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ON IMPULSIVE TIME-VARYING SYSTEMS WITH UNBOUNDED TIME-VARYING POINT DELAYS: STABILITY AND COMPACTNESS OF THE RELEVANT OPERATORS MAPPING THE INPUT SPACE INTO THE STATE AND OUTPUT SPACES

M. DE LA SEN

ABSTRACT. This paper is concerned with time-varying systems with non-necessarily bounded everywhere continuous time-differentiable time-varying point delays. The delay-free and delayed dynamics are assumed to be time-varying and impulsive, in general, and the external input may be impulsive as well. For given initial conditions, the (unique) homogeneous state-trajectory and output trajectory are equivalently constructed from three different auxiliary homogeneous systems, the first one being delay-free and time-invariant, the second one possessing the delay-free dynamics of the current delayed system and the third one being the homogeneous part of the system under study. In this way, the constructed solution trajectories of both the unforced and forced systems are obtained from different (input-state space/output space and state space to output space) operators. The stability of the homogeneous auxiliary system and that of the object system are investigated. Finally, the compactness of some of the various relevant operators involved in the descriptions of the solution trajectories is investigated.

1. Introduction. Time-delay systems have been widely investigated in the last years both in a theoretical context and in that of related applications, see for instance, [**2, 4, 6–12, 14–15, 17, 19–27**]. Those systems become inherently attractive from a theoretical point of view since they are described by (infinite-dimensional) functional equations and because of their interest towards potential applications like, for instance, population growth models, transportation, communications as well as war-peace and agricultural models [**6, 26**]. A wide variety of both dependent and independent (of delay) results exist, see for

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instance, [**2, 9–11, 17, 20, 24, 25, 27**] obtained via Liapunov stability theory or frequency-domain analysis tools, the second one being only useful in the time-invariant case. Most of the available results are restricted to time-invariant systems with constant delays. However, nowadays, the extensions to the nonlinear and time-varying systems as well as to systems described by partial-derivative equations are receiving increasing interest in the literature, see for instance, [**4, 9, 14, 15, 17, 20**]. In this paper, a very general class of time-varying systems is considered whose delay is time-varying and not necessarily everywhere bounded time-differentiable while the delay-free and delayed dynamics are, in general, time-varying and impulsive, and whose external input is impulsive as well what is of interest in some applications. Furthermore, the time-delay is not necessarily bounded and time-differentiable for all time.

The paper is organized as follows. Section 2 contains the basic notation issues used in the paper. Section 3 introduces three different unforced auxiliary systems related to the current unforced delay dynamic system. Those systems are then used to build explicitly in three different ways the unique state and output solution trajectories for any given admissible initial conditions. They are also used to obtain a set of global stability results for the current system based on their stability properties. The first of such unforced auxiliary systems is a delay free time-invariant one whose evolution operator is a C_0-semigroup with an infinitesimal generator. The second one is a delay-free (in general time-varying) homogeneous dynamic system which contains the delay-free dynamics of the whole current system which is, in general, impulsive and its parametrization is subject to bounded discontinuities of first and second class. Its evolution operator is bounded and almost everywhere time-differentiable. Finally, the third auxiliary system contains all the dynamics of the unforced time-delay system under study. Its bounded evolution operator involves explicitly the delay function and it is almost everywhere time-differentiable. All the relevant associated state-state/output and input-state/output operators defined to build the trajectory solutions are characterized from the above evolution operators and their associate evolution equations. Section 4 is devoted to establish and prove a set of results on global exponential and asymptotic stability of the homogeneous and forced time-delay systems based on some intermediate related previously derived results for the auxiliary systems which are obtained via Gronwall's lemma or Lacunae's stability theory, [**6, 16**], by taking into account that the parametrization has bounded discontinuities, [**1, 18**] and has, in general, impulsive terms in general while the input is impulsive as well, [**10**], under the assumption that the impulsive-free partly dynamics is stable and varies at a sufficiently slow rate with time. Section 5 is devoted to investigate the compactness, see for instance, [**3, 5, 13**] for definitions and relevant properties, of the various operators linking the input, state and output Banach's spaces under the key assumption that the input is square-integrable on $[0, \infty)$. That investigation is performed, keeping in mind the relevance of compact operators in the approximation theory in Banach's (and then in Hilbert) spaces since they map bounded sets into totally bounded sets (since any operator is compact in a reflexive space if and only if it is completely continuous). Another relevant property which identifies compact operators in topological or metric spaces (and that implies and it is implied by the above one) is that they map weakly convergent sequences into strongly convergent ones. Thus, the state/output trajectories are either finite-dimensional or arbitrarily close to finite dimension functions if the input is square-integrable on $[0, \infty)$ and the input-state, respectively, input-output operator is compact.

2. Notation. $\mathbf{R}_0^+(\mathbf{Z}_0^+) = \mathbf{R}^+ \cup \{0\}$ $(\mathbf{Z}^+ \cup \{0\})$ and $\mathbf{R}_0^-(\mathbf{Z}_0^-) = \mathbf{R}^- \cup \{0\}$ $(\mathbf{Z}^- \cup \{0\})$ are the (disjoint) sets of nonnegative and negative real (integer) numbers in the real field $\mathbf{R}$ (integer ring $\mathbf{Z}$).

The complement of a subset $\mathbf{S} \subseteq \mathbf{R}$ in $\mathbf{R}$ is denoted as $\overline{\mathbf{S}}$.

The exponential function of "f" is denoted indistinctly as "$\exp(f)$" or "$\mathbf{e}^f$" with the main criterion of notation choice being reading quality.

$\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximum and minimum eigenvalue of the square real (symmetric) matrix $P = P^T$. The notation $P > Q$ $(P \geq Q)$ means that $(P - Q)$ is positive definite denoted by $P - Q > 0$ ($(P - Q)$ being positive semi-definite is denoted by $P - Q \geq 0$) provided that Q is symmetric of the same order as P.

I denotes the identity matrix of any order (depending on context or specified as a subscript when necessary).

$X \subseteq \mathbf{R}^n$, $U \subseteq \mathbf{R}^m$ and $Y \subseteq \mathbf{R}^p$ are, respectively, the state, input and output real spaces of the time-delay dynamic system of respective dimensions n, m and p so that the state, input and output real vectors are, respectively, X, U and Y for all $t \geq 0$.

The real n-vector function $\varphi = \varphi_0 + \tilde{\varphi} + \tilde{\varphi}_{\mathrm{imp}}$ of the time-delay system is the set of admissible initial conditions $\mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ where $\infty > \bar{r} \geq r(0) \geq 0$ and the time-varying delay function $\mathbf{r} : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ so that $\varphi_0 : [-\bar{r}, 0) \to \mathbf{R}^n$ is absolutely continuous, $\tilde{\varphi}(t) = 0$, for all $t \in (t_i, t_{i+1})$ with $\tilde{\varphi}(t_i^-) \neq \tilde{\varphi}(t_i^+)$ being isolated bounded jump discontinuities at $t_i \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ with $\mathrm{Max}_{t_i \in \mathrm{Supp}\,(\tilde{\varphi})}(\|\tilde{\varphi}(t_i^+)\|) \leq M < \infty$ and $\int_{-r}^{0} \tilde{\varphi}(\tau)\, d\tau = 0$. A notation convention adopted is that at any discontinuity points t_i, $\tilde{\varphi}(t_i^+)$ $(\tilde{\varphi}(t_i^-))$ denotes the value of $\varphi(t_i)$ to the right (to the left) of t_i.

$\tilde{\varphi}_{\mathrm{imp}}(t) = \sum_{t_i \in TN_{\mathrm{imp}}} \tilde{\varphi}_{\mathrm{imp}}(t_i)\delta(t - t_i)$ is a real n-vector function from $[-\bar{r}, 0] \cap \mathbf{R}^+$ to $\mathbf{R}^n$ taking nonzero values at set $TN_{\mathrm{imp}} \subset [-\bar{r}, 0] \cap \mathbf{R}^+$, i.e., $\tilde{\varphi}_{\mathrm{imp}}(0) = 0$, of finite cardinal and zero measure with $\delta(t)$ being the Dirac distribution function. That is,

$$\lim_{t \to \infty} \left(\int_{-t}^{t} g(\sigma - \tau)\delta(\tau)\, d\tau \right) = \lim_{t \to 0^+} \left(\int_{\sigma - t}^{\sigma + t} g(\sigma - \tau)\delta(\tau)\, d\tau \right) = g(\sigma).$$

Thus, $\int_{-r}^{0} \tilde{\varphi}_{\mathrm{imp}}(\tau)\, d\tau = \sum_{i=1}^{N} \tilde{\varphi}_{\mathrm{imp}}(t_i)$, where $N_{\mathrm{imp}} := \mathrm{Card}\,(TN_{\mathrm{imp}})$ of $TN_{\mathrm{imp}} \subset [-\bar{r}, 0] \subset \mathbf{R}$.

L_∞^m is the set of essentially bounded m-vector functions from $\mathbf{R}_0^+$ to $\mathbf{R}_0^+$.

$L_2^m(a, b) \equiv L_2((a, b), \mathbf{R}^m)$ is the Hilbert space of the real m-vector functions $f : (a, b) \to \mathbf{R}^m$ which are square-integrable on (a, b) with the inner product denoted by $\langle ., . \rangle$ and the (semi-) norm

$$\|f\|_{L_2^m(a,b)} := \langle f, f \rangle_{L_2^m(a,b)}^{1/2} = \left(\int_a^b f^T(\tau) f(\tau)\, d\tau \right)^{1/2} d\tau < \infty,$$
$$\forall\, f \in L_2^m(a, b).$$

L_2^m is an abbreviated notation to denote $L_2^m(0, \infty) \equiv L_2^m((0, \infty), \mathbf{R}^m)$. We use the notation $M \in L_2^{n \times m} \equiv L_2((0, \infty), \mathbf{R}^{n \times m})$ for any matrix function $M = (m_{ij})$ of real entries m_{ij} in L_2 with m rows and n columns. Since impulsive functions are widely used thoroughly, closed and one-side closed real intervals $[a, b]$, respectively $(a, b]$, $[a, b)$, are used when necessary, as well as related simplified notations for Lebesgue integrals:

$\int_{a^+}^{b^+}(\cdot)$, $\int_{a^-}^{b^-}(\cdot)$, $\int_{a^+}^{b^-}(\cdot)$, $\int_{a^-}^{b^+}(\cdot)$, meaning respectively $\lim_{\varepsilon\to 0^+}(\int_{a+\varepsilon}^{b+\varepsilon}(\cdot))$, $\lim_{\varepsilon\to 0^+}(\int_{a-\varepsilon}^{b-\varepsilon}(\cdot))$, $\lim_{\varepsilon\to 0^+}(\int_{a+\varepsilon}^{b-\varepsilon}(\cdot))$ and $\lim_{\varepsilon\to 0^+}(\int_{a-\varepsilon}^{b+\varepsilon}(\cdot))$. For instance, if $f(t) = g(t) + k_a\delta(t-a) + k_b\delta(t-b)$ is a real function of domain $(a,b)\cap\mathbf{R}$ with $\delta(t)$ being the Dirac distribution and g being a real Lebesgue integrable function on (a,b), then:

$$\int_{a^-}^{b^+} f(\tau)\,d\tau = \int_a^b g(\tau)\,d\tau + k_a + k_b;$$
$$\int_{a^+}^{b^+} f(\tau)\,d\tau = \int_a^b g(\tau)\,d\tau + k_b;$$
$$\int_{a^-}^{b^-} f(\tau)\,d\tau = \int_a^b g(\tau)\,d\tau + k_a;$$

and

$$\int_{a^+}^{b^-} f(\tau)\,d\tau = \int_a^b g(\tau)\,d\tau.$$

The same delta symbol with integer subscripts, i.e., δ_{ij} is unity if and only if $i = j$ and zero otherwise, instead of a time argument, will be used for the Kronecker delta through Section 5.

The set of real absolutely integrable m-vector functions of domain $(a,b)\cap\mathbf{R}$ is denoted by $L_1^m(a,b)$ with $L_1^m \equiv L_1^m(0,\infty) = L_1((0,\infty),\mathbf{R}^m)$.

A (truncated) function $f_t(\tau)$ of $f(\tau)$ on $[0,t]\subset\mathbf{R}$ is defined as $f_t(\tau) := f(\tau)\,[\mathbf{1}(\tau) - \mathbf{1}(\tau - t)]$, which equates $f(\tau)$ on $[0,t]$ and is zero for all $\tau \neq [0,t]$, where $\mathbf{1}(t) = 1$ for all $t \geq 0$ and $\mathbf{1}(t) = 0$ for $t < 0$ is the unity step (Heaviside) function. The space of truncated square-integrable real m-vector functions on $(0,\infty)$ is denoted by $L_{2e}^m \equiv L_{2e}^m((0,\infty),\mathbf{R}^m)$, defined as $L_{2e}^m := \{f_t \in L_2^m$, all finite $t \geq 0\} = \cup_{0<t<\infty} L_2^m(0,t)$ and endowed with an inner product $\langle .,.\rangle_{L_{2e}^m}^{1/2} := \mathrm{Sup}_{0\leq t<\infty}(\int_{-\infty}^{\infty} f_t^T(\tau) f_t(\tau)\,d\tau)$ and associate (semi-) norm:

$$\|f\|_{L_{2e}^m} := \langle f,f\rangle_{L_{2e}^m}^{1/2} = \mathop{\mathrm{Sup}}_{0\leq t<\infty}(\|f_t\|_{L_2^m}) := \mathop{\mathrm{Sup}}_{0\leq t<\infty}(\langle f_t,f_t\rangle_{L_2^m}^{1/2})$$
$$= \mathop{\mathrm{Sup}}_{0\leq t<\infty}\left(\int_{-\infty}^{\infty} f_t^T(\tau) f_t(\tau)\,d\tau\right)^{1/2} = \mathop{\mathrm{Sup}}_{0\leq t<\infty}\left(\int_0^t f^T(\tau) f(\tau)\,d\tau\right)^{1/2},$$

for all $f \in L_{2e}^m$ (or, equivalently, for all $f_t \in L_2^m$) for all finite $t \geq 0$. The usefulness of the L_{2e}^m-space in the formalism is that truncated functions of nonsquare integrable functions are square-integrable, in general. Thus, if $f \notin L_2^m$ but $f_t \in L_2^m$ for all $0 \leq t < \infty$, then $f \in L_{2e}^m$ and most of the properties of the Hilbert space L_2^m may be invoked for f (via f_t) for all finite real intervals $[0, t]$. The notation $f \in L_2^m$ (or $f \notin L_{2e}^m$) may be further specified as $f \in L_2((0,\infty) \cap \mathbf{R},\, F \subseteq \mathbf{R}^m)$ (or $f \in L_{2e}((0,\infty) \cap \mathbf{R},\, F \subseteq \mathbf{R}^m)$ to indicate that $f \in F$ for a real f with definition domain $[0, \infty)$ (or $f_t \in F$ for all finite $t \geq 0$).

