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DOMAINS OF DIMENSION 1 WITH INFINITELY MANY SINGULAR MAXIMAL IDEALS

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ABSTRACT. Let Λ be a Noetherian domain of dimension 1 with normalization Γ, and let $\mathfrak{m}$ range through the maximal ideals of Λ. We study the set of possible factorizations of $\Gamma\mathfrak{m}$ into products of maximal ideals of Γ, in the situation where infinitely many such $\mathfrak{m}$ can be "singular," that is, $\Lambda_\mathfrak{m} \neq \Gamma_\mathfrak{m}$.

1. Introduction. Let Λ be a Noetherian domain of (Krull) dimension 1 with normalization Γ, necessarily a Dedekind domain. Then, for each maximal ideal $\mathfrak{m}$ of Λ, we have a factorization $\Gamma\mathfrak{m} = \prod_{j=1}^{g} \mathfrak{m}_j^{e_j}$ into a product of powers of distinct maximal ideals $\mathfrak{m}_j$ of Γ. The integer e_j is sometimes called the *ramification degree* of $\mathfrak{m}_j$ (over $\mathfrak{m}$ or over Λ). Let f_j denote the *residue degree* of $\mathfrak{m}_j$ (over $\mathfrak{m}$ or over Λ), that is, the dimension of $\Gamma/\Gamma\mathfrak{m}_j$ considered as a vector space over $\Lambda/\mathfrak{m}$. Then we have the *efg*-relation

$$\sum_{i=1}^{g} e_j f_j = \lambda_\Lambda(\Gamma/\Gamma\mathfrak{m}) \tag{1.0.1}$$

where $\lambda_\Lambda(\dots)$ denotes the composition length of the Λ-module $(\dots)$.

In the classical situation—where Γ is module-finite over Λ—the localization $\Lambda_\mathfrak{m}$ equals $\Gamma_\mathfrak{m}$ for *almost all* $\mathfrak{m}$, i.e., for all but finitely many $\mathfrak{m}$. In other words, the *efg*-relation associated with almost every maximal ideal $\mathfrak{m}$ of Λ is *trivial* ($e_j = f_j = g = 1$).

Suppose now that Γ is not module-finite over Λ. Then infinitely many maximal ideals of Λ can have nontrivial *efg*-relations. An instance of this was given by Hochster [**2**]. Our main result is that there is no restriction whatsoever on the *efg*-relations that can occur in this non-finite situation (Theorem 3.2). Note that each individual *efg*-relation

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(1.0.1) is local data, that is, it is determined by $\Lambda_{\mathfrak{m}}$. Our proof is in three parts. In Section 2 we describe a method of building an integral domain with infinitely many suitably specified localizations, and an easy way of obtaining a local ring that realizes a single *efg*-relation. Then we apply this, proving our main result (Theorem 3.2).

Original motivation. A series of papers by Klingler and Levy [**4**–**6**] contains the description of the category of finitely generated Λ-modules (including its direct-sum relations), where Λ is any reduced Noetherian ring not of wild representation type. They call such rings Λ "Dedekind-like" because they generalize Dedekind domains. These rings form a rather small class of rings of dimension 1. However, Dedekind-like *domains* seem to be the only Noetherian domains whose module category has been described since Steinitz gave his description in the case of (what are now called) Dedekind domains in [**8** (1912)]. The localizations of Dedekind-like rings Λ at maximal ideals always have finite normalization, but Klingler and Levy could not determine whether this is true of Λ itself.

We show that the answer is "not necessarily." In fact, we show that the set of singular maximal ideals of Λ can have arbitrarily large cardinality (Section 4).

Throughout this paper, *ring* means "commutative ring" and *local ring* means "Noetherian local ring".

