

## INEQUALITIES OF GENERALIZED HYPERBOLIC METRICS

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ABSTRACT. In this paper inequalities between two generalizations of the hyperbolic metric and the  $j_G$  metric are derived. We also prove inequalities between generalized versions of the  $j_G$  metric and Seittenranta's metric.

**1. Introduction.** In contrast to the situation in the complex plane, the well-known Poincaré hyperbolic metric is defined only in balls and half-spaces in  $\mathbf{R}^n$  when  $n \geq 3$ . Many researchers have proposed metrics that could take the place of the hyperbolic metric in analysis in higher dimensions. Probably the most used one is the quasihyperbolic metric introduced by Gehring and Palka in [5]. This metric has the slight disadvantage in that it does not equal the hyperbolic metric in a ball, but rather may be off by a multiplicative constant of 2. For accurate estimates, for instance asymptotically sharp inequalities, this might pose a problem.

Several metrics have also been proposed that are generalizations of the hyperbolic metric in the sense that they equal the hyperbolic metric if the domain of definition is a ball or a half-space. Some examples are the Apollonian metric introduced by Beardon in [2], the Ferrand metric [3], the Kulkani-Pinkall metric [8] and Seittenranta's metric [9].

The generalizations of the hyperbolic metric studied in this paper are based on directly using two simple closed form formulae for the hyperbolic metric in balls. This approach was suggested by Vuorinen in [11] and it yields generalized hyperbolic metrics that have the desirable property that they equal the hyperbolic metric in balls and half-spaces in all dimensions.

In what follows all topological operations are with respect to  $\overline{\mathbf{R}^n}$  (see Section 2; for further reference, e.g., [11]).

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We will always denote by  $G \subsetneq \overline{\mathbf{R}^n}$  a domain, i.e., open and connected set, with at least two boundary points and by  $x$  and  $y$  points in  $G$ ; similarly for  $G'$ ,  $x'$  and  $y'$ .

We now give the precise definitions of the metrics. The generalized hyperbolic metric,

$$(1) \quad \rho_G(x, y) := \sup_{a, b \in \partial G} \cosh^{-1}(1 + |a, x, b, y| |a, y, b, x|/2),$$

where  $|a, x, b, y|$  denotes the cross-ratio, see equation (3), was introduced by Vuorinen in [11, (3.28)], and proven to be a metric in domains with at least two boundary points in [7]. Seittenranta's metric, from [9, Definition 1.1], is defined by

$$\delta_G(x, y) := \sup_{a, b \in \partial G} \log(1 + |a, x, b, y|).$$

We cite some basic desirable properties of the generalized hyperbolic metrics, as this may help motivate studying them.

**Lemma 1** ([9, Theorem 3.1 and Remark 3.2(2)], [11, 3.25, 3.26, 8.38(3)]). *The metrics  $\rho_G$  and  $\delta_G$  have the following properties:*

- (i) *they are Möbius invariant;*
- (ii) *they are monotone in the domain of definition;*
- (iii) *for  $G = B^n$  and  $G = H^n$ , they equal the hyperbolic metric;*
- (iv)  *$\rho_G(x, y) \geq \cosh((q(\partial G)q(x, y))^2) - 1$  and  $\delta_G(x, y) \geq \exp((q(\partial G) \times q(x, y)) - 1)$ , where  $q$  denotes the chordal metric.*

The well-known  $j_G$  metric, which is a modification from [10] of a metric from [4], is defined for  $G \subsetneq \mathbf{R}^n$  by

$$j_G(x, y) := \log \left( 1 + \frac{|x - y|}{\min\{d(x), d(y)\}} \right),$$

where  $d(x) := d(x, \partial G)$ . The first main results are the following inequalities relating these metrics.

**Theorem 1.** *Let  $G$  be a domain with  $\text{card } \partial G \geq 2$ . Then*

- (i)  $\delta_G \leq \rho_G \leq \cosh^{-1}(3)/\log(3) \delta_G$ .