The space L_{1e}^m of (truncated) absolutely integrable real m-vector functions is defined in an analogous way related to the space $L_1^m \equiv L_1((0,\infty), \mathbf{R}^m) = \{f : [0,\infty) \cap \mathbf{R} \to \mathbf{R}^m : \int_0^\infty (f^T(\tau) f(\tau)\, d\tau)^{1/2} < \infty$.

$x_{[t]}$ is a strip of the solution trajectory, i.e., $x_{[t]} \equiv x : [t - r(t), t] \to \infty$ for $t \geq 0$ of the dynamic time-delay system of point time-varying delay $r(t)$.

The set of linear operators Γ from the linear space X to the linear space Y is denoted by $\mathbf{L}(X, Y)$.

The same norm symbol $\|\cdot\|$ is used for vector and (induced) matrix norms in Euclidean spaces as that used for the spaces L_∞^m, L_2^m and L_1^m, i.e., if $z \in \mathbf{R}^m$ and $Z \in \mathbf{R}^{n \times m}$, then

$$\|Z\| := \operatorname*{Sup}_{z \neq 0} \left(\frac{\|Zz\|}{\|z\|} \right) = \operatorname*{Sup}_{0 < \|z\| \leq 1} \left(\frac{\|Zz\|}{\|z\|} \right) = \operatorname*{Sup}_{\|z\|=1} \left(\frac{\|Zz\|}{\|z\|} \right) = \operatorname*{Sup}_{z=1}(\|Zz\|)$$

for any (vector) norm. Each particular norm symbol is interpreted without difficulty depending upon context. When necessary for clarity, the norm symbol is appropriately subscripted or described. Linear operators $Z \in \mathbf{L}(L_2^p, L_2^q)$ from the Banach space L_2^p to the Banach space L_2^q are usually defined pointwise as $(Zf)(t) : [0, t] \times \mathbf{R}^p \to \mathbf{R}^q$ for each $f \in L_2^p$ in the definition domain of the M-operator. The norm of the M-operator is

$$\begin{aligned} \|Z\| &:= \operatorname*{Sup}_{\|f\|=1} \left(\frac{\|Zf\|}{\|f\|} : f \in L_2^p \right) \\ &= \{\operatorname{Inf} k \in \mathbf{R}_0^+ : \|Zf\| \leq k,\ \forall\, f \in L_2^p \text{ s.t. } \|f\| \leq 1\}, \end{aligned}$$

while its adjoint is $Z^* \in \mathbf{L}(L_2^q, L_2^p)$ with norm defined accordingly.

Indicator binary functions with domain $\mathbf{R}_0^+$ are used to evaluate time-integral functions containing impulses. For instance,

$$\begin{aligned}
&\int_{0^-}^{t^-}\Big(f(\tau)+\sum_{t_i\in TN} g(\tau)\delta(\tau-t_i)\Big)\,d\tau\\
&\qquad = h(t)+\sum_{t_i\in TN(0,t)} g(t_i) = h(t)+\int_{0^-}^{t^-}\mu(\tau)g(\tau)\,d\tau\\
&\qquad = \int_{0^-}^{t^-}\big(f(\tau)+\mu(\tau)g(\tau)\big)\,d\tau,\quad t_i\in TN(0,t),
\end{aligned}$$

with $h(t) := \int_0^t f(\tau)\,d\tau$ if $f:[0,t]\to\mathbf{R}$ is Lebesgue integrable where $TN(0,t) := \{\tau\in TN : 0\le\tau<t\}$ is the support (of zero Lebesgue measure) of the real function $g:[0,t)\to\mathbf{R}$, provided that $\sum_{t_i\in TN(0,t)} g(t_i)<\infty$, while $\mu:(0,t)\to\{0,1\}$ is a (binary) indicator function of $TN(0,t)$ defined as $\mu(t)=0$ if $t\notin TN(0,t)$ and $\mu(t)=1$ if $t\in TN(0,t)$.

3. Time variant time-delay differential system. Consider the dynamic system:

$$\text{(1.a)}\qquad \mathbf{S}: \dot{x}(t) = A(t)x(t)+A_d x(t-r(t))+B(t)u(t)$$

$$\text{(1.b)}\qquad y(t) = C(t)x(t)+D(t)u(t)$$

where u,x and y are the input, state and output real vector functions satisfying $u\in L_2((0,\infty),U)\subset L_2^m$, $x\in L_{2e}((0,\infty),X)\subset L_{2e}^n$ and $y\in L_{2e}((0,\infty),Y)\subset L_{2e}^p$, respectively; and U,X and Y are the m-dimensional real input, n-dimensional real state and p-dimensional real output linear spaces, respectively. That is, $u(t)\in U$, $x(t)\in X$ and $y(t)\in Y$ for all $t\ge 0$. The system (1) is subject to any function of initial conditions $\varphi\in \mathrm{IC}\,([-\bar{r},0],\mathbf{R}^n)$, of the form defined in the notation section, where $r(t)$ is a (not necessarily bounded) time-delay function $r:[0,\infty)\to\mathbf{R}_0^+$ satisfying $0\le r(t)\le t+\bar{r}$ $(0\le r(0)\le\bar{r}<\infty)$

for all $t \in \mathbf{R}_0^+$, and

$$A(t) := A'(t) + \sum_{t_i \in TN} A''(t)\delta(t - t_i) \tag{2.a}$$
$$= A_0(t) + \tilde{A}'(t) + \sum_{t_i \in TN} A''(t)\delta(t - t_i)$$

$$A_d(t) := A'_d(t) + \sum_{t_i \in TN_d} A''_d(t)\delta(t - t_i) \tag{2.b}$$

are, in general, impulsive delay-free and delayed real matrix functions of dynamics from $[0, \infty)$ to $\mathbf{R}^{n\times n}$, respectively, where: DD and $D\overline{D}$ are, respectively, discrete real subsets of time instants where the time-delay function is discontinuous and continuous non-differentiable, respectively. $A_0 \in \mathbf{R}^{n\times n}$ is a constant real n-matrix, and $A'(t)$ and $\tilde{A}'(t) := A'(t) - A_0$; and $A'_d(t)$ have piecewise bounded continuous entries with isolated jump bounded discontinuities at time instants TD and TD_d, respectively.

$A''(t)$ and $A''_d(t)$ are bounded matrix functions from $[0, \infty)$ to $\mathbf{R}^{n\times n}$ of support of zero Lebesgue measure consisting of the set of impulses located at the time instants TN and TN_d, respectively. By convenience for evaluation of time integrals, continuous binary indicator real functions $\mu : [0, \infty) \to \{0, 1\}$ and $\mu_d : [0, \infty) \to \{0, 1\}$ will be used when necessary defined as $\mu(t) = 1$ ($\mu_d(t) = 1$) if $t \in TN$ ($t \in TN_D$); and $\mu(t) = 0$ ($\mu_d(t) = 0$) if $t \notin TN$ ($t \notin TN_d$). $B : [0, \infty) \to \mathbf{R}^{m\times n}$; $C : [0, \infty) \to \mathbf{R}^{n\times p}$ and $D : [0, \infty) \to \mathbf{R}^{p\times p}$ are, respectively, the control, output and interconnection real matrix functions of continuous bounded entries.

The input u in $L_2([0, \infty), U \subseteq \mathbf{R}^m)$ may also be impulsive and possess discontinuities of second class (so-called jump discontinuities) at the set TU; i.e., $u(t) = u'(t) + \sum_{t_i \in TU} u''(t)\delta(t - t_i)$ with $u \in L_2^m$, $u' \in L_2^m$ with $t_{i+1} - t_i \geq T_{0u} > 0$. $\mu_u : [0, \infty) \to \{0, 1\}$ is defined as $\mu_u(t) = 1$ if $t \in TU$ and $\mu_u(t) = 0$ ($t \notin TU$) is a binary indicator of TU. Note that DD, $D\overline{D}$, TN, TD, TD_d, TN_d and TU are strictly ordered sets in $\mathbf{R}_0^+$ and, respectively, in $\mathbf{Z}_0^+$ with respect to the "less than" binary relation '$<$,' i.e., satisfying the anti-reflexive, anti-symmetric and transitive properties. The notation $D(a, b) := \{t_i \in D : a \leq t_i < b\}$ for given $a, b \in \mathbf{R}_0^+$ applies to fixed elements of any set D (being, in particular, DD, $D\overline{D}$, TN, TD, TD_d, TN_d or TU) in $[a, b) \cap \mathbf{R}_0^+$.

3.1 *The use and usefulness of the indicator sets and functions.* When considering the state-trajectory solution of (1.a) for $t \geq 0$ subject to initial conditions $\varphi \in \mathrm{IC}([-\bar{r}, 0], \mathbf{R}^n)$, note that

$$\begin{aligned}
\dot{x}(t^-) &= A'(t^-)x(t^-) + A'_d(t^-)x(t^- - r(t)) + B(t)u'(t^-)\\
x(t^+) &= x(t^-) + \int_{t_-}^{t^+} \dot{x}(\tau)\, d\tau\\
&= (I + \mu(t)A''(t))x(t^-) + \mu_d(t)A''_d(t)x(t^- - r(t))\\
&\quad + \mu_u(t)B(t)u''(t)\\
\dot{x}(t^+) &= A'(t^+)x(t^+) + A'_d(t^+)x(t^+ - r(t)) + B(t)u'(t^+)\\
&= A'(t^+)((I + \mu(t)A''(t))x(t^-)\\
&\quad + \mu_d(t)A''_d(t)x(t^- - r(t)) + \mu_u(t)B(t)u''(t))\\
&\quad + A'_d(t^+)x(t^+ - r(t)) + B(t)u'(t^+)
\end{aligned}$$

so that $x(t^+) = x(t^-)$ if $\mu(t) = \mu_d(t) = \mu_u(t) = 0$, i.e., if $t \notin (TN \cup TN_d \cup TU)$. Another useful notation with indicators is $x(t_i^+) = x(t_i^-)$ if $t_i \notin (TN \cup TN_d \cup TU)$; and

$$x(t_i^+) = (I + A''(t_i))x(t_i^-) + A''_d(t_i)x(t_i^- - r(t)) + B(t_i^+)u''(t_i)$$

if $t_i \in (TN \cap TN_d \cap TU)$. The sets TN, TN_d, TU, etc., are useful to include the contribution of the impulses to the dynamics through time. For instance, assume zero initial conditions, i.e., $\varphi \equiv 0$, then the (forced) solution trajectory of (1.a) at t^- may be expressed as

$$\begin{aligned}
x(t^-) &= \int_{0^-}^{t^-} e^{A_0(t-\tau)}\,[A_d(\tau)x(t-\tau(t)) + B(\tau)u'(\tau)]\, d\tau\\
&\quad + \sum_{t_i \in TU(0,t)} e^{A_0(t-t_i)}B(t_i)u''(t_i).
\end{aligned}$$

The last right-hand side may also be denoted by using the sum over the indicator $I(TU(0,t))$ as $\sum_{t_i \in TU(0,t)} e^{A_0(t-t_i)}B(t_i)u''(t_i)$ or, alternatively, it may be included in the integrand as $e^{A_0(t-\tau)}\mu_u(\tau)B(\tau)\,\delta(\tau) \times u''(\tau)\, d\tau$ by using the binary indicator function $\mu_u : \mathbf{R}_0^+ \to \{0, 1\}$. The choice of each notation is made according to convenience criteria.

3.2 *Auxiliary homogeneous dynamic systems.* It is now discussed how the unique state and output trajectories $x \in L_{2e}^n$ and $y \in L_{2e}^p$ may be equivalently built from three different homogeneous, i.e., unforced, dynamic systems for each $\varphi \in \mathrm{IC}([-\bar{r}, 0], \mathbf{R}^n)$, two being delay-free, with one of them being in addition time-invariant, while the third one is unforced (1.a). This allows to highlight the decomposition of the trajectory solutions into parts and then to discuss stability results based on different conditions and assumptions from (1) as well as the compactness of the relevant operators associated with the trajectories. The three auxiliary homogeneous systems are:

S1: $\dot{z}_{A0}(t) = A_0 z_{A0}(t)$; $z_{A0}(0) = z_0 \in X \subset \mathbf{R}^n$

for any arbitrary constant matrix $A_0 \in \mathbf{R}^{n \times n}$ such that $A'(t) = A_0 + \tilde{A}'(t)$; $A(t) = A_0 + \tilde{A}(t)$ with $\tilde{A}'(t) = A'(t) - A_0$ (prescribed after fixing A_0) and $\tilde{A}(t) = \tilde{A}'(t) + A''(t)$.