2. Specifying infinitely many localizations.

Theorem 2.1. *Let K be a field and I a nonempty set. For each $i \in I$, let $(\Lambda_i, \mathfrak{n}(i))$ be a local integral domain of dimension* 1 *whose quotient field $Q(\Lambda_i)$ equals K. Suppose*:

(i) ("Finite character"). *Every nonzero element of K is a unit in almost all Λ_i; and*

(ii) ("Independence"). *For every pair of distinct indices $h \neq j$ there exist elements $x \in \Lambda_h$ and $y \in \Lambda_j$ such that*: (a) *x and y are nonunits in Λ_h and Λ_j respectively*; (b) *x is a unit in Λ_i when $i \neq h$, and y is a unit in Λ_i when $i \neq j$*; (c) *$x + y$ is a unit in every Λ_i.*

Then the ring $\Lambda = \cap_i \Lambda_i$ *is Noetherian of dimension* 1*, its distinct maximal ideals are* $\mathfrak{m}(i) = \mathfrak{n}(i) \cap \Lambda$*, and each* $\Lambda_i = \Lambda_{\mathfrak{m}(i)}$ *(localization at* $\mathfrak{m}(i)$*).*

Proof. We call a multiplicatively closed set of a ring *proper* if it does not contain 0.

Claim 1. For any proper multiplicatively closed subset S of any local domain R of dimension 1, the localization R_S equals $Q(R)$ if S contains a nonunit of R, and equals R otherwise. (This holds because localizations are determined by the prime ideals that survive.)

Claim 2. For any proper multiplicatively closed subset S of Λ we have $\Lambda_S = \cap_i (\Lambda_i)_S$.

To prove the nontrivial inclusion ($\supseteq$), take any nonzero $x \in \cap_i (\Lambda_i)_S$. Then for each i there is an expression $x = \lambda_i/s_i$ ($\lambda_i \in \Lambda_i,\ s_i \in S$). In fact, by the finite character hypothesis, x is a unit in almost every Λ_i and hence we can take each such $s_i = 1$. Let s be the product of the remaining finite number of s_i. Then $sx \in \cap_i \Lambda_i$ and therefore $x \in \Lambda_S$.

Notation. For any subset $J \subseteq I$ let $\Lambda_J = \cap_{i \in J} \Lambda_i$. We adopt the convention that $\Lambda_\varnothing = K$, where $\varnothing$ denotes the empty set.

Claim 3. Let S be a proper multiplicatively closed subset of Λ and set $H = \{h \in I \mid S \text{ contains a nonunit of } \Lambda_h\}$. Then, in the above notation, $\Lambda_S = \Lambda_{(I-H)}$. Moreover $\Lambda_H \neq \Lambda_J$ when $H \neq J$.

The first assertion follows from Claims 2 and 1. In proving the second assertion, we may assume that H contains an index h that is not in J. By independence conditions (a) and (b), there is a nonunit x of Λ_h that becomes a unit in every other Λ_i. Then $x \in \Lambda$, x is a nonunit in Λ_H but x is a unit in Λ_J.

Claim 4. $Q(\Lambda) = K$. Let S be the set of all nonzero elements of Λ, so that $Q(\Lambda) = \Lambda_S$. By independence condition (a), S contains a nonunit of every Λ_i. Therefore, in the notation of Claim 3, we have $\Lambda_S = \Lambda_\varnothing = K$.

Claim 5. Λ_J is not a local ring if J contains more than one element.

Let h, j be distinct elements of J, and pick x, y according to independence conditions (a) and (c). Then x and y are nonunits of Λ whose sum is a unit, hence Λ_J is not local.

Claim 6. Λ has dimension 1, its distinct maximal ideals are the ideals $\mathfrak{m}(i) = \mathfrak{n}(i) \cap \Lambda$, and $\Lambda_{\mathfrak{m}(i)} = \Lambda_i$.

