H_*(\mathrm{Rat}_k(\mathbf{C}P^n);\mathbf{Z}) \to H_*(\Omega^2 S^{2n+1};\mathbf{Z})$ is injective.

A map $f : S^2 \to \mathbf{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbf{C}P^n$. If f is given by a rational function in (1.1), then f is full if and only if the polynomials $p_i(z)$, $0 \leq i \leq n$, are linearly independent in $\mathbf{C}[z]$. Let $\mathrm{F}_k(\mathbf{C}P^n)$ be the subspace of $\mathrm{Rat}_k(\mathbf{C}P^n)$ consisting of full maps. Particular examples are: $\mathrm{F}_k(\mathbf{C}P^n) = \varnothing$ when $k < n$; and $\mathrm{F}_n(\mathbf{C}P^n) \cong \mathbf{C}^n \times GL(n,\mathbf{C})$. The space $\mathrm{F}_k(\mathbf{C}P^n)$ has a certain significance in connection with harmonic maps. In fact, it is known how to construct harmonic maps $S^2 \to \mathbf{C}P^n$ out of full holomorphic maps. Motivated by this, Crawford studied the topology of $\mathrm{F}_k(\mathbf{C}P^n)$ in [**6**]. He proved that the inclusion $\mathrm{F}_k(\mathbf{C}P^n) \hookrightarrow \mathrm{Rat}_k(\mathbf{C}P^n)$ is a homotopy equivalence up to dimension $2(k-n)+1$. Moreover, $H_*(\mathrm{F}_k(\mathbf{C}P^2);\mathbf{Z}/p)$ was determined for all primes p. The result shows that the inclusion $\mathrm{F}_k(\mathbf{C}P^2) \hookrightarrow \mathrm{Rat}_k(\mathbf{C}P^2)$ has a nontrivial kernel in homology in dimensions above the range of stability.

We denote by $\mathrm{RRat}_k(\mathbf{C}P^n)$ the subspace of $\mathrm{Rat}_k(\mathbf{C}P^n)$ of maps which commute with complex conjugation. An element $(p_0(z),\ldots,p_n(z)) \in \mathrm{Rat}_k(\mathbf{C}P^n)$ belongs to $\mathrm{RRat}_k(\mathbf{C}P^n)$ if and only if each $p_i(z)$ has real coefficients. Hence, in particular, $\mathrm{RRat}_1(\mathbf{C}P^n) \cong \mathbf{R} \times (\mathbf{R}^n)^* \simeq S^{n-1}$. Next we set $\mathrm{RF}_k(\mathbf{C}P^n) = \mathrm{RRat}_k(\mathbf{C}P^n) \cap \mathrm{F}_k(\mathbf{C}P^n)$.

The purpose of this paper is to study the topology of $\mathrm{RRat}_k(\mathbf{C}P^n)$ and $\mathrm{RF}_k(\mathbf{C}P^n)$. There are inclusions

$$i_k : \mathrm{RRat}_k(\mathbf{C}P^n) \hookrightarrow \Omega S^n \times \Omega^2 S^{2n+1} \tag{1.3}$$

(compare Lemma 2.1) and

$$j_k : \mathrm{RF}_k(\mathbf{C}P^n) \hookrightarrow \mathrm{RRat}_k(\mathbf{C}P^n). \tag{1.4}$$

Brockett and Segal ([**2, 9**]) showed that

$$\mathrm{RRat}_k(\mathbf{C}P^1) \cong \coprod_{i=0}^{k} \mathbf{C}^{|k-2i|} \times \mathrm{Rat}_{\min(i,k-i)}(\mathbf{C}P^1). \tag{1.5}$$

But the homology of $\mathrm{RRat}_k(\mathbf{C}P^n)$ is not known for $n \geq 2$. On the other hand, about $\mathrm{RF}_k(\mathbf{C}P^n)$, we have the following:

Example 1.6. (i) For $1 \leq k$, $\mathrm{RF}_k(\mathbf{C}P^1) = \mathrm{RRat}_k(\mathbf{C}P^1)$.

(ii) For $k < n$, $\mathrm{RF}_k(\mathbf{C}P^n) = \varnothing$.