Assume additionally that  $G \subsetneq \mathbf{R}^n$ . Then

$$(ii) \ j_G \leq \rho_G \leq \cosh^{-1}(3)/\log(2) \ j_G.$$

Both inequalities in (i) and the first inequality in (ii) are sharp.

*Remark 1.* Note that the term *sharp* means that the constant in an inequality cannot be improved, i.e., there exist a domain  $G$  and points  $x_i, y_i \in G, i = 1, 2, \dots$ , such that

$$\lim_{i \rightarrow \infty} d_1(x_i, y_i)/d_2(x_i, y_i) = c,$$

for the inequality  $d_1 \leq cd_2$ .

It was shown in [6, Corollary 6.1] that  $\delta_G$  can be embedded in the following family of metrics,  $0 < p < \infty$ :

$$\begin{aligned} \delta_G^p(x, y) &:= \sup_{a, b \in \partial G} \log\{1 + (|x, a, y, b|^p + |x, b, y, a|^p)^{1/p}\}, \\ \delta_G^\infty(x, y) &:= \lim_{p \rightarrow \infty} \delta_G^p(x, y). \end{aligned}$$

With this notation  $\delta_G^\infty = \delta_G$ , Seittenranta's metric. It likewise follows directly from [6, Lemma 6.1 and Remark 6.1] that for  $G \subsetneq \mathbf{R}^n$ ,  $j_G$  can be embedded in the family

$$\begin{aligned} j_G^p(x, y) &:= \sup_{a \in \partial G} \log \left( 1 + \left( \frac{|x - y|^p}{|x - a|^p} + \frac{|x - y|^p}{|y - a|^p} \right)^{1/p} \right), \\ j_G^\infty(x, y) &:= \lim_{p \rightarrow \infty} j_G^p(x, y), \end{aligned}$$

where  $0 < p < \infty$ . Here  $j_G^\infty = j_G$ , the previously defined  $j_G$  metric.

In this paper we prove the following inequalities of the generalized  $j_G$  and  $\delta_G$  metrics. Note that inequality (iii) is a generalization of [9, Theorem 3.4].

**Theorem 2.** *Let  $G$  be a domain with at least two boundary points and let  $0 < q \leq p \leq \infty$ .*

$$(i) \ \text{Then } \delta_G^p \leq \delta_G^q \leq 2^{1/q-1/p} \delta_G^p.$$

(ii) If additionally  $G \subsetneq \mathbf{R}^n$ , then  $j_G^p \leq j_G^q \leq 2^{1/q-1/p} j_G^p$ .

(iii) If  $p \in [1, \infty]$  and  $G \subsetneq \mathbf{R}^n$ , then  $j_G^p \leq \delta_G^p \leq 2j_G^p$ .

All the inequalities are sharp.

The structure of the rest of this paper is as follows. Section 2 describes the notation used in this paper, which conforms to that in [11]. The two main theorems are proved in Sections 3 and 4, respectively.

**2. Notation.** We denote by  $\{e_1, e_2, \dots, e_n\}$  the standard basis of  $\mathbf{R}^n$  and by  $n$  the dimension of the Euclidean space under consideration and assume that  $n \geq 2$ . For  $x \in \mathbf{R}^n$  we denote by  $x_i$  the  $i$ th coordinate of  $x$ . The following notation will be used for balls and the upper half-space:

$$B^n(x, r) := \{y \in \mathbf{R}^n : |x - y| < r\} \quad \text{and} \quad H^n := \{x \in \mathbf{R}^n : x_n > 0\}.$$

We will use the notation  $\overline{\mathbf{R}^n} := \mathbf{R}^n \cup \{\infty\}$  for the one point compactification of  $\mathbf{R}^n$ . By  $\partial G$  we will denote the boundary and by  $G^c$  the complement of  $G$  in  $\overline{\mathbf{R}^n}$ . We define the chordal metric  $q$  in  $\overline{\mathbf{R}^n}$  by means of the canonical projection onto the Riemann sphere,  $S^{n-1}(e_n/2, 1/2)$ , hence

$$(2) \quad q(x, y) := \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad q(x, \infty) := \frac{1}{\sqrt{1 + |x|^2}}.$$