Let $\mathfrak{m}$ be any maximal ideal of Λ. By Claim 3 with $S = \Lambda - \mathfrak{m}$, we have $\Lambda_\mathfrak{m} = \Lambda_J$ for some J, and by Claim 5, J consists of a single element. Say $\Lambda_\mathfrak{m} = \Lambda_j$. Since Λ_j has dimension 1 we conclude that every maximal ideal of Λ has height 1. In particular Λ has dimension 1 as claimed; and the prime ideal $\mathfrak{m}(i)$, which contains $\mathfrak{m}$, must equal $\mathfrak{m}$. Also, since every maximal ideal of Λ has height 1, each prime ideal $\mathfrak{m}(i)$ of Λ is maximal; and hence the set of maximal ideals of Λ coincides with the set of ideals $\mathfrak{m}(i)$.

It remains to prove that $i \neq j$ implies $\mathfrak{m}(i) \neq \mathfrak{m}(j)$. This follows from independence condition (b).

Claim 7. Λ is Noetherian. It suffices to show that, for every nonzero $x \in \Lambda$, the Λ-module $\Lambda/\Lambda x$ has finite length [**3**, Theorem 90]. By Claim 6 the localizations of $\Lambda/\Lambda x$ at the maximal ideals of Λ are the Λ_i-modules $\Lambda_i/\Lambda_i x$. Almost all of these are zero by our finite-character hypothesis. The remaining finite number of them have finite length since every Λ_i is Noetherian of dimension 1. Since every Λ-module is contained in the full direct product of its localizations at maximal ideals, this proves that the composition length of $\Lambda/\Lambda x$ is at most the sum of the composition lengths of its nonzero localizations $\Lambda_i/\Lambda_i x$, and hence is finite. □

The next lemma gives a realization of a single *efg*-relation. Suitable choices of the field k and localizations $\Upsilon_{\mathfrak{p}}$ of the ring Υ provided by this lemma will then provide the local rings Λ_i to use in Theorem 2.1.

Lemma 2.2. *Let k be a field, $p_1, \ldots, p_g$ distinct monic irreducible polynomials in $k[x]$, $\mathfrak{p}_j = k[x]{\cdot}p_j$, and $e_1, \ldots, e_g$ positive integers. Let $p = \prod_j p_j^{e_j}$ and $\Upsilon = k + k[x]{\cdot}p$. Then Υ is Noetherian, $Q(\Upsilon) = k(x)$ and the following properties hold.*

(i) *$\mathfrak{p} = \prod_{j=1}^{g} \mathfrak{p}_j^{e_j}$ is a maximal ideal of Υ (and an ideal of $k[x]$).*

(ii) *Each $\mathfrak{p}_j$ has residue degree $f_j = \deg(p_j)$ over $\mathfrak{p}$.*

(iii) *$k[x]$ is a finitely generated Υ-module and is the normalization of Υ.*

Proof. Redefine Υ to be the pullback of the following "conductor square."

(2.2.1)
$$\begin{array}{ccc} \Upsilon & \xrightarrow{\text{incl}} & \Omega = k[x] \\ \big\downarrow & & \big\downarrow (\ker(\nu) = \Omega p) \\ k & \xrightarrow{\text{diag}} & \overline{\Omega} = \bigoplus_i k[x]/(p_i^{e_i}) \end{array}$$

In more detail, the upper horizontal arrow of this commutative square denotes inclusion, the lower horizontal arrow denotes the "diagonal" map, and the right-hand vertical map ν denotes a surjective ring homomorphism whose kernel is as shown. Let Υ be the pullback of these maps, that is, the inverse image of k in Ω. The left-hand vertical map denotes the restriction of ν to Υ. Thus both vertical maps are surjections. We have $\Upsilon = k + \Omega p$ by (2.2.1).

We claim that Υ is Noetherian. By Eakin's theorem [**1**] and the fact that the ring $k[x]$ is Noetherian, it suffices to show that the Υ-module $k[x]$ is finitely generated; and for this it suffices to show that the Υ-module $k[x]/\Upsilon$ is finitely generated. But by (2.2.1), this module is isomorphic to $\overline{\Omega}/k$ which is a finitely generated Υ-module because it is a finite-dimensional k-vector space.