S2: $\dot{z}_A(t) = A(t) z_A(t)$; $z_A(0) = z_0 \in X \subset \mathbf{R}^n$

S3: $\dot{z}(t) = A(t)z(t) + A_d(t)z(t - r(t))$; $z(0) = z_0 \in X \subset \mathbf{R}^n$.

Note that **S3** is the unforced system (1.a) and also a forced system with the forcing term $A_d(t)z_A(t - r(t))$. Also, $(A(t) - A_0)z_{A0}(t) + A_d(t)z_{A0}(t - r(t))$ is a forcing term to calculate the solution of **S3** from that of the homogeneous **S1**. Note that **S1, S2** may be established from a physical insight. In that way, **S1** is an unforced delay-free time invariant system which may be taken as a reference value for the delay-free dynamics. For instance, the stability of the system (1.a) may be formulated in terms of sufficiency conditions with respect to a stability reference matrix A_0 or A_0 may be the delay-free average delay-free dynamics in the case when $A'(t)$ is slowly time-variant. **S2** is an unforced delay-free, in general, time variant system which becomes identical to the unforced (1.a) when the delayed dynamics is identically zero for all time.

3.3 *Main result of Section* 3. The following result holds concerning the unique state/output trajectory of (1) from the auxiliary systems **S1, S2** on $\mathbf{R}_0^+$ with initial conditions

$$z(0^-) = z_{A0}(0^-) = z_A(0^-) = \varphi_0(0) + \tilde{\varphi}(0^-)$$
$$z(0^+) = z_{A0}(0^+) = z_A(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$$

and **S3** subject to $z(t) \equiv \varphi(t)$ for all $t \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ with any $\varphi \in \mathrm{IC}([-\bar{r}, 0], \mathbf{R}^n)$ satisfying the above constraints at $t = 0$. The following result holds:

Theorem 1. *The unique state/output trajectories of* S*, equation* (1)*, on* $\mathbf{R}_0^+$ *such that* $z(t) \equiv \varphi(t)$ *for* $t \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ *and* $r(t) \in [0, t + \bar{r}]$ *are uniquely defined for all* $t \geq 0$*, by any of the three sets of evolution equations below*:

(i) *Evolution equation* 1 (EE1) *from* **S1**.

$$\text{(3.a)} \quad x(t^-) = (S_{A0}\varphi)(t) + (S'_{A0}x_{[t]})(t^-) + (S^u_{A0}u)(t^-)$$
$$\text{(3.b)} \quad = (\overline{S}_{A0}\bar{x}_0)(t) + (\overline{S}'_{A0}x_{[t]})(t^-) + (S^u_{A0}u)(t^-)$$

$$\text{(3.c)} \quad x(t^+) = (S_{A0}\varphi)(t) + (S'_{A0}x_{[t]})(t^+) + (S^u_{A0}u)(t^+)$$
$$\text{(3.d)} \quad = (\overline{S}_{A0}\bar{x}_0)(t) + (\overline{S}'_{A0}x_{[t]})(t^+) + (S^u_{A0}u)(t^+)$$
$$\text{(3.e)} \quad = (I + \mu(t)A''(t))x(t^-) + \mu_d(t)A''_d(t)x(t^- - r(t)) + \mu_u(t)B(t)u''(t)$$

$$\text{(3.f)} \quad y(t^-) = (MA_0\varphi)(t) + (M'_{A0}x_{[t]})(t^-) + (M^u_{A0}u)(t^-)$$
$$\text{(3.g)} \quad = (\overline{M}_{A0}\bar{x}_0)(t) + (\overline{M}'_{A0}x_{[t]})(t^-) + (M^u_{A0}u)(t^-)$$

$$\text{(3.h)} \quad y(t^+) = (M_{A0}\varphi)(t) + (M'_{A0}x_{[t]})(t^+) + (M^u_{A0}u)(t^+)$$
$$\text{(3.i)} \quad = (\overline{M}_{A0}\bar{x}_0)(t) + (\overline{M}'_{A0}x_{[t]})(t^+) + (\overline{M}^u_{A0}u)(t^+)$$
$$\text{(3.j)} \quad = C(t)\big((I + \mu(t)A''(t))x(t^-) + \mu_d(t)A''_d(t)x(t^- - r(t)) + \mu_u(t)B(t)u''(t)\big) + \mu_u(t)D(t)u''(t)\delta(0)$$

where $S_{A0} \in \mathbf{L}(IC, L^n_{2e})$*,* $S'_{A0} \in \mathbf{L}(L^n_{2e}, L^n_{2e})$*,* $\overline{S}_{A0} \in \mathbf{L}(L^m_2, L^n_{2e})$*,* $\overline{S}'_{A0} \in \mathbf{L}(L^n_{2e}, L^n_{2e})$*,* $S^u_{A0} \in \mathbf{L}(L^m_{2e}, L^n_{2e})$*,* $M_{A0} \in \mathbf{L}(IC, L^p_{2e})$*,* $M'_{A0} \in \mathbf{L}(L^n_{2e}, L^p_{2e})$

and $M_{A0}^{u} \in \mathbf{L}(L_{2e}^{m}, L_{2e}^{p})$ *are defined point-wise at* t^{-} *via*:

$$(4.a)\quad (S_{A0}\varphi)(t) := e^{A_0 t}\left(x_0^{+} + \int_{I_{1t}} e^{-A_0\tau} A_d(\tau)\varphi(\tau - r(\tau))\,d\tau\right), x(0^{+}) = x_0^{+} = \varphi_0(0) + \tilde{\varphi}(0^{+})$$

$$(4.b)\quad (S'_{A0}x_{[t]})(t^{-}) := e^{A_0 t}\Bigg[\int_{I_{2t}} e^{-A_0\tau} A_d(\tau)x(\tau - r(\tau))\,d\tau + \int_{0^{-}}^{t^{-}} e^{-A_0\tau}\tilde{A}(\tau)x(\tau)\,d\tau + \int_{\bar{r}}^{t^{-}} e^{-A_0\tau} A_d(\tau)x(\tau - r(\tau)) \times [\mathbf{1}(\tau - r(\tau))(\mathbf{1}(\tau) - \mathbf{1}(\tau - t))]\,d\tau\Bigg]$$

$$(4.c)\quad (\overline{S}_{A0}\bar{x}_0)(t) := e^{A_0 t}\bar{x}_0;\quad \bar{x}_0 := x_0^{+} + \int_{0^{-}}^{\bar{r}} e^{-A_0\tau} A_d(\tau)\varphi(t - r(\tau))[\mathbf{1}(\tau) - \mathbf{1}(\tau - r(\tau))]\,d\tau$$

$$(4.d)\quad (\overline{S}'_{A0}\bar{x}_0)(t^{-}) := e^{A_0 t}\Bigg[\int_{0^{-}}^{t^{-}} e^{-A_0\tau}\tilde{A}(\tau)x(\tau)\,d\tau + \int_{I_2 t} e^{-A_0\tau} A_d(\tau)x(t - r(\tau))\,d\tau - \int_{I'_{1t}} e^{-A_0\tau} A_d(\tau)x(\tau - r(\tau))\,d\tau + \int_{\bar{r}}^{t^{-}} e^{-A_0\tau} A_d(\tau)x(\tau - r(\tau)) \times [\mathbf{1}(\tau - r(\tau))(\mathbf{1}(\tau) - \mathbf{1}(\tau - t))]\,d\tau\Bigg]$$

where $I_t := (I_{1t} \cup I_{2t}) \cap \mathbf{R} = (0, \mathrm{Min}\,(\bar{r}, t) \cap \mathbf{R})$ *is a real interval of measure* $\mathrm{Min}\,(\bar{r}, t)$ *for each* $t \geq 0$,

$$(4.e)\quad I'_{1t} := [0, \bar{r}] \cap \mathbf{R} \setminus I_{1t} = \{\tau \in \mathbf{R}_0^{+} : \bar{r} \geq \tau > r(\tau)\} \quad (\equiv I_{2t} \text{ if } 0 \leq t \leq \bar{r})$$

$$\begin{aligned} I_{1t} &:= \{\tau \in [0, \mathrm{Min}\,(\bar{r}, t)] \cap \mathbf{R} : \tau \le r(\tau)\} \\ I_{2t} &:= \{\tau \in [0, \mathrm{Min}\,(\bar{r}, t)] \cap \mathbf{R} : \tau > r(\tau)\} \\ &= [0, \mathrm{Min}\,(\bar{r}, t)] \cap \mathbf{R} \setminus I_{1t} \end{aligned}$$

$$\text{(4.f)} \qquad \begin{aligned} (\overline{S}^u_{A0} u)(t^-) &:= \int_{0^-}^{t^-} e^{A_0(t-\tau)} B(\tau) u(\tau)\, d\tau \\ &= \int_{0^-}^{t^-} e^{A_0(t-\tau)} B(\tau) u(\tau)\, d\tau \\ &\quad + \sum_{t_i \in TU(0,t)} e^{-A_0(t-t_i)} B(t_i) u''(t_i) \end{aligned}$$

$$\text{(4.g)} \qquad \begin{aligned} (M_{A0}\varphi)(t) &:= C(t)(S_{A0}\varphi)(t); \\ (M'_{A0} x_{[t]})(t) &:= C(t)(S'_{A0} x_{[t]})(t^-) \end{aligned}$$

$$\text{(4.h)} \qquad \begin{aligned} (\overline{M}_{A0}\bar{x}_0)(t) &:= C(t)(\overline{S}_{A0}\bar{x}_0)(t); \\ (\overline{M}'_{A0} x_{[t]})(t^-) &:= C(t)(\overline{S}'_{A0} x_{[t]})(t^-) \end{aligned}$$

$$\text{(4.i)} \qquad (M^u_{A0} u)(t^-) := C(t)(S^u_{A0} u)(t^-) + D(t) u'(t^-)$$

and $(S'_{A0}x_{[t]})(t^+)$, $(\overline{S}'_{A0}x_{[t]})(t^+)$, $(M'_{A0}x_{[t]})(t^+)$, $(\overline{M}'_{A0}x_{[t]})(t^+)$, $(S^u_{A0}u)$ (t^+) *and* $(M^u_{A0}x_{[t]})(t^+)$ *are defined similarly as their counterparts at* t^- *by replacing* t^- *with* t^+ *in the corresponding definitions.*

(ii) *Evolution equations* 2 (EE2) *from* **S2**.

$$\text{(5.a)} \quad x(t^-) = (S_A\varphi)(t) + (S'_A x_{[t]})(t^-) + (S^u_A u)(t^-)$$

$$\text{(5.b)} \quad \begin{aligned} x(t^+) &= (S_A\varphi)(t^+) + (S'_A x_{[t]})(t^+) + (S^u_A u)(t^+) \\ &= (I + \mu(t)A''(t))x(t^-) + \mu_d(t) A''_d(t) x(t^- - r(t)) \\ &\quad + \mu_u(t) B(t) u''(t) \end{aligned}$$

$$\text{(5.c)} \quad \begin{aligned} y(t^-) &= (M_A\varphi)(t^-) + (M'_A x_{[t]})(t^-) + (M^u_A u)(t^-) \\ &= C(t)((S_A\varphi)(t^-) + (S'_A x_{[t]})(t^-) + (S^u_A u)(t^-)) + D(t)u'(t^-) \\ &= C(t)x(t^-) + D(t)u'(t^-) \end{aligned}$$

$$\text{(5.d)} \quad \begin{aligned} y(t^+) &= (M_A\varphi)(t^+) + (M'_A x_{[t]})(t^+) + (M^u_A u)(t^+) \\ &= C(t)((S_A\varphi)(t^+) + (S'_A x_{[t]})(t^+) + (S^u_A u)(t^+)) \end{aligned}$$

$$
\begin{aligned}
&+ D(t)u'(t^+) + \mu_u(t)D(t)u''(t)\delta(0) \\
&= C(t)((I+\mu(t)A''(t))x(t^+) + \mu_d(t)A''_d(t)x(t^+ - r(t)) \\
&+ \mu_u(t)B(t)u''(t)) + D(t)u'(t^+) + \mu_u(t)D(t)u''(t)\delta(0)
\end{aligned}
$$

where the linear operators $S_A \in \mathbf{L}(L_{2e}^n, L_{2e}^n)$, $S'_A \in \mathbf{L}(L_{2e}^n, L_{2e}^n)$, $S_A^u \in \mathbf{L}(L_2^m, L_{2e}^n)$, $M_A \in \mathbf{L}(\mathrm{IC}, L_{2e}^p)$, $M'_A \in \mathbf{L}(L_{2e}^n, L_{2e}^p)$ *and* $M_A^u \in \mathbf{L}(L_2^m, L_{2e}^p)$ *are defined point-wise at* t^- *via*:

(6.a)
$$
(S_A\varphi)(t^-) := \Psi_A(t^-,0)\bigg(x_0^+ + \int_{0^-}^{\bar r} \Psi_A(t^-,\tau)A_d(\tau)\varphi(\tau - r((\tau)) \times (\mathbf{1}(\tau) - \mathbf{1}(\tau - t))\, d\tau)\bigg),
$$
$$
x(0^+) = \varphi_0(0) + \tilde\varphi(0^+)
$$