We will consider  $\overline{\mathbf{R}^n}$  as the metric space  $(\overline{\mathbf{R}^n}, q)$ , hence its balls are the (open) balls of  $\mathbf{R}^n$  and complements of closed balls of  $\mathbf{R}^n$  as well as half-spaces. The cross-ratio  $|a, b, c, d|$  is defined by

$$(3) \quad |a, b, c, d| := \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)} \left( = \frac{|a - c| |b - d|}{|a - b| |c - d|} \right),$$

for  $a, b, c, d \in \overline{\mathbf{R}^n}$ ,  $a \neq b$  and  $c \neq d$ , where the second expression is valid if  $a, b, c, d \in \mathbf{R}^n$ . A mapping  $f: \overline{\mathbf{R}^n} \rightarrow \overline{\mathbf{R}^n}$  is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple  $a, b, c, d \in \overline{\mathbf{R}^n}$  with  $a \neq b$  and  $c \neq d$  ([1, p. 32]).

**3. The proof of Theorem 1.**

*Proof of Theorem 1(i).* We start by proving the first inequality,  $\delta_G \leq \rho_G$ . Fix the points  $x, y \in G$ . Let  $a, b \in \partial G$  be such that  $\delta_G(x, y) = \log(1 + |a, x, b, y|)$ . The points  $a$  and  $b$  can be chosen, since  $\partial G$  is a compact set in  $\mathbf{R}^n$ . Then it suffices to prove the first inequality in

$$(4) \quad \log(1 + |a, x, b, y|) \leq \cosh^{-1} (1 + |a, x, b, y| |a, y, b, x|/2) \leq \rho_G(x, y),$$

since the second follows directly from the definition of  $\rho_G$ . Moreover, since the cross-ratio is Möbius invariant we may assume that  $a = \infty$  and  $b = 0$ . Denote  $s := |x - y|/\sqrt{|x||y|}$  and  $k := \sqrt{|x|/|y|}$  and assume by symmetry that  $|x| \geq |y|$ . Then the first inequality in (4) becomes

$$\log(1 + ks) \leq \log \left( 1 + s^2/2 + \sqrt{s^4/4 + s^2} \right)$$

which reduces to  $k - s/2 \leq \sqrt{s^2/4 + 1}$ . Squaring this, we see that the inequality holds, since  $s \geq k - 1/k$  by the definitions of  $k$  and  $s$  using the Euclidean triangle inequality. We see that there is equality for  $G = \mathbf{R}^n \setminus \{0\}$ ,  $x = e_1$  and  $y = re_1$ ,  $r > 0$ .

In proving the second inequality it again suffices to assume  $a = \infty$  and  $b = 0$ . Let  $s$  and  $k$  be as before and set  $c := \cosh^{-1} (3)/\log(3)$ . The second inequality is equivalent to

$$(5) \quad c \log(1 + ks) - \log(1 + s^2/2 + \sqrt{s^4/4 + s^2}) \geq 0.$$

The partial derivative with respect to  $s$  of the left-hand side of the above inequality equals

$$\frac{c}{s + 1/k} - \frac{1}{\sqrt{s^2/4 + 1}} = \frac{1}{s + 1/k} \left( c - \frac{s + 1/k}{\sqrt{s^2/4 + 1}} \right).$$

Since the term in the parenthesis is decreasing in  $s$ , the derivative has at most one zero, which is a maximum. Therefore we need only check that (5) holds at the end points,  $s = 0$  and  $s = k + 1/k$ , which correspond to  $|x - y| = 0$  and  $|x - y| = |x| + |y|$ , respectively. For  $s = 0$  the inequality

(5) obviously holds. In the case  $s = k + 1/k$ , since  $k = (s + \sqrt{s^2 - 4})/2$ , we need to show that

$$c \log(1 + s^2/2 + \sqrt{s^4/4 - s^2}) \geq \log(1 + s^2/2 + \sqrt{s^4/4 + s^2}).$$