We have $Q(\Upsilon) = k(x)$ because $x = xp/p \in Q(\Upsilon)$. Since Ω is a finitely generated Υ-module, Ω is integral over Υ; and since Ω is its own normalization, we conclude that Ω is the normalization of Υ.

The ideal $\mathfrak{p} = \Omega p$ of Ω defined in statement (i) equals $\ker(\Upsilon \to k)$ in (2.2.1), and is therefore an ideal of Υ, a maximal ideal since k is a field. Statement (ii) of the lemma is now clear. □

3. Specifying *efg*-relations.

Lemma 3.1. *Let $p \in k[x]$ be a monic irreducible polynomial over an infinite field k, and let Y be a set of indeterminates over $k(x)$. Then p remains irreducible over the field $k(Y)$.*

Proof. Suppose that there is a proper factorization $p = f_1 f_2$ in $k(Y)[x]$. We may suppose that each f is monic and x actually appears in it. Consider the elements of $k(Y)$ that occur as coefficients of the powers of x that occur in f_1 and f_2. They can be written as fractions s/t with $s, t \in k[Y]$. Since k is an infinite field there must be at least one substitution in k for the indeterminates Y such that every t is nonzero. Making that substitution into the equation $p = f_1 f_2$ then yields a proper factorization of p in $k[x]$, a contradiction that proves the lemma. □

A classical reference for Lemma 3.1 is [**9**, Theorem 35, p. 223]. The result also holds in the case where the field k is finite.

The next proof uses the fact that "normalizations localize"; that is, if Λ is a reduced Noetherian ring with normalization Γ, then the normalization of $\Lambda_{\mathfrak{m}}$ is $\Gamma_{\mathfrak{m}}$. This is a consequence, for example, of Serre's characterization of normal rings [**7**, Theorem 23.8].

Theorem 3.2. *Let I be an index set, and for each $i \in I$ let a numerical expression of the following form be given*

$$\sum_{j=1}^{g(i)} e(i)_j f(i)_j \tag{3.2.1}$$

where each e, f, g is a positive integer. Then there is a Noetherian *domain Λ of dimension 1 whose maximal ideals $\mathfrak{m}(i)$ are parameterized by I and satisfy the following efg-properties with respect to the normalization Γ of Λ.*

(i) *For each i there is a factorization $\Gamma\cdot\mathfrak{m}(i) = \prod_{j=1}^{g_j} \mathfrak{m}(i)_j^{e(i)_j}$, where the $\mathfrak{m}(i)_j$ are distinct maximal ideals of Γ.*

(ii) *The residue degree of each $\mathfrak{m}(i)_j$ equals $f(i)_j$.*

Proof. Let F be any infinite field that has irreducible polynomials of all degrees, and hence infinitely many irreducible polynomials of all degrees, e.g., the field of rational numbers. For each index $i \in I$, we want to define a local domain Λ_i whose maximal ideal $\mathfrak{n}(i)$ realizes the given formal *efg*-expression (3.2.1) as the actual *efg*-relation (1.0.1) associated with $\mathfrak{n}(i)$. We do this in such a way that the set of maximal ideals of the ring Λ provided by Theorem 2.1 is parameterized by I and has the specified set of *efg*-relations.

Choose a set of indeterminates $X = \{x_i \mid i \in I\}$ over F, and let the field K in Theorem 2.1 be $F(X)$.

Temporarily fix i. For each j in the formal *efg*-expression (3.2.1) corresponding to this i, choose a distinct monic irreducible polynomial $p(i)_j$ of degree $f(i)_j$ in the ring $F[x_i]$. Let K_i denote the field obtained by adjoining every x_k *other than* x_i to F. By Lemma 3.1 each $p(i)_j$ remains irreducible in $K_i[x_i]$.

Note that $Q(K_i[x_i]) = K$ for all i. We continue with the temporarily fixed i.