(iii) $\mathrm{RF}_n(\mathbf{C}P^n) \cong \mathbf{R}^n \times GL(n, \mathbf{R})$. Hence, $\mathrm{RF}_n(\mathbf{C}P^n) \simeq O(n)$.

In fact, (i) and (ii) are clear. We prove (iii) in Section 3.

Now we state our results. We first determine $H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p)$ for all primes p. Since the topological type of $\mathrm{RRat}_k(\mathbf{C}P^1)$ is known in (1.5), we assume $n \geq 2$. Recall that $H_*(\Omega S^n; \mathbf{Z}/p) \cong \mathbf{Z}/p\,[u_{n-1}]$. As usual, we set $\mathrm{w}(u_{n-1}) = 1$, where w denotes the weight. On the other hand, we define the weight of an element of $H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/p)$ to be twice the usual one. In particular, for the generator $\iota_{2n-1} \in H_{2n-1}(\Omega^2 S^{2n+1}; \mathbf{Z}/p)$, we set $\mathrm{w}(Q_1^d(\iota_{2n-1})) = 2p^d$.

Theorem A. *Let $n \geq 2$. Then, as a vector space, $H_*(\mathrm{RRat}_k(\mathbf{C}P^n);$ $\mathbf{Z}/p)$ is isomorphic to the subspace of $H_*(\Omega S^n \times \Omega^2 S^{2n+1}; \mathbf{Z}/p)$ spanned by monomials of weight $\leq k$.*

Remark. When $n = 1$, let us understand $\Omega S^n \times \Omega^2 S^{2n+1}$ in Theorem A as $\{0, 1, 2, \dots\} \times \Omega^2 S^3$, where $\{0, 1, 2, \dots\}$ is a discrete set with $\mathrm{w}(j) = j$. (Here w denotes the weight.) Then (1.5) implies that Theorem A remains valid for $n = 1$.

Theorem A implies that $i_{k*} : H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}) \to H_*(\Omega S^n \times \Omega^2 S^{2n+1}; \mathbf{Z})$ is injective, as in the inclusion $\mathrm{Rat}_k(\mathbf{C}P^n) \hookrightarrow \Omega^2 S^{2n+1}$. (Compare (1.2).) We have the following analogue of Segal's theorem.

Corollary B. *The inclusion i_k in* (1.3) *satisfies the following properties*:

(i) *For $n \geq 2$, i_k induces isomorphisms in homology groups in dimensions $\leq (k+1)(n-1) - 1$.*

(ii) *For $n \geq 3$, i_k is a homotopy equivalence up to dimension $(k+1)(n-1) - 1$.*

Remark. Recall that the stable homotopy type of $\mathrm{Rat}_k(\mathbf{C}P^n)$ is described in (1.2) in terms of stable summands in $\Omega^2 S^{2n+1}$. Similarly, it is possible to prove a stable homotopy equivalence between $\mathrm{RRat}_k(\mathbf{C}P^n)$ and the collection of stable summands in $\Omega S^n \times \Omega^2 S^{2n+1}$ of weight $\leq k$. In a subsequent paper [**7**], we shall prove this.

The following theorem asserts the stability of the map j_k in (1.4).

Theorem C. *The inclusion j_k is a homotopy equivalence up to dimension $k - n$.*

The following theorem is more useful than Theorem C when $k \leq 2n - 1$.

Theorem D. *Let $SO(k)/SO(k-n)$ be the Stiefel manifold of orthonormal n-frames in $\mathbf{R}^k$. (When $k = n$, we understand this as $O(n)$.) Then there is a map $\alpha_{k,n} : \mathrm{RF}_k(\mathbf{C}P^n) \to SO(k)/SO(k-n)$ so that $\alpha_{k,n}$ is a homotopy equivalence up to dimension $n - 1$.*

In particular, when $k = n + 1$, we have the following:

Corollary E. *We set $SO = \cup_{1 \leq n} SO(n)$ and let $\iota(n+1) : SO(n+1) \hookrightarrow SO$ be the inclusion. Then $\iota(n+1) \circ \alpha_{n+1,n} : \mathrm{RF}_{n+1}(\mathbf{C}P^n) \to SO$ is a homotopy equivalence up to dimension $n - 1$.*

It is possible to determine $H_*(\mathrm{RF}_k(\mathbf{C}P^2); \mathbf{Z}/p)$ by a similar argument to the calculations of $H_*(\mathrm{F}_k(\mathbf{C}P^2); \mathbf{Z}/p)$ in [**6**]. But the results are rather complicated. Hence we omit them.