Clearly equality holds for  $s = 2$ . The claim then follows when we show that the left-hand side has greater derivative than the right-hand side for  $s \geq 2$ . Let us change the variable,  $t = s^2$ , and differentiate with respect to  $t$ :

$$c \frac{t^2 + t\sqrt{t^2 - 4} - 1}{\sqrt{t^2 - 4}(2t^2 + 1)} \geq \frac{1}{\sqrt{t^2 + 4}}.$$

Since  $c \geq 1$ , we may drop it. Multiplying by  $\sqrt{t^2 - 4}(2t^2 + 1)\sqrt{t^2 + 4}$  and squaring gives, after rearranging and dividing by 2,

$$t(t^2 - 1)(t^2 + 4)\sqrt{t^2 - 4} \geq t^6 - 8t^4 + 4t^2 - 4.$$

To see that this holds, observe the following chain of inequalities (note that  $t \geq 4$ ):

$$t(t^2 - 1)(t^2 + 4)\sqrt{t^2 - 4} \geq t^5\sqrt{t^2 - 4} \geq t^6 - 7t^4 \geq t^6 - 8t^4 + 4t^2 - 4.$$

For the sharpness of this inequality we choose  $G = \mathbf{R}^n \setminus \{0\}$ ,  $x = 1$  and  $y = -1$ . Then there is equality in the inequality, and hence the constant cannot be improved.  $\square$

*Proof of Theorem 1(ii).* The first inequality follows from Theorem 1(i) ( $\delta_G \leq \rho_G$ ) and [9, Theorem 3.4] ( $j_G \leq \delta_G$ ). Its sharpness follows by letting  $G = H^n$ ,  $x = se_n$  and  $y = re_n$ , see [9, Remark 3.5].

We turn to the second inequality. The metric  $j_G$  as it is normally defined is not Möbius invariant, and indeed  $\infty$  is a special point in the sense that it may not belong to the domain  $G$  in which the metric is defined. We may, however, think of  $j_G$  as the member  $j_{G,\infty}$  of the following family:

$$(6) \quad j_{G,b}(x, y) := \sup_{a \in \partial G} \log(1 + \max\{|x, a, y, b|, |x, b, y, a|\}),$$

where  $G \subsetneq \overline{\mathbf{R}^n}$  is a domain not containing  $b$  with at least two boundary points. Since  $j_{G,b}$  is defined in terms of cross ratios, it is clear that it is

Möbius invariant. Hence we may apply an auxiliary Möbius transform to both sides of the inequality  $\rho_G \leq (\cosh^{-1} 3/\log 2)j_{G,\infty}$ , as long as we keep track of where  $\infty$  is mapped and use the appropriate  $j_{G,b}$ .

As before we may then assume that the boundary points  $a$  and  $b$  occurring in the definition of  $\rho_G$  equal 0 and  $\infty$ . We need to prove

$$\begin{aligned} (7) \quad & \cosh^{-1} \left( 1 + \frac{|x' - y'|^2}{2|x'||y|} \right) \\ & \leq \frac{\cosh^{-1} 3}{\log 2} \sup_{a \in \partial G'} \log (1 + \max\{|x', a, y', b|, |x', b, y', a|\}) \\ & = \cosh^{-1} 3 \log 2 \sup_{a \in \partial G'} \log \left( 1 + \frac{|x' - y'| |b - a|}{\min\{|x' - b| |y' - a|, |x' - a| |y' - b|\}} \right), \end{aligned}$$

where the supremum is over the boundary point  $a$  only;  $b$  is some fixed point in the complement of  $G$ . If  $b = 0$  or  $b = \infty$ , we may proceed exactly as in the proof of Theorem 1(i) and arrive at the better constant  $\cosh^{-1} (3)/\log(3)$ . Assume then that  $b \notin \{0, \infty\}$ . We may assume without loss of generality that  $b = e_1$  by scaling and rotating. Since both sides are Möbius invariant, we may assume that  $|x'| |y'| \leq 1$  by performing an inversion in the unit sphere, since this leaves  $b$  fixed. We then forget about the original  $x$  and  $y$  and denote  $x'$  by  $x$  and  $y'$  by  $y$ , to simplify the notation.