Apply Lemma 2.2 with k replaced by the field K_i (over which each $p(i)_j$ is irreducible). Note that $p(i) = \prod_j p(i)_j^{e(i)_j}$ is a polynomial in x_i over the smaller field F. Consider the ring Υ, which we now call $\Upsilon(i)$, furnished by Lemma 2.2. Then the normalization and quotient field of $\Upsilon(i)$ are $K_i[X_i]$ and $Q(K_i[x_i]) = K$, respectively. Also, by this lemma, there is maximal ideal $\mathfrak{p}(i)$ of $\Upsilon(i)$ with a factorization:

$$\mathfrak{p}(i) = \prod_j \mathfrak{p}(i)_j^{e(i)_j} \tag{3.2.2}$$

into a product of powers of distinct maximal ideals $\mathfrak{p}(i)_j$ of $K_i[x_i]$ and with respective residue degrees $f(i)_j$.

Let $\Lambda_i = \Upsilon(i)_{\mathfrak{p}(i)}$, a local domain with maximal ideal $\mathfrak{n}(i) = \mathfrak{p}(i)_{\mathfrak{p}(i)}$, quotient field $Q(K_i[x_i]) = K$, and normalization $K_i[x_i]_{\mathfrak{p}(i)}$. Localizing (3.2.2.) at $\mathfrak{p}(i)$ yields the factorization

$$\mathfrak{n}(i) = \prod_j \mathfrak{n}(i)_j^{e(i)_j} \tag{3.2.3}$$

into a product of powers of distinct maximal ideals of the normalization of Λ_i, again with respective residue degrees $f(i)_j$.

In order to apply Theorem 2.1, we now check its compatibility conditions.

Finite character. This holds because any individual polynomial involves only finitely many indeterminates x_i.

Independence (a) *and* (b). For each i the polynomial $p(i)$ is a polynomial in x_i alone and is therefore a nonunit in Λ_i. It is a unit in every other Λ_j because $p(i) \in K_j \subseteq \Upsilon(j) \subseteq \Lambda_j$.

Independence (c). Choose distinct indices h, j. Then $p(h)$ and $p(j)$ are nonunits in Λ_h and Λ_j, respectively, and units elsewhere, as already noted. Consider $p(h)+p(j)$. This is a unit in the local ring Λ_h because $p(h)$ is a nonunit there and $p(j)$ is a unit there. Similarly, $p(h)+p(j)$ is a unit in Λ_j. And $p(h)+p(j)$ is a unit in every other Λ_i since x_i does not appear in $p(h)+p(j)$.

Theorem 2.1 now yields the Noetherian domain $\Lambda = \cap_i \Lambda_i$ of dimension 1 whose distinct maximal ideals are the ideals $\mathfrak{m}(i) = \mathfrak{n}(i) \cap \Lambda$ and such that each $\Lambda_i = \Lambda_{\mathfrak{m}(i)}$.

The normalization Γ of Λ is a Dedekind domain since Λ has dimension 1, and therefore each ideal $\Gamma \cdot \mathfrak{m}(i)$ of Γ has a unique factorization into a product of powers of distinct maximal ideals of Γ. Since normalizations localize, localizing this factorization of $\Gamma \cdot \mathfrak{m}(i)$ yields the unique factorization of $\mathfrak{n}(i)$ in (3.2.3) and shows that our factorization of $\Gamma \cdot \mathfrak{m}(i)$ is as described in statements (i) and (ii) of the theorem we have just proved. $\square$

Addendum 3.3. For use in the next section, we show that the ring Λ constructed in Theorem 3.2 has the following two additional properties. We make no claim that these properties are consequences of the theorem itself.

(i) For every maximal ideal $\mathfrak{m}$ of Λ the normalization $\Gamma_{\mathfrak{m}}$ of $\Lambda_{\mathfrak{m}}$ is a finitely generated $\Lambda_{\mathfrak{m}}$-module.

(ii) For every maximal ideal $\mathfrak{m}$ of Γ, the ideal $\mathfrak{m}_{\mathfrak{m}}$ of $\Lambda_{\mathfrak{m}}$ is also an ideal of $\Gamma_{\mathfrak{m}}$.