(6.b)
$$
\begin{aligned}
(S'_A x_{[t]})(t^-) := \bigg[&\int_{\bar r}^{t^-} \Psi_A(t^-,\tau)A_d(\tau)x(\tau - r(\tau))\mathbf{1}(t-\tau)\, d\tau \\
&+ \int_{I_{2t}} \Psi_A(t^-,\tau)A_d(\tau)x(\tau - r(\tau))\, d\tau \\
&- \int_{I_{1t}}^{t^-} \Psi_A(t^-,\tau)A_d(\tau)x(\tau - r(\tau))\, d\tau \bigg]
\end{aligned}
$$

(6.c)
$$
\begin{aligned}
(\overline{S}_A^u u)(t^-) &:= \int_{0^-}^{t^-} \Psi_A(t^-,\tau)B(\tau)u(\tau)\, d\tau \\
&= \int_{0^-}^{t^-} \Psi_A(t^-,\tau)B(\tau)u'(\tau)\, d\tau \\
&\quad + \sum_{t_i \in TU(0,t)} \Psi_A(t^-,t_i)B(t_i)u''(t_i)
\end{aligned}
$$

(6.d)
$$
\begin{aligned}
(\overline{S}_A^u u)(t^+) &:= [\textit{the above definition (6.c) with the replacement } t^- \to t^+] \\
&= (I+\mu(t)A''(t))(S_u u)(t^-) + \mu_u(t)B(t)u''(t)
\end{aligned}
$$

(6.e)
$$
(M_A\varphi)(t^-) := C(t)(S_A\varphi)(t^-); \quad (M'_A x_{[t]})(t^-) := C(t)(S'_A x_{[t]})(t^-)
$$

(6.f)
$$
(M_A u)(t^-) := C(t)(S_A^u u)(t^-) + D(t)u'(t^-)
$$

$$(6.g) \qquad (M_A u)(t^+) := C(t)(S_A^u u)(t^+) + D(t)u'(t^+) + \mu_u(t)D(t)u''(t)$$

and $(S_A\varphi)(t^+)$, $(S'_A x_{[t]})(t^+)$, $(M'_A x_{[t]})(t^+)$ *and* $(S_A^u u)(t^+)$ *are defined similarly as their counterparts at* t^- *by replacing* t^- *with* t^+ *in the corresponding definitions and the evolution operator* $\Psi_A(t^-,0)$ *satisfies the first-order differential system*:

(7)

$$\begin{aligned}&\dot{\Psi}_A(t^-,0) = A(t^-)\Psi_A(t^-,0) = A'(t^-)\Psi_A(t^-,0) \quad \textit{with } \Psi_A(0,0) = I;\\ &\Psi_A(t^-,\tau) = 0 \quad \textit{for all} \quad \tau > t \geq 0\end{aligned}$$

and it is defined explicitly as follows:

$$\begin{aligned}(8.a)\quad \Psi_A(t^-,0) &= \Psi_{A0}(t,0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)\tilde{A}(\tau)\Psi_{A0}(\tau,0)\,d\tau\right]\\ (8.b)\qquad &= \Psi_{A0}(t,0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)\tilde{A}'(\tau)\Psi_{A0}(\tau,0)\,d\tau\right]\\ &\quad + \sum_{t_i \in TN(0,t)} \Psi_{A0}(0,t_i)A''(t_i)\Psi_{A0}(t_i^-,0) \quad \textit{for} \quad t \geq 0\end{aligned}$$

with $\tilde{A}(t) = A(t) - A_0 = \tilde{A}'(t) + A''(t)$ *and* $\Psi_{A0}(t,\tau) := e^{A_0(t-\tau)}$, *for all* t *and* τ, *is a* $\mathbf{C}_0$*-semigroup of infinitesimal generator* A_0, *which is the evolution operator of* **S1**, *and*

$$(9) \qquad \Psi_A(t^+,0) = (I + \mu(t)A''(t))\Psi_A(t^-,0).$$

(iii) *Evolution equations* 2 (EE3) *from* **S3**.

$$(10.a) \qquad x(t^-) = (S\varphi)(t) + (S^u u)(t^-)$$

$$\begin{aligned}(10.b) \qquad x(t^+) &= (S\varphi)(t^+) + (S^u u)(t^+)\\ &= (I + \mu(t)A''(t))x(t^-) + \mu_d(t)A''_d(t)x(t^- - r(t))\\ &\quad + \mu_u(t)B(t)u''(t)\end{aligned}$$

$$(10.c) \qquad y(t^-) = (M\varphi)(t^-) + (M^u u)(t^-)$$

$$\begin{aligned}(10.d) \qquad y(t^+) &= (M\varphi)(t^+) + (M^u u)(t^+)\\ &= C(t)x(t^+) + D(t)(u'(t) + \mu_u(t)u''(t)\delta(0))\end{aligned}$$

where the linear operators $S \in \mathbf{L}(\mathrm{IC}, L^n_{2e})$, $S^u \in \mathbf{L}(L^m_2, L^n_{2e})$, $M \in \mathbf{L}(\mathrm{IC}, L^p_{2e})$ *and* $M^u \in \mathbf{L}(L^m_2, L^p_{2e})$ *are defined point-wise via:*

$$(11.a)\quad (S\varphi)(t^-) := T(t^-, 0)x_0 + \int_{-\bar{r}}^{0^+} T(t^-, -\tau)A_d(\tau + r(\tau))\varphi(\tau)\, d\tau,$$
$$x_0 = x(0^+) - \varphi_{\mathrm{imp}}(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$$

$$\begin{aligned}(11.b)\quad (S\varphi)(t^+) &:= T(t^+, 0)x_0 + \int_{-\bar{r}}^{0^+} T(t^+, -\tau)A_d(\tau + r(\tau))\varphi(\tau)\, d\tau \\ &= (I + \mu(t)A''(t))(S\varphi)(t^-) + \mu_d(t)A''_d(t)x(t^- - r(t))\end{aligned}$$

$$\begin{aligned}(11.c)\quad (S^u u)(t^-) &:= \int_{0^-}^{t^-} T(t^-, \tau)B(\tau)u(\tau)\, d\tau \\ &= \int_{0^-}^{t^-} T(t^-, \tau)B(\tau)u'(\tau)\, d\tau + \sum_{t_i \in TU} T(t^-, t_i)B(t_i)u''(t_i)\end{aligned}$$

$$\begin{aligned}(11.d)\quad (S^u u)(t^+) &:= \int_{0^-}^{t^+} T(t^+, \tau)B(\tau)u(\tau)\, d\tau \\ &= \int_{0^-}^{t^-} T(t^+, \tau)B(\tau)u'(\tau)\, d\tau \\ &\quad + \sum_{t_i \in TU} T(t^+, t_i)B(t_i)u''(t_i) + \mu_u(t)B(t)u''(t) \\ &= (I + \mu(t)A''(t))(S_u u)(t^-) + \mu_u(t)B(t)u''(t)\end{aligned}$$

$$(11.e)\quad (M\varphi)(t^-) := C(t)(S\varphi)(t^-); \quad (M\varphi)(t^+) := C(t)S(\varphi)(t^+)$$

$$(11.f)\quad (Mu)(t^-) := C(t)(S^u u)(t^-) + D(t)u'(t^-)$$

$$\begin{aligned}(11.g)\quad (Mu)(t^+) &:= C(t)(S^u u)(t^+) + D(t)(u'(t^+) + \mu_u(t)u''(t)\delta(0)) \\ &= C(t)[(I + \mu(t)A''(t))(S_u u)(t^-) + \mu_u(t)B(t)u''(t)] \\ &\quad + D(t)(u'(t^+) + \mu_u(t)u''(t)\delta(0)) \\ &= C(t)\bigg[\int_{0^-}^{t^-} T(t^+, \tau)B(\tau)u'(\tau)\, d\tau\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{t_i \in TU} T(t^+, t_i) B(t_i) u''(t_i) + \mu_u(t) B(t) u''(t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \mu_u(t) B(t) u''(t)\Big] \\
&+ D(t)(u'(t^+) + \mu_u(t) u''(t)\delta(0))
\end{aligned}
$$

and $T(t,0)$ is an almost everywhere time-differentiable evolution operator that satisfies the differential system:

$$
\begin{aligned}
\dot{T}(t^-,0) &= A'(t^-)T(t^-,0) + A_d(t^-)T(t^- - r(t),0) \\
\dot{T}(t^+,0) &= A(t^+)T(t^+,0) + A_d(t^+)T(t^+ - r(t),0) \\
&= (A'(t^+) + \mu(t)A''(t^+))T(t^+,0) \\
&\quad + (A'_d(t^+) + \mu_d(t)A''_d(t))T(t^+ - r(t),0)
\end{aligned}
\tag{12.a}
$$

$$\textit{with } T(0,0) = I;\ T(t^-,\tau) = 0 \textit{ for all } \tau > t \geq 0$$

and it is defined explicitly as follows for $t \geq 0$:

$$
\begin{aligned}
T(t^-,0) &= \Psi_{A0}(t,0)\Big[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)\tilde{A}(\tau)T(\tau,0)\,d\tau \\
&\qquad + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)A_d(\tau)T(\tau - r(\tau),0)\mathbf{1}(t-\tau)\,d\tau\Big]
\end{aligned}
\tag{12.b}
$$

$$
\begin{aligned}
&= \Psi_{A0}(t,0)\Big[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)\tilde{A}'(\tau)T(\tau,0)\,d\tau \\
&\qquad + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)A'_d(\tau)T(\tau - r(\tau),0)\mathbf{1}(t-\tau)\,d\tau\Big] \\
&\quad + \sum_{t_i \in TN(0,t)} \Psi_{A0}(t,t_i)A''(t_i)T(t_i^-,0) \\
&\quad + \sum_{t_i \in TN_d(0,t)} \Psi_{A0}(t,t_i)A''_d(t_i)T(t_i^- - r(t_i),0)
\end{aligned}
\tag{12.c}
$$

$$
= \Psi_A(t^-,0) + \int_{0^-}^{t^-} \Psi_A(t^-,\tau)A_d(\tau)T(\tau - r(\tau),0)\mathbf{1}(t-\tau)\,d\tau
\tag{12.d}
$$

$$
= \Psi_A(t^-,0) + \int_{0^-}^{t^-} \Psi_A(t^-,\tau)A''_d(t_i)T(t_i^- - r(t_i),0)
\tag{12.e}
$$

and

$$T(t^+,0) = (I+\mu(t)A''(t))T(t^-,0) + \mu_d(t)A_d''(t)T(t^- - r(t),0). \tag{13}$$

Proof [*Part* (i)]. Note that $\Psi_{A0}(t,\tau) = e^{A_0(t-\tau)}$, for all t,τ is a (C_0-semigroup) evolution operator for **S1** with infinitesimal generator A_0 possessing the well-known properties $\Psi_{A0}(t,t) = I$ and $\Psi_{A0}(t,\tau) = \Psi_{A0}(t-\tau) = \Psi_{A0}^{-1}(\tau,t) = e^{A_0(t-\tau)}$ and $f(t) = (A(t)-A_0)x(t) + A_d(t)x(t-r(t)) + B(t)u(t)$ is a forcing term in (1.a) with respect to the homogeneous system **S1** for any $\varphi \in \mathrm{IC}([-\bar{r},0],\mathbf{R}^n)$; i.e., $x(t) \equiv z_{A0}(t)$ for any real bounded $x(0+) = z_{A0}(0) = x_0 = \varphi_0(0^+) + \tilde{\varphi}(0)$ if $f \equiv 0$ and $\varphi(t) = 0$, $t \in [\bar{r},0)$. Thus, the unique state and output solution trajectories of (1) satisfy the integral identities for $t > 0$:

$$x(t^-) = e^{A_0 t}\left[x_0^+ + \int_{0^-}^{t^-} e^{-A_0\tau} f(\tau)\,d\tau\right]; \tag{14.a}$$
$$y(t^-) = C(t)x(t^-) + D(t)u'(t)$$

$$\begin{aligned} x(t^+) &= e^{A_0 t}\left[x_0^+ + \int_{0^-}^{t^+} e^{-A_0\tau} f(\tau)\,d\tau\right] \\ &= (I+\mu(t)A''(t))x(t^-) + \mu_d(t)A_d''(t)x(t^- - r(t)) \\ &\quad + \mu_u(t)B(t)u''(t) \end{aligned} \tag{14.b}$$

$$y(t^+) = C(t)x(t^+) + D(t)(u'(t) + \mu_u(t)u''(t)\delta(0)) \tag{14.c}$$

since $u(t^-) = u'(t^-)$ and $u(t^+) = u'(t^+) + \mu_u(t)u''(t)\delta(0)$. Equations (14.a) are directly obtained by constructing the solution of (1.a) via the homogeneous auxiliary system **S1** and the use of (1.c) from the integral equalities $\int_{0^-}^{t^-}(.) = \int_{0^-}^{\bar{r}}(.) + \int_{\bar{r}}^{t^-}(.)$ if $t \geq \bar{r}$ and $\int_{0^-}^{t^-}(.) = \int_{I_{1t}}(.) + \int_{I_{2t}}(.)$ for all $t \geq 0$ with I_{1t} and I_{2t} being connected real intervals if $r : [0, \mathrm{Min}\,(\bar{r},t)] \to \mathbf{R}_0^+$ is continuous, and $\int_{0^-}^{t^-}(.) = \int_{I_{1t}}(.) + \int_{I'_{1t}}$ if $0 \leq t \leq \bar{r}$ by equating $x(t) \equiv \varphi(t)$ for all $t \in [r(t)-\bar{r},0]$ since $\int_{\bar{r}}^{t}(.)$ is annihilated for all $0 \leq \tau \leq r(\tau)$ for any given function $\varphi \in \mathrm{IC}\,([-\bar{r},0],\mathbf{R}^n)$ since the function $\mathbf{1}(\tau)(\mathbf{1}(\tau) - \mathbf{1}(\tau - t))$ becomes zero. The proofs concerning (14.b), (14.c) follow directly from calculating $x(t^+) = x(t^-) + \int_{t^-}^{t^+} \dot{x}(\tau)\,d\tau$ via (14.a) and (1.b), the given function of initial conditions.