We may restrict the supremum from  $\partial G$  to its subset  $\{0, \infty\}$  on the right-hand side of (7), since this only makes the supremum smaller. Moreover, we can move the supremum to the denominator of the fraction inside the logarithm, changing it to minimum, since it is taken over finitely many terms. In other words

$$\begin{aligned} & \log \left( 1 + \frac{|x - y|}{\min\{|x - e_1| |y|, |x| |y - e_1|, |y - e_1|, |x - e_1|\}} \right) \\ & \leq \sup_{a \in \partial G} \log \left( 1 + \frac{|x - y| |a - e_1|}{\min\{|x - e_1| |y - a|, |x - a| |y - e_1|\}} \right). \end{aligned}$$

Let us estimate  $|y - e_1| \leq |y| + 1$  and  $|x - e_1| \leq |x| + 1$ . Then

$$\begin{aligned} & \min\{|x - e_1| |y|, |x| |y - e_1|, |y - e_1|, |x - e_1|\} \\ & \leq \min\{\min\{1, |y|\}(1 + |x|), \min\{1, |x|\}(1 + |y|)\} \\ & = \min\{|x|, |y|\} + \min\{1, |x| |y|\}. \end{aligned}$$

Recall that we assumed that  $|x||y| \leq 1$ . By symmetry, we may assume that  $|x| \leq |y|$ . Then we need to prove

$$(8) \quad \cosh^{-1} \left( 1 + \frac{|x-y|^2}{2|x||y|} \right) \leq \frac{\cosh^{-1} 3}{\log 2} \log \left( 1 + \frac{|x-y|}{|x|+|x||y|} \right).$$

Denote  $S_c := \{z \in \mathbf{R}^n : |z-y| = c|z|\}$ . For fixed  $y$  and  $c > 0$  consider how the inequality (8) varies as  $x$  varies over  $S_c$ :

$$(9) \quad \cosh^{-1} \left( 1 + \frac{c|x-y|}{2|y|} \right) \leq \frac{\cosh^{-1} 3}{\log 2} \log \left( 1 + \frac{c}{1+|y|} \right).$$

We see that the right-hand side does not depend on  $|x-y|$ , which means that it suffices to consider points  $x \in S_c$  which maximize this distance, since this yields the hardest inequality.

Observe that for all  $c > 0$  the sphere  $S_c$  intersects the segment  $[0, y]$  and  $S_c$  encloses  $y$  if and only if  $c \in (0, 1)$  and 0 if and only if  $c > 1$ . Note also that  $S_c$  is a subset of  $B^n(0, |y|)$  if and only if  $c > 2$ . Since we need only consider points  $x$  that satisfy  $|x| \leq |y|$ , we see that for  $c \in (0, 2]$ , the distance  $|x-y|$  is maximized by some  $x$  satisfying  $|x| = |y|$ . If  $c > 2$ , then  $|x-y|$  is maximized by the choice  $x = -y/(c-1)$ .

Let  $\lambda := \sqrt{|x||y|} \leq 1$ . If  $\lambda < 1$ , then we can consider the points  $x' := x/\lambda$  and  $y' := y/\lambda$ . The left-hand side of (8) is the same for the points  $x$  and  $y$  as for the points  $x'$  and  $y'$ , however the right-hand side is smaller for the latter points. Hence we see that it suffices to prove (8) for points  $x$  and  $y$  with  $|x||y| = 1$ .

Combining the conclusions of the previous two paragraphs, we see that if  $c \leq 2$  we need to consider only the case  $|x| = |y| = 1$ , i.e.,

$$\cosh^{-1} (1 + s^2/2) \leq \frac{\cosh^{-1} 3}{\log 2} \log(1 + s/2),$$

where we have denoted  $s := |x-y|$ . For  $s = 0$  there is equality in the inequality, and since the left-hand side has lesser derivative than the right-hand side the inequality holds for larger  $s$ , also.