Proof. As observed at the end of the proof of the Theorem, every maximal ideal of Λ has the form $\mathfrak{m}(i) = \mathfrak{n}(i) \cap \Lambda$ where $\mathfrak{n}(i)$ is the maximal ideal of the local ring $\Lambda_i = \Lambda_{\mathfrak{m}(i)}$.

Let $\Upsilon(i)$ be the ring defined above (3.2.2). This ring was obtained from Lemma 2.2, which states that the normalization $k[x]$ of $\Upsilon = \Upsilon(i)$ is a finitely generated Υ-module and $\mathfrak{p} = \mathfrak{p}(i)$ is a maximal ideal of Υ as well as an ideal of $k[x]$.

Equation (3.2.3) was obtained by localizing $\Upsilon(i)$ and (3.2.2) at $\mathfrak{p}(i)$. Therefore the normalization of $\Upsilon(i)_{\mathfrak{p}(i)}$ is a finitely generated $\Upsilon(i)_{\mathfrak{p}(i)}$-module, and the maximal ideal $\mathfrak{n}(i) = \mathfrak{p}(i)_{\mathfrak{p}(i)}$ of $\Upsilon(i)_{\mathfrak{p}(i)}$ is also an ideal of the normalization of $\Upsilon(i)_{\mathfrak{p}(i)}$. Since $\Lambda_i = \Upsilon(i)_{\mathfrak{p}(i)}$ (defined above (3.2.3)), the proof of the addendum is complete. □

4. Dedekind-like domains. Dedekind-like rings are defined in [**6**, Section 10]. For some natural examples of these rings see [**6**, Examples 2.2]. For the purposes of this section, it is more useful to take as our definition the following characterization, given in [**6**, Corollary 10.7].

Lemma 4.1. *A Noetherian ring Λ is Dedekind-like if and only if all localizations at maximal ideals are Dedekind-like.*

We therefore need to define "local Dedekind-like ring." The notation $\mu_\Lambda(M)$ denotes the minimum number of elements needed to generate a Λ-module M. The next definition comes from [**4**, Definition 2.5].

Definition 4.2. Let $(\Lambda, \mathfrak{m}, k)$ be a local ring. We call Λ a *Dedekind-like* ring if Λ is reduced and its normalization Γ has the following properties: Γ is a direct sum of principal ideal domains (necessarily semi-local), $\mathfrak{m} = \operatorname{rad}(\Gamma)$ (the Jacobson radical of Γ) and $\mu_\Lambda(\Gamma) \leq 2$.

We do not consider fields to be principal ideal domains. Therefore Γ *and* Λ *have dimension* 1.

Next we give the connection of local Dedekind-like rings with *efg*-properties.

Lemma 4.3. *Let* $(\Lambda, \mathfrak{m}, k)$ *be a local domain with normalization* Γ*. Then* Λ *is Dedekind-like if and only if* $\mathfrak{m} = \operatorname{rad}(\Gamma) \neq 0$ *and one of the following conditions holds.*

(U) $\mathfrak{m}$ *is the unique maximal ideal of* Γ*, and* $\Gamma/\mathfrak{m}$ *is a two-dimensional field extension of* $k = m\Lambda/\mathfrak{m}$. (*Here we call* Λ *"unsplit."*)

(S) $\mathfrak{m}$ *is the product of two distinct maximal ideals of* Γ*, each with residue field* k. (*Here we call* Λ *"nonstrictly split."*)

(N) $\Lambda = \Gamma$. (*Here* Λ *is a DVR.*)

Proof. Suppose that Λ is Dedekind-like. Since $\mathfrak{m}$ is an ideal of Γ as well as Λ, and $\mu_\Lambda(\Gamma) \leq 2$, the k-vector space $\Gamma/\mathfrak{m}$ must have dimension ≤ 2. In addition, $\Gamma/\mathfrak{m} = \Gamma/\operatorname{rad}(\Gamma)$, a reduced ring. Therefore, the ideal $\mathfrak{m}$ of Γ is the product of *distinct* maximal ideals, at most two such ideals since the k-dimension of $\Gamma/\mathfrak{m}$ is ≤ 2.