This paper is organized as follows. In Section 2 we prove Theorem A and Corollary B. Theorem A is proved by considering the spectral sequence of the Vassiliev type. In Section 3 we prove Theorems C, D and Corollary E. The proofs are mostly general position argument.

2. Proofs of Theorem A and Corollary B. Let $\mathrm{Map}_k^T(\mathbf{C}P^1, \mathbf{C}P^n)$ denote the space of continuous basepoint preserving conjugation-equivariant maps of degree k from $\mathbf{C}P^1$ to $\mathbf{C}P^n$. There is an inclusion

$$\mathrm{RRat}_k(\mathbf{C}P^n) \hookrightarrow \mathrm{Map}_k^T(\mathbf{C}P^1, \mathbf{C}P^n).$$

It is easy to prove the following lemma, compare [**7**].

Lemma 2.1. *For $n \geq 1$, there is a homotopy equivalence*

$$\mathrm{Map}_k^T(\mathbf{C}P^1, \mathbf{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}.$$

Here, when $n = 1$, we understand ΩS^n as $\mathbf{Z}$ so that $\mathbf{Z}$ is parametrized by the degree of maps $\mathbf{R}P^1 \to \mathbf{R}P^1$ which are restrictions of elements of $\mathrm{Map}_k^T(\mathbf{C}P^1, \mathbf{C}P^1)$ to the real line. Moreover, under the inclusion $\mathrm{RRat}_k(\mathbf{C}P^1) \hookrightarrow \mathrm{Map}_k^T(\mathbf{C}P^1, \mathbf{C}P^1)$, the connected component indexed by i, $0 \leq i \leq k$, in (1.5) *is mapped to $(k-2i) \times \Omega^2 S^3 \in \mathbf{Z} \times \Omega^2 S^3$.*

Theorem A is proved as follows. First, by constructing homology classes explicitly, we find a lower bound for the mod$p$ homology of $\mathrm{RRat}_k(\mathbf{C}P^n)$. (Compare Proposition 2.2.) Next, considering a geometrical resolution of a resultant, we construct a spectral sequence of the Vassiliev type. The spectral sequence converges to the mod$p$ homology of $\mathrm{RRat}_k(\mathbf{C}P^n)$ and the E^1-term coincides with the lower bound. Hence, the spectral sequence collapses at the E^1-term and the lower bound is actually an upper bound. (Compare Proposition 2.3.)

Proposition 2.2. *Let L_k be the subspace of $H_*(\Omega S^n \times \Omega^2 S^{2n+1}; \mathbf{Z}/p)$ spanned by monomials of weight $\leq k$. Then every element of L_k is in the image of i_{k*}, where i_k is defined in* (1.3). *Hence, these elements are a lower bound for $H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p)$.*

Proof. We recall the structure of $H_*(\Omega S^n \times \Omega^2 S^{2n+1}; \mathbf{Z}/p)$. First,

$$H_*(\Omega S^n; \mathbf{Z}/p) \cong \mathbf{Z}/p\,[u_{n-1}]\,.$$

Next, there is a (torsion free) generator $\iota_{2n-1} \in H_{2n-1}(\Omega^2 S^{2n+1}; \mathbf{Z}/p) \cong \mathbf{Z}/p$, and the following hold. (Compare [**4**].)

(i) For $p = 2$,

$$H_*(\Omega^2 S^{2n+1}; \mathbf{Z}/2) \cong \mathbf{Z}/2[\iota_{2n-1}, Q_1(\iota_{2n-1}), \ldots, Q_1 \cdots Q_1(\iota_{2n-1}), \ldots].$$

(ii) For an odd prime p,

$$H_*(\Omega^2 S^{2n+1};\mathbf{Z}/p) \cong \bigwedge(\iota_{2n-1}, Q_1(\iota_{2n-1}),\dots, Q_1\cdots Q_1(\iota_{2n-1}),\dots) \otimes \mathbf{Z}/p[\beta Q_1(\iota_{2n-1}),\dots,\beta Q_1\cdots Q_1(\iota_{2n-1}),\dots].$$

In (i) and (ii), Q_1 is the first Dyer-Lashof operation (it takes a class of dimension d to a class of dimension $dp+p-1$) and β is the mod$p$ Bockstein operation.