In the case  $c < 2$  we need to consider points  $x$  and  $y$  with  $|x||y| = 1$  such that  $x, 0$  and  $y$  lie on some line in this order. Hence we need to show that

$$\cosh^{-1} \left( 1 + \frac{(|x|+|y|)^2}{2|x||y|} \right) \leq \frac{\cosh^{-1} 3}{\log 2} \log \left( 1 + \frac{|x|+|y|}{|x|+|x||y|} \right).$$

Let us write  $t := |y| = 1/|x| \geq 1$ . The previous inequality becomes

$$\cosh^{-1} \left( 1 + (t+1/t)^2/2 \right) \leq \frac{\cosh^{-1} 3}{\log 2} \log \left( 1 + (t^2+1)/(t+1) \right).$$

For  $t = 1$  there is clearly equality in this inequality. We show that the right-hand side has larger derivative than the left-hand side for all  $t > 1$ , which is equivalent to

$$\frac{2}{t} \frac{t^2 - 1}{\sqrt{t^4 + 6t^2 + 1}} \leq \frac{\cosh^{-1} 3}{\log 2} \frac{t^2 + 2t - 1}{(t+1)(t^2 + t + 2)}.$$

We use the estimate  $\cosh^{-1}(3)/(2 \log 2) \geq 5/4$  and multiply both sides by the denominators:

$$4(t^2 - 1)(t+1)(t^2 + t + 2) \leq 5(t^2 + 2t - 1)t\sqrt{t^4 + 6t^2 + 1}.$$

We then use the estimates  $t^2 + 2t - 1 \geq t(t+1)$  and  $\sqrt{t^4 + 6t^2 + 1} \geq t^2 + 1$  and cancel the term  $t+1$  from both sides:

$$4(t^4 + t^3 + t^2 - t - 2) \leq 5(t^4 + t^2).$$

With the substitution  $u := t+1$  this is equivalent to  $u^4 - 5u^2 - 2u + 10 \geq 0$ . Since  $2u \leq u^2 + 1$  we have  $u^4 - 5u^2 - 2u + 10 \geq u^4 - 6u^2 + 9 \geq (u^2 - 3)^2 \geq 0$ .  $\square$

#### 4. The proof of Theorem 2.

*Proof of Theorem 2(i) and (ii).* In this proof we use the convention that  $1/p = 0$  and  $(x^p + y^p)^{1/p} = \max\{x, y\}$  if  $p = \infty$ .

It suffices to prove each of the claims for some fixed boundary point(s), since we may choose it (them) to correspond to the point(s) where the supremum is attained in the quantity whose upper bound we want to establish. Hence it suffices to prove the real-number inequality

$$\log(1+(x^p+y^p)^{1/p}) \leq \log(1+(x^q+y^q)^{1/q}) \leq 2^{1/p-1/q} \log(1+(x^p+y^p)^{1/p})$$

in order to prove both of the claims. Since  $(x^p + y^p)^{1/p} \leq (x^q + y^q)^{1/q}$  the first inequality is clear. Let us denote  $s := 2^{1/q-1/p} \geq 1$ . Then

$\log(1 + xs) \leq s \log(1 + x)$  for  $x \geq 0$  by the Bernoulli inequality. Hence it suffices to prove the first inequality in

$$\log(1 + (x^q + y^q)^{1/q}) \leq \log(1 + s(x^p + y^p)^{1/p}) \leq s \log(1 + (x^p + y^p)^{1/p}).$$

However, this is immediately clear, since  $(x^q + y^q)^{1/q} \leq s(x^p + y^p)^{1/p}$  by the power-mean inequality.