If the dimension of $\Gamma/\mathfrak{m}$ equals 2, then the only possibilities are those in conditions (U) and (S). If the dimension equals 1, then (N) holds.

Conversely, suppose that the stated conditions hold. For any nonzero element $x \in \mathfrak{m}$ we have $x\Gamma \subseteq \Lambda$. Since the ring Λ is Noetherian, this shows that Γ is a finitely generated Λ-module. In any of the three situations (U), (S), or (N) the k-dimension of $\Gamma/\mathfrak{m}$ is ≤ 2; and therefore $\mu_\Lambda(\Gamma) \leq 2$ by Nakayama's lemma. Therefore Λ is Dedekind-like. □

There is a fourth type of local Dedekind-like ring, called "strictly split." However such rings are never integral domains, and so do not interest us here. (See [**4**, Definition 2.5].)

We call a maximal ideal $\mathfrak{m}$ of a Dedekind-like domain Λ *unsplit, nonstrictly split or nonsingular* according to whether the local Dedekind-like ring $\Lambda_\mathfrak{m}$ is respectively unsplit, nonstrictly split or a DVR.

We now show that a Dedekind-like domain can have arbitrarily many maximal ideals of each of the three types.

Theorem 4.4. *Let a set I be the union of three disjoint subsets $I = U \cup S \cup N$ (some, but not all of which, can be empty). Then there is a Dedekind-like domain Λ whose maximal ideals $m(i)$ are paramaterized by I and such that each $\mathfrak{m}(i)$ is unsplit, nonstrictly split or nonsingular according as i is an element of U, S, or N respectively.*

Proof. Let Γ be the normalization of Λ. By Theorem 3.2 there is a Noetherian domain Λ of dimension 1 whose maximal ideals $\mathfrak{m}(i)$ are paramaterized by I, and the following *efg*-properties hold.

(4.4.1) (U) If $i \in U$, then $\Gamma{\cdot}\mathfrak{m}(i)$ is a maximal ideal of Γ and has residue degree 2 over Λ; (S) If $i \in S$, then $\Gamma{\cdot}\mathfrak{m}(i)$ is a product of two distinct maximal ideals of Γ, each with residue degree 1 over Λ; (N) If $i \in N$, then $\Gamma{\cdot}\mathfrak{m}(i)$ is a maximal ideal with residue degree 1 over Λ.

Moreover, every $\Gamma_{\mathfrak{m}}$ is the normalization of $\Lambda_{\mathfrak{m}}$ (see the paragraph before Theorem 3.2) and, by Addendum 3.3, the construction can be carried out so that, for every maximal ideal $\mathfrak{m}$ of Λ, $\Gamma_{\mathfrak{m}}$ is a finitely generated $\Lambda_{\mathfrak{m}}$-module and $\mathfrak{m}_{\mathfrak{m}}$ is an ideal of $\Gamma_{\mathfrak{m}}$.

To complete the proof, we show that Λ is Dedekind-like. By Lemma 4.1, we may assume that Λ is a local domain, $(\Lambda, \mathfrak{m}, k)$. Since Λ is a local domain of dimension 1, its normalization Γ is a semi-local principal ideal domain, and every maximal ideal of Γ lies over $\mathfrak{m}$, and therefore must appear in the factorization of $\Gamma\mathfrak{m}$ into a product of maximal ideals of Γ. Therefore, by properties (4.4.1), $\Gamma\mathfrak{m}$ is the product of the one or two maximal ideals of Γ. Also, $\Gamma\mathfrak{m} = \mathfrak{m}$ since $\mathfrak{m}$ is also an ideal of Γ. We conclude that $\mathfrak{m} = \operatorname{rad}(\Gamma) \neq 0$. It follows from Lemma 4.3 that Λ is Dedekind-like and its sets of unsplit, strictly split, and nonsingular maximal ideals are indexed by U, S and N, respectively. □

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