We construct the following three maps:

(1) *The inclusion* $\eta_q : \mathrm{Rat}_q(\mathbf{C}P^n) \hookrightarrow \mathrm{RRat}_{2q}(\mathbf{C}P^n)$,

(2) *Loop sum* $* : \mathrm{RRat}_{k_1}(\mathbf{C}P^n)\times\mathrm{RRat}_{k_2}(\mathbf{C}P^n) \to \mathrm{RRat}_{k_1+k_2}(\mathbf{C}P^n)$,

and

(3) *Stabilization map* $s : \mathrm{RRat}_k(\mathbf{C}P^n) \hookrightarrow \mathrm{RRat}_{k+1}(\mathbf{C}P^n)$.

One can construct the maps (2) and (3) in the same way as in [**1**]. On the other hand, the map (1) is constructed as follows: We fix a homeomorphism $h : \mathbf{C} \overset{\cong}{\to} H_+$, where H_+ denotes the open upper half-plane. For $(p_0(z),\dots,p_n(z)) \in \mathrm{Rat}_q(\mathbf{C}P^n)$, we write $p_j(z) = \prod_{s=1}^q (z-\alpha_{s,j})$. Then we set

$$\eta_q(p_0(z),\dots,p_n(z)) = \left(\prod_{s=1}^{q} z - h(\alpha_{s,0}))(z-\overline{h(\alpha_{s,0})}),\dots,\prod_{s=1}^{q}(z-h(\alpha_{s,n}))(z-\overline{h\alpha_{s,n}})\right).$$

Now let $\alpha \in L_k$. We write $\alpha = u_{n-1}^i \otimes \xi$, where $\xi \in H_*(\Omega^2 S^{2n+1};\mathbf{Z}/p)$. The fact that $\mathrm{RRat}_1(\mathbf{C}P^n) \cong \mathbf{R}\times(\mathbf{R}^n)^* \simeq S^{n-1}$ shows that there is an element $v_{n-1} \in H_{n-1}(\mathrm{RRat}_1(\mathbf{C}P^n);\mathbf{Z}/p)$ so that

$$i_{1*}(v_{n-1}) = u_{n-1}.$$

Let $\overline{\mathrm{w}}$ be the usual weight on $H_*(\Omega^2 S^{2n+1};\mathbf{Z}/p)$. Then, from (1.2), we have $\xi \in H_*(\mathrm{Rat}_{\overline{\mathrm{w}}(\xi)}(\mathbf{C}P^n);\mathbf{Z}/p)$, hence

$$\eta_{\overline{\mathrm{w}}(\xi)*}(\xi) \in H_*(\mathrm{RRat}_{2\overline{\mathrm{w}}(\xi)}(\mathbf{C}P^n);\mathbf{Z}/p),$$

where the inclusion $\eta_{\overline{\mathrm{w}}(\xi)}$ is defined in (1). Using the loop sum in (2), we have

$$v_{n-1}^i * \eta_{\overline{\mathrm{w}}(\xi)*}(\xi) \in H_*(\mathrm{RRat}_{\mathrm{w}(\alpha)}(\mathbf{C}P^n);\mathbf{Z}/p),$$

where w is the weight in Theorem A, i.e., $\mathrm{w}(\alpha) = i + 2\overline{\mathrm{w}}(\xi)$. Since $\mathrm{w}(\alpha) \le k$, using the stabilization map in (3), we can regard this as an element of $H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p)$. This completes the proof of Proposition 2.2. □

Proposition 2.3. *The lower bound of Proposition* 2.2 *is actually an upper bound.*

Proof. We prove the proposition along the lines of [**10**, p. 151]. For a locally compact space X, let $\overline{X}$ denote the one-point compactification of X, $\overline{X} = X \cup \{\infty\}$, and let $\overline{H}_*(X; \mathbf{Z})$ be the Borel-Moore homology group $\overline{H}_*(X; \mathbf{Z}) = \widetilde{H}_*(\overline{X}; \mathbf{Z})$.