[*Part* (ii)]. By comparing the auxiliary systems **S2** and **S1** with $\tilde{A}(t) = A(t) - A_0 = \tilde{A}'(t) + A''(t)$. Since **S2** has a forcing term $\tilde{A}(t)z_A(t)$ with respect to **S1**, the unique state-trajectory solution of **S2** for $t > 0$ for any $z_0^+ = z_A(0^+) = \varphi_0(0^+) + \tilde{\varphi}(0)$ is given by

$$\begin{aligned} z_A(t^-) &= \Psi_A(t^-, 0)z_0^+ \\ &= \Psi_{A0}(t^-, 0)z_0^+ + \int_{0^-}^{t^-} \Psi_{A0}(t, \tau)\tilde{A}(\tau)z(\tau)\, d\tau \\ &= \Psi_{A0}(t^-, 0)\left(I + \int_{0^-}^{t^-} \Psi_{A0}(0, \tau)\tilde{A}(\tau)\Psi(\tau, 0)\, d\tau\right)z_0^+ \end{aligned} \tag{15}$$

since $\Psi_{A0}(t, \tau) = \Psi_{A0}(t, 0)\Psi_{A0}(\tau, 0) = e^{A_0(t-\tau)}$ for any t, τ provided that $z_A(\tau) = \Psi_A(\tau, 0)z_0^+$ is the unique solution of **S2** on $[0, t)$ for any $z_0^+ \in \mathbf{R}^n$ if and only if (8) holds for $t \geq 0$ with $\Psi_A(t, \tau) \equiv 0$ and $\Psi_A(t, t) = I$ for all $\tau > t \geq 0$. Direct calculation with (8) yields

$$\begin{aligned} \dot{\Psi}_A(t^-, 0) &= A_0\left\{\Psi_{A0}(t, 0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0, \tau)\tilde{A}(\tau)\, d\tau\right]\right\} + \tilde{A}(t)\Psi(t^-, 0) \\ &= (A_0 + \tilde{A}(t))z(t) = A(t)\Psi(t^-, 0)z_0^+ \end{aligned}$$

since $\dot{\Psi}_{A0}(t, \tau) = A_0\Psi_{A0}(t, \tau) = A_0 e^{A_0(t-\tau)}$, for all t, τ, provided that (8) holds, which satisfies the differential system describing **S2**. On the other hand,

$$\begin{aligned} z_A(t^+) &= \Psi_A(t^+, 0)z_0^+ = \Psi_A(t^-, 0)z_0^+ \\ &\quad + \left(\int_{t^-}^{t^+} A''(\tau)\delta(\tau - t)\Psi_A(\tau, 0)\, d\tau\right)z_0^+ \\ &= (I + \mu(t)A''(t))\Psi_A(t^-, 0)z_0^+ \\ &= (I + \mu(t)A''(t))z_A(t^-) \end{aligned} \tag{16}$$

which holds for any z_0^+ if and only if (9) holds. It has been proved that the evolution operator of **S2** satisfies (8), (9). Now, since **S3** has a forcing term $A_d(t)z(t - r(t))$ with respect to **S2**, the unique state trajectory solution of **S3** for any given $\varphi \in \mathrm{IC} \in ([-\bar{r}, 0], \mathbf{R}^n)$, which is also the unique solution of the homogeneous equation (1.a) for such an initial condition for all $t > 0$, is by construction:

$$\begin{aligned}
\text{(17.a)}\quad z(t^-) &= \Psi_A(t^-,0)z_0^+ + \int_{0^-}^{t^-} \Psi^A(t^-,\tau)A_d(\tau)z(\tau - r(\tau))\,d\tau \\
&= \Psi_A(t^-,0)z_0^+ + \int_{0^-}^{t^-} \Psi_A(t^-,\tau)A'_d(\tau)z(\tau - r(\tau))\,d\tau \\
&\quad + \sum_{t_i \in TN_d} \Psi_A(t^-,t^i)A''_d(t_i)z(t_i^- - r(t_i))
\end{aligned}$$

$$\begin{aligned}
\text{(17.b)}\quad z(t^+) &= \Psi_A(t^+,0)z_0^+ + \int_{0^-}^{t^+} \Psi_A(t^+,\tau)A_d(\tau)z(\tau - r(\tau))\,d\tau \\
&= \Psi_A(t^+,0)z_0^+ + \int_{0^-}^{t^+} \Psi_A(t^+,\tau)A'_d(\tau)z(\tau - r(\tau))\,d\tau \\
&\quad + \sum_{t_i \in TN_d} \Psi_A(t^+,t_i)A''_d(t_i)z(t_i^+ - r(t_i))
\end{aligned}$$

$$\begin{aligned}
\text{(17.c)}\qquad &= (I+\mu(t)A''(t))z(t^-) + \int_{t^-}^{t^+} \Psi_A(t^+,\tau)A''_d(\tau)z(\tau - r(\tau))\,d\tau \\
&= (I+\mu(t)A''(t))z(t^-) + \mu_d(t)A''_d(t)z(\tau - r(\tau))\,d\tau.
\end{aligned}$$

Now, from the integral identity $(\int_0^r(.)-) + \int_{I_{2t}}(.) = \int_{I_{1t}}(.) + \int_{I_{2t}}(.)$, it follows that the homogeneous solution of **S3**, i.e., that of (1.a) for $u \equiv 0$, for $t \geq 0$ for any given $\varphi \in \mathrm{IC} \in ([-\bar{r},0],\mathbf{R}^n)$ is:

$$z(t^-) = (S_A\varphi)(t^-) + (S'_A z_{[t]})(t^-); \quad z(t^+) = (S_A\varphi)(t^+) + (S'_A z_{[t]})(t^+),$$

while that of **S2** is $z_A(t^-) = (S_A z_0^+)(t^-)$; $z_A(t^-) = (S_A z_0^+)(t^+)$ with $z_0^+ = z_A(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$. Then, the unique state-trajectory solution of **S** from (1.a) is then uniquely given by construction by (5.a), (5.b) subject to (6.a)–(6.d) since

$$\sum_{t_i \in TU(0,t)} \Psi_A(t^-,t_i)B(t_i)u''(t_i) = \sum_{t_i \in TU} \Psi_A(t^-,t_i)B(t_i)u''(t_i)$$

since $\Psi_A(t,\tau) = 0$ for $\tau > t$. Combining (1.b) with (5.a), (5.b) yields directly the output-trajectory solution (5.c), (5.d) on $[0,\infty)$ with the operator definitions (6.e)–(6.g) and the replacements $t^- \to t^+$ referred to previously.

[*Part* (iii)]. If the state-trajectory solution satisfies (10.a) with $z(t) \equiv \varphi(t)$ for any given $\varphi \in \mathrm{IC}([-\bar{r}, 0], \mathbf{R}^n)$ satisfying the corresponding equations (11). Then, for any $z(0^+) = z_0^+ = \varphi_0 + \tilde{\varphi}(0^+)$ and all $t > 0$,

(18)
$$\begin{aligned}
\dot{z}(t^-) &= \dot{T}(t^-, 0)z_0^+ + \int_{-\bar{r}}^{0^-} \dot{T}(t, -\tau)A_d(\tau + r(\tau))\varphi(\tau)\,d\tau \\
&= A(t^-)\left[T(t^-, 0)z_0^+ + \int_{-\bar{r}}^{0^-} T(t^-, -\tau)A_d(\tau + r(\tau))\varphi(\tau)\,d\tau\right] \\
&\quad + A_d(t^-)\left[T(t^- - r(t))z_0^+ + \int_{-\bar{r}}^{0^-} T(t^- - r(t) - \tau, 0)\right. \\
&\qquad\qquad\qquad\qquad \left. \times A_d(\tau + r(\tau))\varphi(\tau)\,d\tau\right] \\
&= A(t^-)z(t^-) + A_d(t^-)z(t^-)
\end{aligned}$$

by using the definition of $T(t^-, 0)$ in (12) and $T(t, \tau) = 0$ for $\tau > t$. Similarly, the definition of $T(t^+, 0)$ and a similar derivation as that of (18) yield:

$$\dot{z}(t^+) = A(t^+)z(t^+) + A_d(t^+)z(t^+ - r(t)) \tag{19}$$

and direct calculation yields

$$\begin{aligned}
z(t^+) = (S\varphi)(t^+) &= z(t^-) + \int_{t^-}^{t^+} \dot{z}(\tau)\,d\tau \\
&= (I + \mu(t)A''(t))(S\varphi)(t^-) + \mu_d(t)A_d''(t)(S\varphi)(t^- - r(t)).
\end{aligned}$$

Then, the state trajectory solution of **S3**, and thus that of **S**, equation (1.a), is satisfied on $(0, \infty)$ for $u \equiv 0$ by (10.a), (10.b) via the corresponding definitions of the operators $(S\varphi)(t^-)$, $(S\varphi)(t^+)$, $T(t^-, 0)$ and $T(t^+, 0)$ via (11), (12) provided that $z(t) \equiv \varphi(t)$, $t \in [-\bar{r}, 0]$, for all $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$.

The forced state-trajectory solution (10.a), (10.b) follows by direct construction from the homogeneous solution of **S3**. The output trajectory solution (10.c), (10.d) follows directly from (1.b) via (10.a), (10.b) by replacing the operators $S \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$ and $S^u \in \mathbf{L}(L_2^m, L_{2e}^n)$ by $M \in \mathbf{L}(\mathrm{IC}, L_{2e}^p)$ and $S^u \in \mathbf{L}(L_2^m, L_{2e}^p)$ defined in (11), respectively. $\square$

4. Stability. The following result is concerned with sufficient conditions for global exponential stability (GES) of the System **S** (via obtaining related properties for the auxiliary systems **S1, S2**), which implies global asymptotic stability (GAS) in the sense that the state trajectory of the unforced system vanishes exponentially, respectively, asymptotically with time for any $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ for any bounded $z(0^+) \equiv \varphi(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$. Such conditions also imply the global stability (GE) of the forced system **S** and that its output $y \in L_2^p$, for any input $u \in L_2^m$, and that $y \in L_2^p \cap L_\infty^p$ if $u \in L_2^m \cap L_\infty^m$, i.e., when u is square-integrable but not impulsive. The conditions concerning the "smallness" of the absolute values of certain parameters referred to in the result statement are then made explicit in the corresponding parts of the proof.

4.1 *Main stability result.*

Theorem 2. *The following items hold*: (i) *Let A_0 be a stability matrix with fundamental matrix of the associated differential system being the evolution C_0-semigroup $\Psi_{A0}(t, \tau)$ satisfying $\|\Psi_{A0}(t, \tau)\| \leq k_0 e^{-\rho_0(t-\tau)}$ for all t, τ, some real (norm-dependent) constants $k_0 \geq 1$ and all real constants $\rho_0 \in (0, \rho^*)$ where $(-\rho^*) < 0$ is the stability abscissa of A_0 (i.e., the largest real part of its eigenvalues. If the eigenvalue of largest real part is simple then $\rho \in (0, \rho^*]$). Thus,* **S2** *is* GES *if $0 \notin TN$ and*

$$(20) \quad \rho_0 \geq \rho_0 := \operatorname*{Sup}_{t \in R_0^+} \left(\frac{k_0}{t} \left[\int_{0^-}^{t^-} \|\tilde{A}'(\tau)\| \, d\tau + \sum_{t_i \in TN(0,t)} \|A''(t_i)\| \right] \right)$$

for any (vector-induced) matrix norm $\|(.)\|$.

(ii) *Assume that $0 \notin TN$ and that there exist finite nonnegative real constants $a := \operatorname{ess\,Sup}_{t \in R_0^+} (\|\tilde{A}'(t)\|)$,*

$$b \geq \operatorname*{Sup}_{t \in R_0^+} \left(\frac{1}{t} \sum_{t_i \in TN(0,t)} \|A''(t_i)\| \right).$$

Then, **S2** *is* GES *if $\rho_0 > k_0(a + b)$.*

(iii) *Assume that* $A'(t)$ *has uniformly bounded entries on* $[0,\infty)$ *and eigenvalues satisfying* $\mathrm{Re}\,[\lambda_i(A'(t)) \le -\sigma < 0]$, *for all* $t \ge 0$, *and that positive real constants* $T_1 > 0$ *and* $T > 0$ *exist such that*:

• *Any two consecutive* $t_i, t_{i+1} \in TN$ *satisfy* $t_{i+1} - t_i \ge T_1$, *i.e.,* $TN(t, t+T_1)$ *contains at most a* $t_i \in TN$ (*otherwise, it is empty*), *for all* $t \ge 0$.

• *Any two consecutive* $t_i, t_{i+1} \in TD$ *satisfy* $t_{i+1} - t_i \ge T_2$, *i.e.,* $TD(t, t+T_2)$ *contains at most a* $t_i \in TD$ (*otherwise, it is empty*), *for all* $t \ge 0$.