We still need to show that the inequalities are sharp: Let  $G := \mathbf{R}^n \setminus \{0\}$ . Then

$$\delta_G^p(x, y) = j_G^p(x, y) = \log \left( 1 + \left( \frac{|x - y|^p}{|x|^p} + \frac{|x - y|^p}{|y|^p} \right)^{1/p} \right).$$

Fix  $y$  and let  $x \rightarrow \infty$ . Then

$$\lim_{x \rightarrow \infty} \frac{j_G^p(x, y)}{\log |x|} \longrightarrow \log 2$$

irrespective of the value of  $p$ , which shows that the first inequalities are sharp. If  $|x| = |y|$ , then

$$j_G^p(x, y) = \log(1 + 2^{1/p}|x - y|/|x|).$$

As  $y \rightarrow x$  we see that the second inequalities are also sharp.  $\square$

*Proof of Theorem 2(iii).* It is easy to see that the supremum in the definition of  $\delta_G$  can be taken over the complement of  $G$  instead of over the boundary without affecting the value of  $\delta_G$ . Since  $\infty$  is either a boundary or exterior point of  $G$ , it is clear that  $\delta_G^p \geq j_G^p$ , as taking  $b = \infty$  in the expression of  $\delta_G$  gives the expression for  $j_G$ . In the domain  $\mathbf{R}^n \setminus \{0\}$  we have  $\delta_G^p(x, y) = j_G^p(x, y)$  for every pair of points  $x, y \in G$ , hence the inequality is sharp. It remains to consider the second inequality.

Fix  $x$  and  $y$  in  $G$  and the boundary points  $a$  and  $b$  for which the supremum is attained. We may assume without loss of generality that  $|x - y| = 1$ . Then

$$\begin{aligned} \delta_G^p(x, y) &= \log(1 + (|x, a, y, b|^p + |x, b, y, a|^p)^{1/p}) \\ &\leq \log(1 + ((s + t + st)^p + (u + v + uv)^p)^{1/p}), \end{aligned}$$

where we have denoted

$$s := \frac{1}{|x-a|}, \quad t := \frac{1}{|y-b|}, \quad u := \frac{1}{|x-b|}, \quad v := \frac{1}{|y-a|},$$

and used the estimates

$$|a-b| \leq |a-x| + |x-y| + |y-b| \quad \text{and} \quad |a-b| \leq |a-y| + |y-x| + |x-b|$$

in  $|x, a, y, b|$  and  $|x, b, y, a|$ , respectively. Now

$$\begin{aligned} j_G^p(x, y) &\geq \sup_{w \in \{a, b\}} \log(1 + (|x-w|^{-p} + |y-w|^{-p})^{1/p}) \\ &= \log(1 + \max\{s^p + v^p, t^p + u^p\}^{1/p}). \end{aligned}$$

By symmetry we may assume that the  $t^p + u^p \leq s^p + v^p$ . If we apply the exponential function to both sides of the second inequality in

$$\begin{aligned} \delta_G^p(x, y) &\leq \log(1 + ((s+t+st)^p + (u+v+uv)^p)^{1/p}) \\ &\leq 2 \log(1 + \max\{s^p + v^p, t^p + u^p\}^{1/p}) \leq 2j_G^p(x, y), \end{aligned}$$

we see that it suffices to show that

$$(10) \quad 1 + ((s+t+st)^p + (u+v+uv)^p)^{1/p} \leq (1 + (s^p + v^p)^{1/p})^2.$$

We see that the left-hand side can be increased by increasing  $t$  while keeping the right-hand side constant if  $t^p + u^p < s^p + v^p$ . Hence we may assume that  $t^p + u^p = s^p + v^p =: \alpha^p$ .

We will show that (10) holds for every quadruple  $s, t, u, v \in \mathbf{R}^+$  with  $t^p + u^p = s^p + v^p$  for  $p \geq 1$ . For fixed  $s, t, u$  and  $v$  let us consider how the inequality varies under the transformation  $x \mapsto wx, y \mapsto wy, u \mapsto wu$  and  $v \mapsto wv$ . Then the inequality (10) becomes, after we divide it by the common factor  $w$ ,