We regard $\mathbf{R}^{k(n+1)}$ as the space consisting of all $(n+1)$-tuples $(p_0(z), \dots, p_n(z))$ of monic polynomials over $\mathbf{R}$ of degree k. Let Σ_k^n be the complement of $\mathrm{RRat}_k(\mathbf{C}P^n)$ in $\mathbf{R}^{k(n+1)}$. Thus

$$\Sigma_k^n = \{(p_0(z), \dots, p_n(z)) \in \mathbf{R}^{k(n+1)} : p_0(\alpha) = \cdots = p_n(\alpha) = 0 \quad \text{for some } \alpha \in \mathbf{C}\}.$$

From the Alexander duality, there is a natural isomorphism

$$\widetilde{H}^*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}) \cong \overline{H}_{k(n+1)-1-*}(\Sigma_k^n; \mathbf{Z})$$

and so we study $\overline{H}_*(\Sigma_k^n; \mathbf{Z})$.

Let $I : \mathbf{C} \to \mathbf{C}^k$ be the Veronese embedding $I(z) = (z, z^2, \dots, z^k)$. Let $f = (p_0(z), \dots, p_n(z)) \in \Sigma_k^n$, and suppose that $p_0(z), \dots, p_n(z)$ have at least i distinct common real roots $r_1, \dots, r_i$ and j distinct common roots $\zeta_1, \dots, \zeta_j$ in H_+ (hence $\overline{\zeta}_1, \dots, \overline{\zeta}_j$ are common roots in H_- since polynomials are real). We denote by $\Delta(f, \{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\}) \subset \mathbf{C}^k$ the open simplex in $\mathbf{C}^k$ with vertices

$$\{I(r_1), \dots, I(r_i), I(\zeta_1), \dots, I(\zeta_j)\}.$$

(Note that since $i + 2j \le k$, the points $\{I(r_1), \dots, I(r_i), I(\zeta_1), \dots, I(\zeta_j)\}$ are in general position.) Define a geometrical resolution $\widetilde{\Sigma}_k^n$ of Σ_k^n by

$$\widetilde{\Sigma}_k^n = \bigcup_{f \in \Sigma_k^n; \{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\}} \{f\} \times \Delta(f, \{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\}) \subset \Sigma_k^n \times \mathbf{C}^k.$$

The first projection defines an open proper map $\pi : \widetilde{\Sigma}_k^n \to \Sigma_k^n$, and this induces a map between the one-point compactification spaces $\overline{\pi} : \overline{\widetilde{\Sigma}_k^n} \to \overline{\Sigma_k^n}$. It is known [**10**] that the map $\overline{\pi}$ is a homotopy equivalence. Define subspaces $F_s \subset \overline{\widetilde{\Sigma}_k^n}$ by

$$F_s = \begin{cases} \{\infty\} \cup \bigcup_{f \in \Sigma_k^n; \{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\}, i+2j \leq s} \{f\} \\ \qquad \times \Delta(f, \{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\}) & \text{if } s \geq 1 \\ \{\infty\} & \text{if } s = 0. \end{cases}$$

There is an increasing filtration

$$F_0 = \{\infty\} \subset F_1 \subset F_2 \subset \cdots \subset F_k = \overline{\widetilde{\Sigma}_k^n} \simeq \overline{\Sigma_k^n},$$

and this induces a spectral sequence

$$E^1_{s,t} = \overline{H}_{s+t}(F_s - F_{s-1}; \mathbf{Z}) \Longrightarrow \overline{H}_{s+t}(\widetilde{\Sigma}_k^n; \mathbf{Z}) \cong \overline{H}_{s+t}(\Sigma_k^n; \mathbf{Z}).$$

$F_s - F_{s-1}$ has connected components indexed by nonnegative integers (i, j) with $i + 2j = s$. The connected component indexed by (i, j) is a fibered product of the following two fiber bundles: They have a common base $C_i(\mathbf{R}) \times C_j(H_+) \cong \mathbf{R}^i \times C_j(\mathbf{C})$, where $C_r(X)$ denotes the configuration space of unordered r-tuples of distinct points in X.