• *Real constants* α_0 *and* α_1 *exist such that* $\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\|\, d\tau \le \alpha_1 T + \alpha_0$, *for all* $t \ge 0$. *Then,* **S2** *is* GES *for all real constants* $\alpha_1 \in [0, \alpha_1^*)$ *and* $\varepsilon \in [0, \varepsilon^*)$, *some sufficiently small* α_0, $\alpha_1^* \in \mathbf{R}^+$ *and* $\varepsilon \in \mathbf{R}^+$ *where* $\varepsilon := \mathrm{Max}\,(\varepsilon_1 + \varepsilon_2, \varepsilon_3)$ *with*

(21)

$$\varepsilon_1 := \operatorname*{Sup}_{t \in S_1(t,t+T)} \left(\ln \frac{z_A^T(t^+)P(t^-)z_A(t^+)}{z_A^T(t^-)P(t^-)z_A(t^-)} \right)$$
$$\le \varepsilon_1^* := \operatorname*{Sup}_{t \in S_1(t,t+T)} \left(\ln \frac{\lambda_{\min}[(I+\mu(t)A''^T(t))P(t^-)(I+\mu(t)A''(t))]}{\lambda_{\max}(P(t^-))} \right)$$

$$\varepsilon_2 := \operatorname*{Sup}_{t \in S_2(t,t+T)} \left(\ln \frac{z_A^T(t)P(t^+)z_A(t)}{z_A^T(t)P(t^-)z_A(t)} \right)$$
$$\le \varepsilon_2^* := \operatorname*{Sup}_{t \in S_2(t,t+T)} \left(\ln \frac{\lambda_{\min}(P(t^+))}{\lambda_{\max}(P(t^-))} \right)$$

$$\varepsilon_3 := \operatorname*{Sup}_{t \in S_3(t,t+T)} \left(\ln \frac{z_A^T(t^+)P(t^+)z_A(t^+)}{z_A^T(t^-)P(t^-)z_A(t^-)} \right)$$
$$\le \varepsilon_3^* := \operatorname*{Sup}_{t \in S_3(t,t+T)} \left(\ln \frac{\lambda_{\min}[(I+\mu(t)A''^T(t))P(t^+)(I+\mu(t)A''(t))]}{\lambda_{\max}(P(t^-))} \right)$$

where $S_i(t, t+T)$, $i = 1, 2, 3$, *are the empty or nonempty sets of time instants of impulses, jump discontinuities or combined impulses and jump discontinuities in the real interval* $(t, t+T]$ *defined as*:

$$\begin{aligned} S_1(t,t+T) &:= TN(t,t+T) \cap \overline{TD}(t,t+T); \\ S_2(t,t+T) &:= TD(t,t+T) \cap \overline{TN}(t,t+T); \\ S_3(t,t+T) &:= TN(t,t+T) \cap TD(t,t+T), \end{aligned} \tag{22}$$

and $P(t) = P^T(t) > 0$ *is a real matrix function* $P : \mathbf{R}_0^+ \to \mathbf{R}^{n\times n}$ *that satisfies a Liapunov matrix equation*:

(23)
$$\begin{aligned} A'^T(t)P(t) + P(t)A'(t) &= -I \quad \textit{for} \quad t \notin TD \\ A'^T(t)P(t^+) + P(t^+)A'(t) &= A'^T(t^-)P(t^-) \\ &\quad + P(t^-)A(t^-) = -I \quad \textit{for} \quad t \in TD. \end{aligned}$$

(iv) *If the growing rate condition on* $\|A'(t)\|$ *in* (iii) *is replaced with*

$$\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\|\, d\tau \le \alpha_1 T + \alpha_0 + \alpha_0', \quad \textit{for all} \quad t \ge 0,$$

where
$$\alpha_0' := \operatorname*{Sup}_{t\in\mathbf{R}_0^+} \Bigg(\sum_{t\in TD(t,t+T)} (\|A'(\tau^+) - A'(\tau^-)\|) \Bigg),$$

then **S2** *is* GES *for all real* $\alpha_1 \in [0, \bar{\alpha}_1^*)$, *some* $\bar{\alpha}_1^* \in \mathbf{R}^+$ *if* $\varepsilon \in [0, \varepsilon^*)$, *for some sufficiently small* $(\alpha_0 + \alpha_0')$, $\bar{\alpha}_1^* \in \mathbf{R}$ *and* $\varepsilon^* \in \mathbf{R}^+$ *where* $\varepsilon, \varepsilon_i$, $i = 1, 2, 3$, *are defined in* (iii).

(v) *Items* (iii)–(iv) *still hold under similar conditions if*

$$\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\|^2\, d\tau \le \alpha_1^2 T + \bar{\alpha}_0, \quad \textit{for all} \quad t \ge 0, \tag{24}$$

respectively,

$$\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\|^2\, d\tau \le \alpha_1^2 T + \bar{\alpha}_0 + \bar{\alpha}_0', \quad \textit{for all} \quad t \ge 0, \tag{25}$$

where $\bar{\alpha}_0' := \operatorname{Sup}_{t\in\mathbf{R}_0^+} (\sum_{t\in TD(t,t+T)} \|A'(\tau^+) - A'(\tau^-)\|^2)$ *provided that* $\alpha_1 \in [0, \bar{\alpha}_1^*)$, *respectively* $\alpha_1 \in [0, \bar{\alpha}_1'^*)$, *with* $\bar{\alpha}_1^*$, *respectively* $\bar{\alpha}_1^{*\prime}$, *and* α_0, *respectively* $(\bar{\alpha}_0 + \bar{\alpha}_0')$ *being sufficiently small.*

(vi) *Items* (iii), (iv) *still hold if* $\|\dot{A}'\| \in L_2^{n\times n}$.

(vii) *Assume that* $A'(t)$ *is locally integrable for all* $t \ge 0$ *and that for some integers* n_0, $1 \le n_0 \le n$, *and* n_0', *there is an* $n_0 \times n_0'$ *matrix function* $N(t)$ *such that the Liapunov matrix equation*

$$A'^T(t)P(t) + P(t)A'(t) = -N(t)N^T(t) - q_0 I \tag{26}$$

holds for all real $t \in \mathbf{R}_0^+$, *some real square* n-*matrix* $P = P^T > 0$ *and some real* $q_0 \in \mathbf{R}_0^+$. *Then,* **S2** *is* GES *if* $A''(t_i)$, $t_i \in TN$, *is uniformly bounded, the pair* $[A'(t), N^T(t)]$ *is uniformly completely observable if* $q_0 = 0$ (*not required if* $q_0 > 0$) *and*

(27)
$$\prod_{t_i \in TN(t,t+T)} \left[\frac{z_A^T(t_i^-)(I+A''^T(t_i))P(I+A''(t_i))z_A(t_i^-)}{z_A^T(t_i^-)Pz_A(t_i^-)}\right] < \frac{\lambda_{\max}(P)}{\alpha}$$

where αI *is a lower-bound matrix of the observability Grammian* $G(t, t+T)$ *of* $[A'(t), N^T(t)]$, *i.e.,* $G(t, t+T) - \alpha I \geq 0$.

(viii) *Assume a real sequence* $S_t \equiv \{t_i\}_1^\infty$ *defined by* $S_t := \{t_i \in \mathbf{R}_0^+ : t_{i+1} \geq T_* > 0\}$, *some real fixed* $T_* > 0$. *Thus,* (vii) *still holds if the subsequent Liapunov matrix inequality is satisfied*:

$$A'^T(t)P(t) + P(t)A'(t) \leq -N(t)N^T(t) - q_0(t_i)I, \quad \text{for all} \quad t \in [t_i, t_{i+1})$$

with the real sequence $\{q_0(t_i)\}_1^\infty$ *satisfying* $q_0(t_i) \geq q_0 > 0$ *for all real intervals* $[t_i, t_{i+1})$, $t_i \in S_i$, *provided that* (27) *holds where* $[A'(t), N^T(t)]$ *is not uniformly completely observable.*

(ix) *The auxiliary system* **S3**, *and then the unforced* **S** (1.a), *is* GES *for all* $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ *if* **S2** *is* GES *satisfying all the conditions of* (iii), *or, alternatively, those of* (ix), (v) *or* (vi), *for some positive real constant* T, *with* $0 < q_0 < (4\sigma_A^2/\alpha_A^4)$ *and, furthermore,*

(a) $\|A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))\|$ *is finite for all* $t \in C\overline{D} \cup DD$ *where* $h(t) := t - r(t)$, $C\overline{D}$ *is the set of zero measure where the delay function* $r : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ *is continuous nondifferentiable with respect to time,* DD *is the set of zero measure where it is discontinuous,*

$$\text{(28)} \qquad \text{(b)} \quad \int_{1/t_1^-}^{1/t_2^-} \|A_d'(\tau)\|^2(1 - \dot{r}(\tau))\, d\tau \leq \gamma'(t_2 - t_1) + \gamma_0'(t_1, t_2)$$

for all $t_2 \geq t_1 > 0$ *such that* $(C\overline{D} \cup DD) \cap [t_1, t_2) = \varnothing$ *for some nonnegative finite real constants* $\gamma'(t_2 - t_1)$ *and* $\gamma_0'(t_1, t_2)$. *If* $[t_1, t_2) \cap TN = \varnothing$, *then* $\gamma_0'(t_1, t_2) = 0$.

$$\text{(29)} \qquad \text{(c)} \quad e^{-qT}\left(\prod_{t_i \in D(t,t+T)} [\lambda(t_i)]\right) < 1; \quad \lambda(t_i) := \frac{V'(t_i^+)}{V'(t_i^-)}$$

where $q := q_0(1 - (\alpha_A^4 q_0/4\sigma_A^2)) - \mathrm{Sup}_{t \notin D}\left(\|A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))\|\right) > 0$ *for some real constant* $q_0 \geq q_0 \geq 0$ *with sets* $D(t, t+T) := TN(t, t+T) \cup TN_d(t, t+T) \cup TD(t, t+T) \cup DD(t, t+T)$ *and* $DD(t, t+T) := DD(t) \cap DD(t, t+T)$ *of discontinuities, and*

(30)
$$V'(t) := z^T(t)P(t)z(t) + \int_{t^- - r(t)}^{t^-} z^T(\tau)A_d^T(h^{-1}(\tau))A_d(h^{-1}(\tau))z(\tau)\,d\tau.$$

(x) *Assume that there exist a matrix function* $N(t)$ *and* $q_0 \in \mathbf{R}_0^+$ *satisfying* $q_0 < (4\sigma_A^2/\alpha_A^4)$ *such that the Liapunov matrix inequality below is satisfied*:

$$A'^T(t)P(t) + P(t)A'(t) \leq -q_0 I - N(t)N^T(t) \tag{31}$$

with $A'(t)$ *being locally integrable and* $A''(t)$. *Assume also that the matrix function* $A_d' : \mathbf{R}_0^+ \to \mathbf{R}^{n\times n}$ *satisfies the same conditions as in* (ix) *and* (28) *of* (ix) *is fulfilled. Then* **S3**, *and then the unforced* (1.a) *of* **S**, *is* GES *for all* $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$.

(xi) *Assume that* $A'(t)$ *is locally integrable,* $A''(t)$ *is uniformly bounded and, for some positive real constants* α, β, $\beta \geq \alpha$, *and, in addition,*

$$\beta I \geq \int_{0^-}^{t^-} T^T(t,\tau)N(\tau)N^T(\tau)T(t,\tau)\,d\tau \geq \alpha I, \quad \textit{for all} \quad t \geq 0, \tag{32}$$

for the case when the Liapunov matrix equation (26) *holds with* $q_0 = 0$. *Then* **S3**, *and then the unforced* **S**, *is* GES *for all* $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ *while* **S** *is* GE *for all* $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ *and* $u \in L_2^m$ *and, furthermore,*

$$\frac{\alpha\vartheta_t'(0)}{\lambda_{\max}(P)} > \varepsilon_0; \quad \left(1 - \frac{\alpha\vartheta_T'(t)}{\lambda_{\max}(P)}\right) \prod_{t_i \in D(t, t+T)} [\lambda(t_i)] < 1 - \varepsilon_0 \tag{33}$$

with

$$\begin{gathered} 0 < \vartheta_T'(t) \leq e^{-q_0 T}; \qquad \lambda(t_i) := \frac{z^T(t_i^+)Pz(t_i^+)}{z^T(t_i^-)Pz(t_i^-)}; \\ 0 < \varepsilon_0 \leq \alpha k_\varphi \sup_{t \in R_0^+} \left(\frac{\|T(t,0)\|_2^2 \vartheta_T'(t)}{\lambda_{\max}(P)}\right) < 1 \end{gathered} \tag{34}$$

for some real constant k_φ, dependent on the initial conditions, and for all $t \geq 0$.