$$f(w) := 2\alpha + w\alpha^2 - ((s+t+stw)^p + (u+v+uvw)^p)^{1/p} \geq 0.$$

We will show that  $f$  increases in  $w$ . The derivative  $f'(w)$  equals

$$\begin{aligned} \alpha^2 - \{(s+t+stw)^p + (u+v+uvw)^p\}^{1/p-1} \{(s+t+stw)^{p-1}st \\ + (u+v+uvw)^{p-1}uw\} \\ = \alpha^2 - \{(1+\zeta^p)^{1/p-1}st + (1+\zeta^{-p})^{1/p-1}uw\} =: \alpha^2 - g(\zeta), \end{aligned}$$

where  $\zeta := (u + v + uvw)/(s + t + stw)$ . We will now consider how

$$g(\zeta) = (1 + \zeta^p)^{1/p-1} st + (1 + \zeta^{-p})^{1/p-1} uv$$

varies with  $\zeta$ . The derivative  $g'(\zeta)$  equals

$$\begin{aligned} & -(p-1)((1 + \zeta^p)^{1/p-2} \zeta^{p-1} st - (1 + \zeta^{-p})^{1/p-2} \zeta^{-p-1} uv) \\ & = -(p-1)(1 + \zeta^p)^{1/p-2} \zeta^{p-2} (st\zeta - uv). \end{aligned}$$

We see that  $g$  has a maximum at  $\zeta = uv/(st)$  for  $p > 1$ . Hence

$$\begin{aligned} \frac{df}{dw} & \geq \alpha^2 - g\left(\frac{uv}{st}\right) \\ & = \alpha^2 - \left( \left( \frac{(st)^p + (uv)^p}{(st)^p} \right)^{1/p-1} st + \left( \frac{(st)^p + (uv)^p}{(uv)^p} \right)^{1/p-1} uv \right) \\ & = \alpha^2 - ((st)^p + (uv)^p)^{1/p} \\ & = (t^p + u^p)^{1/p} (s^p + v^p)^{1/p} - ((st)^p + (uv)^p)^{1/p} \geq 0. \end{aligned}$$

Now since  $f$  is increasing in  $w$ , it suffices to show that  $f(0) \geq 0$  in order to obtain  $f(w) \geq 0$ , which is equivalent with (10). In other words we have to show that

$$2(s^p + v^p)^{1/p} - ((s+t)^p + (u+v)^p)^{1/p} \geq 0.$$

Recall that  $t^p + u^p = s^p + v^p =: \alpha^{1/p}$  and denote additionally  $\beta := s+t$ . The previous inequality becomes

$$2\alpha - \left\{ \beta^p + [(\alpha^p - s^p)^{1/p} + (\alpha^p - (\beta - s)^p)^{1/p}]^p \right\}^{1/p}.$$

For fixed  $\alpha$  and  $\beta$ ,  $(\alpha^p - s^p)^{1/p} + (\alpha^p - (\beta - s)^p)^{1/p} \leq 2(\alpha^p - (\beta/2)^p)^{1/p}$  and so it suffices to show that  $2\alpha - (\beta^p + 2^p(\alpha^p - (\beta/2)^p))^{1/p} \geq 0$ , which is obvious.

We still have to show that the inequality is sharp. Consider then the domain  $G = \mathbf{R}^n \setminus \{-e_1, e_1\}$  and the points  $\varepsilon e_2$  and  $-\varepsilon e_2$ . We have

$$\delta_G^p(\varepsilon e_2, -\varepsilon e_2) = \log \left( 1 + \frac{2^{1/p+2}\varepsilon}{\sqrt{1 + \varepsilon^2}} \right)$$

and

$$j_G^p(\varepsilon e_2, -\varepsilon e_2) = \log \left( 1 + \frac{2^{1/p+1} \varepsilon}{\sqrt{1 + \varepsilon^2}} \right).$$

It is then clear that

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_G^p(\varepsilon e_2, -\varepsilon e_2)}{j_G^p(\varepsilon e_2, -\varepsilon e_2)} = 2. \quad \square$$

*Remark 2.* It is not immediately clear whether the inequality from Theorem 2(iii) holds for  $0 < p < 1$  as well. It is clear that (10) does not hold in this case for arbitrary  $s, t, u, v \in \mathbf{R}^+$ , however, these variables are not really arbitrary but rather related by various triangle inequalities.

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