(i) The first bundle has the $(i + j - 1)$-dimensional open simplex as a fiber.

(ii) The second bundle is an affine $\mathbf{R}^{(k-s)(n+1)}$-bundle. The fiber over a collection $\{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j\} \in C_i(\mathbf{R}) \times C_j(H_+)$ consists of $(p_0(z), \dots, p_n(z))$ having common roots $\{r_1, \dots, r_i, \zeta_1, \dots, \zeta_j, \overline{\zeta}_1, \dots, \overline{\zeta}_j\}$. By the Thom and Poincaré isomorphisms,

$$E^1_{s,t} = \begin{cases} \bigoplus_{i+2j=s} H^{(k-s)(n+1)+i+j-t-1}(C_j(\mathbf{C}); \pm\mathbf{Z}) & 1 \leq s \leq k \\ 0 & \text{otherwise,} \end{cases}$$

where $\pm\mathbf{Z}$ denotes the local system locally isomorphic to $\mathbf{Z}$ but changes the orientation over the loops defining odd permutations. For $1 \leq s \leq k$, we can rewrite this as

$$\bigoplus_{j=1}^{[s/2]} \widetilde{H}^{k(n+1)-sn-t-1}(D_j(S^1); \mathbf{Z}) \bigoplus \widetilde{H}^{k(n+1)-sn-t-1}(S^0; \mathbf{Z}).$$

Recall that $D_j(S^{2n-1}) \simeq \Sigma^{2j(n-1)} D_j(S^1)$, compare [**5**]. Hence, this is equivalent to

$$\bigoplus_{j=1}^{[s/2]} \widetilde{H}^{k(n+1)-sn-t-1+2j(n-1)}(D_j(S^{2n-1}); \mathbf{Z}) \bigoplus \widetilde{H}^{k(n+1)-sn-t-1}(S^0; \mathbf{Z}).$$

Let $1 \leq *$. From the Alexander duality, we have

$$\begin{aligned} &\dim H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p) \\ &\quad \leq \sum_{s=2}^{k} \sum_{j=1}^{[s/2]} \dim H_*(\Sigma^{(s-2j)(n-1)} D_j(S^{2n-1}); \mathbf{Z}/p) \\ &\quad\quad + \sum_{s=1}^{k} \dim H_*(S^{s(n-1)}; \mathbf{Z}/p). \end{aligned}$$

Identifying $H_*(\Sigma^{(s-2j)(n-1)} D_j(S^{2n-1}); \mathbf{Z}/p)$ with

$$u_{n-1}^{s-2j} \otimes \widetilde{H}_*(D_j(S^{2n-1}); \mathbf{Z}/p)$$

and

$$H_*(S^{s(n-1)}; \mathbf{Z}/p)$$

with u_{n-1}^s, we see that $H_*(\mathrm{RRat}_k(\mathbf{C}P^n); \mathbf{Z}/p)$ is at most as big as L_k. This completes the proof of Proposition 2.3, and, consequently, of Theorem A. □

Proof of Corollary B. Theorem A implies that among elements of $H_*(\Omega S^n \times \Omega^2 S^{2n+1}; \mathbf{Z}/p)$ which are not contained in $\mathrm{Im}\, i_{k*}$, the element of least degree is u_{n-1}^{k+1}. Hence, (i) follows. Since $\mathrm{RRat}_k(\mathbf{C}P^n)$ and $\Omega S^n \times \Omega^2 S^{2n+1}$ are simply connected for $n \geq 3$, (ii) follows from the Whitehead theorem. □

3. Proofs of Theorems C, D and Corollary E. For $(p_0(z), \dots, p_n(z)) \in \mathrm{RRat}_k(\mathbf{C}P^n)$, we set $q_0(z) = p_0(z)$ and $q_i(z) = p_i(z) - p_0(z)$ for $1 \leq i \leq n$. Then $\mathrm{RRat}_k(\mathbf{C}P^n)$ is identified with the space of $(n+1)$-tuples of polynomials $(q_0(z), q_1(z), \dots, q_n(z))$ which satisfy the following conditions (i) and (ii):

(i) Each $q_i(z)$, $0 \le i \le n$, has the form

$$q_0(z) = z^n + a_{0,1}z^{n-1} + \cdots + a_{0,n}$$

and

$$q_i(z) = a_{i,1}z^{n-1} + \cdots + a_{i,n}, \quad 1 \le i \le n,$$

where $a_{i,j} \in \mathbf{R}$.

(ii) There are no roots common to all $q_i(z)$ for $0 \le i \le n$.

Proof of Theorem C. We set $A_{k,n} = \mathrm{RRat}_k(\mathbf{C}P^n) - \mathrm{RF}_k(\mathbf{C}P^n)$. We claim that the codimension of $A_{k,n}$ in $\mathrm{RRat}_k(\mathbf{C}P^n)$ is $k - n + 1$. Here the codimension means as usual the minimum value of $\dim T_f \mathrm{RRat}_k(\mathbf{C}P^n) - \dim T_f A_{k,n}$ for $f \in A_{k,n}$, where T_f denotes the tangent space at the point f. In fact, let $f = (q_0(z), q_1(z), \ldots, q_n(z)) \in A_{k,n}$. Generically we may assume that $q_n(z)$ is a linear combination of $q_1(z), \ldots, q_{n-1}(z)$. Then $\dim T_f A_{k,n} = kn + n - 1$. Hence the codimension is $k - n + 1$. Now Theorem C follows from general position argument. □

Proof of Theorem D. Let $V_n(\mathbf{R}^k)$ be the Stiefel manifold of, not necessarily orthonormal, n-frames in $\mathbf{R}^k$. We consider $V_n(\mathbf{R}^k)$ as an open set of the set of $n \times k$ matrices. We identify $\mathbf{R}^k \times V_n(\mathbf{R}^k)$ with the space of $(n+1)$-tuples of polynomials $(q_0(z), q_1(z), \ldots, q_n(z))$ which satisfy the above condition (i) and the following condition (iii):

(iii) The polynomials $q_1(z), \ldots, q_n(z)$ are linearly independent.

(More precisely, considering the coefficients of polynomials, we regard $q_0(z) \in \mathbf{R}^k$ and $(q_1(z), \ldots, q_n(z))$ as an $n \times k$ matrix.) We set $B_{k,n} = \mathbf{R}^k \times V_n(\mathbf{R}^k) - \mathrm{RF}_k(\mathbf{C}P^n)$. We claim that the codimension of $B_{k,n}$ in $\mathbf{R}^k \times V_n(\mathbf{R}^k)$ is n. In fact, let $f = (q_0(z), q_1(z), \ldots, q_n(z)) \in B_{k,n}$, and let $\xi \in \mathbf{C}$ be a common root of $q_0(z), q_1(z), \ldots, q_n(z)$. If ξ is a root of a real polynomial, then so is $\overline{\xi}$. Since we need to calculate the maximum value of $\dim T_f B_{k,n}$ for $f \in B_{k,n}$, we may assume that $\xi \in \mathbf{R}$. Then $\dim T_f B_{k,n} = (k-1)(n+1) + 1$. Hence, the codimension is n.

Now the general position argument shows that the inclusion

$$\mathrm{RF}_k(\mathbf{C}P^n) \hookrightarrow \mathbf{R}^k \times V_n(\mathbf{R}^k)$$

is a homotopy equivalence up to dimension $n-1$. Let $\alpha_{k,n} : \mathrm{RF}_k(\mathbf{C}P^n) \to SO(k)/SO(k-n)$ be the composition of the inclusion with a homotopy equivalence $V_n(\mathbf{R}^k) \simeq SO(k)/SO(k-n)$. Then $\alpha_{k,n}$ satisfies the assertion of Theorem D. □

Proof of Example 1.6 (iii). In the proof of Theorem D, when $k = n$, the conditions (i) and (iii) imply the condition (ii). Hence $\mathrm{RF}_n(\mathbf{C}P^n) \cong \mathbf{R}^n \times V_n(\mathbf{R}^n)$. □

Proof of Corollary E. Recall that $\iota(n+1) : SO(n+1) \hookrightarrow SO$ is a homotopy equivalence up to dimension n (see, for example, [**8**, Corollary 3.17]). Then the result follows from Theorem D for $k = n+1$. □

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