(xii) *Assume that $A'(t)$ is locally integrable, $A''(t)$ is uniformly bounded and $[A'(t), N^T(t)]$ is uniformly completely observable, i.e., for some positive real constants α, β, $\beta \geq \alpha$, such that*

(35)
$$\beta I \geq \int_{t^-}^{t^- + T} T^T(t,\tau) N(\tau) N^T(\tau) T(t,\tau)\, d\tau \geq \alpha I, \quad \textit{for all} \quad t \geq 0.$$

Then **S3***, and then the unforced* **S***, is* GES *for all $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ while* **S** *is* GE *for all $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ and $u \in L_2^m$ if the Liapunov equation* (26) *and* (33), (34) *hold with $q_0 = 0$.*

4.2 *Proof of Theorem* 2. (i) One gets directly from the differential system defining **S2**:

$$(36) \quad \|z_A(t^+)\| \leq k_0 \left[e^{-\rho_0 t} \|z_A(0^-)\| + \int_{0^-}^{t^+} e^{-\rho_0 \tau} \|\tilde{A}(\tau)\|\, \|z_A(\tau)\|\, d\tau \right]$$

for all $t \geq 0$ and any vector norm $\|m\|$ and associated induced matrix norm

$$\|M\| = \operatorname*{Sup}_{\|m\| \leq 1} \left(\frac{\|Mm\|}{\|m\|} \right) = \operatorname*{Sup}_{\|m\| = 1} (\|Mm\|),$$
$$\text{for all} \quad m \in \mathbf{R}^n \quad \text{and} \quad M \in \mathbf{R}^{n \times n},$$

since $\|\Psi_{A0}(t,\tau) \leq k_0 e^{-\rho_0 (t-\tau)}$ with some real constants $k_0 \geq 1$ (norm-dependent) and $\rho_0 > 0$ (since A_0 is a stability matrix). It follows from (36) that

$$(37) \quad v(t) := e^{\rho_0 \tau} \|z_A(t^+)\| \leq k_0 \left[\|z_A(0^-)\| + \int_{0^-}^{t^+} \|\tilde{A}(\tau)\| v(\tau)\, d\tau \right],$$

for all $t \geq 0$. Using Gronwall's lemma [**8**], in (37):

$$(38) \quad \|z_A(t^+)\| \leq e^{-\rho_0 t} v(t) \leq k_0 \left[\|z_A(0^-)\| e^{-(\rho_0 t - \int_{0^-}^{t^+} \|\tilde{A}(\tau)\|\, d\tau)} \right]$$

for all $t \geq 0$, but

$$\int_{0^-}^{t^+} \|\tilde{A}(\tau)\| \, d\tau \leq \int_{0^-}^{t^+} \|\tilde{A}'(\tau)\| \, d\tau + \sum_{t_i \in TN} \|A''(t_i)\| \tag{39}$$

and **S2** is GES from(32) for all $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ if $\rho_0 > \underline{\rho_0}$, which is finite since $0 \notin TN$ which implies $A''(0) = 0$.

(ii) $\|z_A(t^+)\| \geq k_0 \|z_A(0^-)\| e^{-(\rho_0 - k_0(a+b)t)}$ for all $t \geq 0$ which yields directly the result.

(iii) Since $A'(t)$ is a stability matrix for any fixed $t \geq 0$, then the Liapunov matrix equation $A'^T(t)P(t) + P(t)A'(t) = -q_0 I$ has a unique solution $P(t) := q_0 \int_0^\infty e^{A'^T(\tau)} e^{A'(\tau)} \, d\tau$ for all $t \notin TN$ and to the left limit t^- of any $t \in TN$, and $\|e^{A'(t)\tau}\| \leq \alpha_A e^{-\sigma_A \tau}$, some (norm-dependent) real constant $\alpha_A \geq 1$ and all real constant $\sigma_A \in [0, \sigma)$ ($\sigma_A \in [0, \sigma)$ if the eigenvalue with largest real part of $A(t)$ is simple). Thus, note that $\int_0^t \|e^{A'(\tau)\tau}\| \, d\tau \leq (\alpha_A/\sigma_A)$. If α_A is related to the l_2-matrix norm $\|P(t)\|_2 = \lambda_{\max}(P(t))$. Then, $\|P(t^-)\|_2 = \lambda_{\max}(P(t)) \leq \beta_2 := (q_0 \alpha_A^2 / 2\sigma_A)$. Taking time-derivatives within any open neighborhood $(t - 2\varepsilon, t - \varepsilon)$ of $t \notin TD$, one gets:

$$A'^T(t^-)\dot{P}(t^-) + \dot{P}(t^-)A'(t^-) = -\,Q(t) := \dot{A}'^T(t^-)P(t^-) + P(t^-)\dot{A}'(t^-) \tag{40}$$

so that, since $A'(t)$ is a stability matrix,

$$\dot{P}(t^-) := \int_0^\infty e^{A'T(\tau)} (\dot{A}'^T(t^-) + P(t^-)A'(t^-)) e^{A'(\tau)} \, d\tau, \quad \text{for all} \quad t \notin TD. \tag{41}$$

From (40), (41),

$$\|\dot{P}(t^-)\|_2 \leq \int_0^t \|e^{A'(t^-)\tau}\|_2^2 \|Q(t^-)\|_2 \, d\tau \leq \beta \|\dot{A}'(t)\|_2, \quad \text{for all} \quad t \geq 0, \tag{42}$$

with $\beta := (\alpha_A^2 \beta_2 / \sigma_A) = (q_0 \alpha_A^4 / 2\sigma_A^2)$ and, for any real $0 < \beta_1 \leq \lambda_{\min}(P)$, one has

$$0 < \beta_1 \leq \lambda_{\min}(P) \leq \beta_2 \leq \frac{q_0 \alpha_A^2}{2\sigma_A}. \tag{43}$$

Then, along the state trajectory of **S2**, the Liapunov function candidate $V(t, z_A(t)) := z_A^T(t)P(t)z_A(t)$, denoted by simplicity as $V(t)$ in what follows, is time-differentiable for all $t \in (t_i, t_{i+1})$ with t_i, t_{i+1}, being two consecutive elements in TN. For all $t \in TN \cup TD$, one has $V(t^+) - V(t^-) = z_A^T(t^+)P(t^+)z_A(t^+) - z_A^T(t^-)P(t^-)z_A(t^-)$ with $P(t^+) = P(t) = P(t^-)$ if $t \in TN \cup \overline{TD}$; $P(t^+) \neq P(t^-)$ if $t \in TD$, $z_A(t^+) \neq z_A(t^-)$ if $t \in TN$. Taking time-derivatives of $V(t)$ in an open neighborhood of $t \notin TN$ and using (43), one gets:

$$\begin{aligned} \dot{V}(t) &= -\|z_A(t)\|_2^2 + z_A^T(t)\dot{P}(t)z_A(t) \\ &= -(\beta_2^{-1}V(t) - \beta\beta_1^{-1}\|\dot{A}'(t)\|_2)V(t) \end{aligned} \tag{44}$$

for all $t \notin TN \cup TD$ and, from (23) and $t_{i+1} - t_i \geq T_1$, for all $t_i \in TN$, one has:

(45)
$$\begin{aligned} \infty > V(0^-) \geq V(t^- + T) &\geq \exp[-(\beta_2^{-1} - \beta\beta_1^{-1}\alpha_1)T] \\ &\times \exp(\beta\beta_1^{-1}\alpha_0) \prod_{t_i \in TN \cup TD} \prod_{t_i \in TN \cup TD} \left[\frac{V(t_i^+)}{V(t_i^-)}\right] V(t^-) \end{aligned}$$

so that $V(t^- + T) \leq e^{-\varepsilon' T}V(t^-) \to 0$ as $t \to \infty$. Thus, the candidate is a Liapunov function provided that

$$0 < \varepsilon' := (\beta_2^{-1} - \beta\beta_1^{-1}\varepsilon)T - \beta\beta_1^{-1}\alpha_0 - \varepsilon \tag{46}$$

such that $e^{-\varepsilon} \geq \prod_{t_i \in TN \cup TD}[V(t_i^+)/V(t_i^-)]$ so that, for some real constant $\varepsilon > 0$,

$$\alpha_1 := \frac{\beta_1}{\beta\beta_2} - \frac{1}{T}\left(\frac{(\varepsilon + \varepsilon')\beta_1}{\beta} + \alpha_0\right) \tag{47}$$

is nonnegative for any prefixed real constant ε' with $0 < \varepsilon' < \mathrm{Min}\,(1 - \varepsilon, (T/\beta_2) - \varepsilon)$ provided that $\varepsilon \in [0, \varepsilon^*)$, $\mathrm{Max}\,(\varepsilon^*, \varepsilon') < (T/\beta_2)$, $\alpha_1 \in [0, \alpha_1^*)$ with

$$\begin{aligned} \alpha_0 &\leq \frac{T}{\beta_2} - (\varepsilon^* + \varepsilon')\,\frac{\beta_1}{\beta}; \\ \alpha_1^* &:= \frac{4\sigma_A^3\,\mathrm{Inf}_{t \geq 0}\,[\lambda_{\min}(P(t^-))]}{q_0^2\,\alpha_A^6} \end{aligned} \tag{48}$$

(iv) The proof is similar to that of (iii) but now the effect of jump discontinuities at TD is taken into account through impulses in $\dot{A}'(t)$ via the replacement $\alpha_0 \to \alpha_0 + \alpha_0'$ to evaluate the upper-bound of $\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\| \, d\tau$. Thus, $\varepsilon^* = \varepsilon_1^*$, and $\prod_{t_i \in S_1(t,t+T)} [V(t_i^+)/V(t_i^-)]$ is replaced with $\prod_{t_i \in TN(t,t+T)} [V(t_i^+)/V(t_i^-)]$.

(v) The part of the proof concerned with condition (24) is as follows from Schwartz's inequality:

$$\begin{aligned} \int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\| \, d\tau &\le \left(\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\|^2 \, d\tau \right)^{1/2} \sqrt{T} \\ &\le (\alpha_1^2 T^2 + \bar{\alpha}_0 T)^{1/2} \le \alpha_1 T + \sqrt{\bar{\alpha}_0 T} \end{aligned} \tag{49}$$

so that

$$\begin{aligned} \infty > V(0^-) \ge V(t^- + T) &\le \exp\left[-\frac{1}{2} (\beta_2^{-1} - \beta\beta_1^{-1}\alpha_1) T \right] \\ &\times \exp\left[-\frac{1}{2} \beta_2^{-1} T + \beta\beta_1^{-1} \sqrt{\alpha_0 T} \right] \\ &\prod_{t_i \in TN(t,t+T) \cup TD(t,t+T)} \left[\frac{V(t_i^+)}{V(t_i^-)} V(t^-) \right] \\ &\le \exp\left[-\left(\frac{1}{2} \beta_2^{-1} - \beta\beta_1^{-1}\alpha_1 \right) T \right] \exp\left[\frac{1}{2} \bar{\alpha}_0 \frac{\beta^2 \beta_2}{\beta_1^2} \right] V(t^-) \\ &\le e^{-\varepsilon' T} V(t^-) \longrightarrow 0 \quad \text{as} \quad t \to \infty, \end{aligned} \tag{50}$$

provided that $\bar{\alpha}_0 < \varepsilon + \varepsilon' < (T/2\beta_2)$ for $\varepsilon \in [0, \varepsilon^*)$, [**26**]. If (25) holds, i.e., the bounded discontinuities in $\dot{A}'(t)$ are considered as impulses in $\dot{A}'(\tau)$ to govern $\int_{t^-}^{t^-+T} \|\dot{A}'(\tau)\| \, d\tau$ then $\bar{\alpha}_0 \to \bar{\alpha}_0 + \bar{\alpha}_0'$ has to fulfil $\bar{\alpha}_0 + \bar{\alpha}_0' \ge (\beta_1/\beta\beta_2)^2 (T - 2\beta_2(\varepsilon + \varepsilon'))$ and the times $t \in TN(t, t+T) \cup TD(t, t+T)$ to evaluate jumps $V(t_i^+)/V(t_i^-)$ in (50) are replaced with $t \in TN(t, t+T)$ leading to the constraint $\alpha_1 \in [0, \bar{\alpha}_1^{*\prime})$ with $\bar{\alpha}_1^{*\prime} := (\beta_1/\beta)((1/2\beta_2) - (\varepsilon_1^*/T))$ for **S2** to be GES for any $\varphi \in \mathrm{IC}\,)[-\bar{r}, 0], \mathbf{R}^n)$.

(vi) Since $\|\dot{A}'\| \in L_2$ implies the integral constraints of $\|\dot{A}'(t)\|$ of items (iii), (iv), the proof follows directly.

(vii) Take a Liapunov function candidate $V(T) := z_A^T(t) P z_A(t)$ with $P = P^T > 0$ being a real symmetric n-matrix. Along any trajectory solution of **S2**, $\dot{V}(T) = -z_A^T(t)[N(t)N^T(t) + q_0 I]z_A(t) \leq 0$ so that **S2** is GES for all bounded $z_0 \in \mathbf{R}^n$. Since $A'(t)$ is a stability matrix for all $t \geq 0$, the solution to (26) for all $t \geq 0$ is:

$$
\begin{aligned}
P &= \int_0^{\infty} \Psi_A^T(\tau, t^-)[N(\tau)N^T(\tau) + q_0 I]\Psi_A(\tau, t^-)\, d\tau \\
&= \int_0^{t^-} \Psi_A^T(\tau, t^-)[N(\tau)N^T(\tau) + q_0 I]\Psi_A(\tau, t^-)\, d\tau,
\end{aligned}
\tag{51}
$$

the second identity arising since $\Psi_A(\tau, t) = 0$ for $\tau > t$, all $t \geq 0$. Direct calculus from (52) yields:

$$
\begin{aligned}
\dot{P}(t^-) = \int_0^{t^-} &[\dot{\Psi}_A^T(\tau, t^-)[N(\tau)N^T(\tau) + q_0 I]\Psi_A(\tau, t^-) \\
&+ \Psi_A^T(\tau, t^-)[N(\tau)N^T(\tau) + q_0 I]\dot{\Psi}_A(\tau, t^-)]\, d\tau \\
&+ \Psi_A^T(t^-, t^-)[N(t^-)N^T(t^-) + q_0 I]\dot{\Psi}_A(t^-, t^-)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&= A^T(t^-)\left\{ \int_0^{\infty} \Psi_A^T(\tau, t^-)[N(\tau)N^T(\tau) + q_0 I]\Psi_A(\tau, t^-)\, d\tau \right\} \\
&\quad + \left\{ \int_0^{\infty} \Psi_A^T(t, t^-)[N(\tau)N^T(\tau) + q_0 I]\Psi_A(\tau, t^-)\, d\tau \right\} A(t^-) \\
&\quad + N(t)N^T(t) + q_0 I \\
&= A'^T(t)P + PA'(t) + N(t)N^T(t) + q_0 I = 0,
\end{aligned}
\tag{53}
$$

by using the properties $\Psi_A(t, t) = I$ and $A'(t)$ and $\Psi_A(t, \tau)$ commute for all t, τ since $\Psi_A(t, \tau)$ is a fundamental matrix of **S2**. Now, since $[A'(t), N^T(t)]$ is uniformly completely observable, if $q_0 = 0$ and $P = P^T > 0$, equations (52), (53) satisfy two Liapunov matrix equations (26); one has from (52) for $q_0 = 0$ for some real $T > 0$, $\alpha > 0$, one has

$$
\begin{aligned}
0 < \alpha I &\leq \int_{t^-}^{t^- + T} \Psi_A^T(\tau, t^-)N(\tau)N^T(\tau)\Psi_A(\tau, t^-)\, d\tau \\
&\leq P = \int_{0^-}^{t^- + T} \Psi_A^T(\tau, t^-)N(\tau)N^T(\tau)\Psi_A(\tau, t^-)\, d\tau \\
&\leq \lambda_{\max}(P) I
\end{aligned}
\tag{54}
$$

so that $\lambda_{\max}(P) \geq \alpha$. If $q_0 > 0$ then $\dot{V}(t) \leq -q_0 V(t)$. Also, one gets directly by defining $\lambda(t) := (V(t_i^+)/V(t_i^-))$ for all $t_i \in TN$:

(55)

$$\begin{aligned}
&\lambda_{\min}(P)\|z_A(t^- + T)\|_2^2 - \lambda_{\max}(P)\|z_A(t^-)\|_2^2 \\
&\quad \leq V(t^- + T) - V(t^-) \\
&\quad \leq -\prod_{t_i \in TN(t,t+T)} [\lambda(t_i)] \Bigg(\sum_{t_i \in TN(t,t+T)} z_A^T(\tau) N(\tau) N^T(\tau) z_A(\tau)\, d\tau \Bigg) \\
&\quad = -\prod_{t_i \in TN(t,t+T)} [\lambda(t_i)] z_A^T(t) \\
&\qquad \times \left(\int_{0^-}^{t^- + T} \Psi_A^T(\tau, t^-) N(\tau) N^T(\tau) \Psi_A(\tau, t^-)\, d\tau \right) z_A(t) \\
&\quad \leq -\alpha \vartheta_T(t) \|z_A(t)\|_2^2 \leq -\alpha \frac{\vartheta_T(t) V(t^-)}{\lambda_{\max}(P)}
\end{aligned}$$

where $\vartheta_T(t) := \prod_{t_i \in TN(t,t+T)} [\lambda(t_i)]$ since $\lambda_{\max}(P) \geq \alpha$. Thus, one gets from (55)

$$V(t^- + T) \leq \left(1 - \frac{\alpha \vartheta_T(t)}{\lambda_{\max}(P)}\right) V(t^-); \quad \text{for all} \quad t \geq 0, \tag{56}$$

$$\|z_A(t^- + T)\|_2^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \left(1 - \frac{\alpha \vartheta_T(t)}{\lambda_{\max}(P)}\right) \|z_A(t^-)\|_2^2 \tag{57}$$

and **S2** is GES for all bounded $z_A(0) = z_0 \in \mathbf{R}^n$ if (27) holds and the proof of (vii) is complete.

Alternative condition via the condition number of P for the l_2-matrix norm $(K_2(P))$.

(58)

$$\begin{aligned}
&\frac{\alpha K_2(P)}{\lambda_{\min}(P)} \prod_{t_i \in TN(t,t+T)} [\lambda_{\max}(I + A''^T(t_i))(I + A''(t_i))] \\
&\quad \leq K_2^{\nu}(P) \prod [\lambda_{\max}((I + A''^T(t_i))(I + A''^T(t_i))(I + A''(t_i)))] < \frac{1}{\alpha}
\end{aligned}$$

where $\nu \in \operatorname{Card}[TN(t, t+T)]$.

(viii) The proof follows directly from that of (vii) since $V(t_{i+1}) \leq \delta(T_*)V(t_i)$ for some $0 < \delta(T_*) < 1$, for all $t_i \in S_i$ if $[A'(t), N^T(t)]$ is completely observable on $[t_i, t_{i+1})$ or, if $q_0(t_i) \geq q_0 > 0$, for all $t_i \in S_t$. Define the sets $D := TN \cup TN_d \cup TD \cup D\overline{D}$ and $D_e := TN \cup TN_d \cup TD \cup (DD \cup C\overline{D})$ where DD and $C\overline{D}$ are the real subsets where the delay is discontinuous and continuous, respectively.

(ix) Consider the Liapunov function candidate (30). It is first proved that $\int_{t^- - r(t)}^{t^-} z^T(\tau)K(\tau)z(\tau)\,d\tau$ does not diverge faster than $r(t)(\mathrm{Sup}_{0\leq\tau\leq t}\,(\|z(\tau)\|_2^2)$ where $K(t) := A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))$. Direct calculus yields, [**9**]:

$$\begin{aligned}\Delta V(t) := V'(t) - V(t) &= \int_{t-r(t)}^{t} \|A_d'(h^{-1}(\tau))z(\tau)\|_2^2\,d\tau \\ &\leq \mathrm{ess}\sup_{t-r(t)\leq\tau\leq t}(\|z(\tau)\|_2^2)\int_{1/(t^- - r(t))}^{1/t^-}\|A_d'(h^{-1}(\tau))\|_2^2\,\dot h(\tau)\,d\tau \\ &\leq \mu_a r(t)\underset{t-r(t)\leq\tau\leq t}{\mathrm{ess}}(\|z(\tau)\|_2^2)\end{aligned}$$

for all $t \notin C\overline{D} \cup DD$ for any vector and associate (induced) matrix norm and some $\mu_a \in \mathbf{R}_0^+$. Since $\|A_d'(1/t)\|$ is bounded if $t \in C\overline{D} \cup DD$, $\mu_b \in \mathbf{R}^+$ exists such that $\Delta V(t) \leq \mu_b r(t)\mathrm{ess}\,\mathrm{Sup}_{t-r(t)\leq\tau\leq t}(\|z(\tau)\|_2^2)$. Also,

$$\begin{aligned}\dot V'(t) &= 2\dot z^T(t)P(t)z(t) + z^T(t)K(t)z(t) \\ &\quad - \dot h(t)z^T(h(t))K(h(t))z(h(t)) - z^T(t)\dot P(t)z(t) \\ &= z^T(t)[A'^T(t)P(t) + P(t)A'(t) + K(t)]z(t) \\ &\quad + 2z^T(t)P(t)A_d'(t)z(h(t)) \\ &\quad - (1 - r'(t))z^T(h(t))K(h(t))z(h(t)) - z^T(t)\dot P(t)z(t)\end{aligned} \tag{59}$$

for all $t \notin C\overline{D}$ where $r'(t) := 1 - \dot r(t)$. Assume that $q \in \mathbf{R}^+$ exists such that

$$q_0 \geq q + \sup_{t\notin C\overline{D}}[A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))] + \frac{q_0^2\alpha_A^4}{4\sigma_A^2} \tag{60}$$

with q_0, σ_A and α_A defined in the proof of (iii). Direct calculus guarantees (60) if for all $t \notin D_e$:

$$0 < q_0 < \frac{4\sigma^2}{\alpha_A^4}; \quad \lambda_{\max}[A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))] < q_0\left(1 - \frac{q_0\alpha_A^4}{4\sigma_A^2}\right). \tag{61}$$

Then,

$$A'^T(t)P(t) + P(t)A'(t) + \dot{P}(t) = -q_0 I \le -\left(qI + P^2(t) + A_d'^T(h^{-1}(t))A_d'(h^{-1}(t))\right) \tag{62}$$

for $t \notin D_e$ provided that (60) holds via (61). Then, the substitution of (62) into (59) and the use of item (iii) yields, provided that (61) holds, that

$$\begin{aligned} \dot{V}(t) &\le -\left(q_0\beta_2^{-1} - \beta\beta_1^{-1}\|\dot{A}'(t)\|_2^2\right) z^T(t)P(t)z(t) \\ &\quad - \left\| \frac{1}{\sqrt{1-r'(t)}}\, P(t)z(t) - \sqrt{1-r'(t)}\, A_d(t)z(h(t)) \right\|_2^2 \\ &\le -(q_0\beta_2^{-1} - \beta\beta_1^{-1}\|\dot{A}'(t)\|_2)\, z^T(t)P(t)z(t) \le 0, \end{aligned} \tag{63}$$

where $\|\cdot\|_2$ denotes the l_2 vector and associated induced matrix norms. Thus, since $V'(t) \ge z^T(t)P(t)z(t)$, by construction:

$$\frac{\dot{V}'(t)}{V'(t)} \le \frac{\dot{V}'(t)}{z^T(t)P(t)z(t)} \le -q < 0, \quad \text{for all} \quad t \notin D_e. \tag{64}$$

If $t_i \in C\overline{D}$ and $C\overline{D} \cap (TN \cup TN_d) = \varnothing$, then $V'(t_i^+) = V'(t_i^-)$ (so that $C\vec{D}$ is irrelevant for stability analysis purposes). If $t_i \in (TN \cup TD \cup \cup DD \cup TN_d)$, then $V'(t_i^+) \ne V'(t_i^-)$, in general, through respective jump discontinuities $\dot{z}(t_i^+) \ne \dot{z}(t_i^-)$; $P(t_i^+) \ne P(t_i^-)$, see item (vi), and

$$\Delta V(t_i^+) = V'(t_i) - z^T(t_i)P(t_i)z(t_i) = \int_{t_i - r(t_i)}^{t_i} \|A_d'(h^{-1}(\tau))z(\tau)\|_2^2\, d\tau.$$

The various discontinuities at $V'(t_i^+)$ may be combined since the above discontinuity sets are not required to be disjoint. Thus, if $e^{-qT}[\prod_{t_i \in D(t,t+T)}[\lambda(t_i)]] < 1$ and taking into account (64),

$$V'(t+T) \le e^{-qT} \prod_{t_i \in D(t,t+T)} [\lambda(t_i)]V'(t_i) \le V'(t^-), \quad \text{for all} \quad t \ge 0$$

with $D(t+T) := \{\tau \in D : t \le \tau < t+T\}$ and $\lambda(t_i) := V'(t_i^+)/V'(t_i^-)$. It has been proved that **S2** is GES if item (iii) holds. The proof is quite similar if **S2** satisfies any of items (iv)–(vi).

(x) It follows directly since the Liapunov matrix inequality of item (viii) implies that of (62) with a constant positive finite symmetric matrix P. Thus, the Liapunov function candidate $V'(t)$ for **S3**, which is well-posed from the assumption on $A'_d(t)$ has a time-derivative satisfying (63) for all $t \notin D$ which leads directly to the proof since $V(t^- + T) \leq e^{-qT} V(t^-) \leq \vartheta_T(t) V(t^-)$, $0 < \vartheta_T(t) < 1$, and

$$q < q^* := q_0 \left(1 - \frac{\alpha_A^4 \, q_0}{4\sigma_A^2} \right) \operatorname*{Sup}_{t \notin D(t,t+T)} [\| A'_d(h^{-1}(t)) \|_2^2],$$

see the proof of (ix).

(xi) Take the Liapunov function candidate $V(t) = z^T(t) P z(t)$ for **S3** so that for all $t \notin D$ along the state trajectory generated from any $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ satisfies $\dot{V}(t) = 2z^T(t) P \dot{z}(t) = -z^T(t) N(t) N^T(t) z(t)$, so that for all $t \geq T > 0$:

$$(65)\quad V(t^-) - V(0^-) = -\,\bar{z}_0^T \left(\int_{0^-}^{t^-} T^T(\tau, 0) N(\tau) N^T(\tau) T(\tau, 0) \, d\tau \right) \bar{z}_0,$$

with $\bar{z}_0 = z_0^- + \int_{-\bar{r}}^0 T^{-1}(t^-, 0) T(t^- - \tau) \varphi(\tau) \, d\tau$. Note that, since $\|T(t, \tau)\|$ is of exponential order and φ is essentially bounded on $[-\bar{r}, 0]$, except at most at a finite set of time instants where impulses take place, as a direct consequence of Theorem 1 (iii), there are finite real constants $\bar{\rho} \geq 0$, $k_\varphi \geq 0$ and $k_T \geq 1$ such that $\|\bar{z}_0\| \leq \|z_0\| + k_{T1} \leq k_\varphi \|\bar{z}_0\|$, some nonnegative $k_{T1} := k_T e^{\bar{r}\bar{\rho}} (\operatorname{ess\,Sup}_{-\bar{r} \leq \tau < 0} (\|\varphi(\tau)\|) + k_1 \varphi) < \infty$.Thus, if $M(t) := \|T(t, 0)\|_2^2$, then

$$\begin{aligned} \|z(t^-)\|_2 &= \|T(t^-, 0) \bar{z}_0\|_2 \leq M^{1/2}(t^-) \|z_0\|_2 \\ &\leq M^{1/2}(t^-) k_\varphi \|\bar{z}_0^-\|_2 \leq \frac{M^{1/2}(t^-) k_\varphi V(0)^{1/2}}{\lambda_{\min}(P)} \end{aligned}$$

so that, from (65),

$$V(t^-) \leq \left(1 - \frac{\alpha k_\varphi M(t) \vartheta'_T(t)}{\lambda_{\max}(P)} \right) V(0^-)$$

so that **S3** is GES for all $\varphi \in \mathrm{IC}\,([-\bar{r}, 0], \mathbf{R}^n)$ and then **S** is globally stable for all $u \in L_2^m$ since from the Cauchy-Buniakovski inequality, [**3,**

22], one has for the l_2-vector and associate induced matrix norm:

$$\begin{aligned}\Big\| \int_0^\infty T(t,\tau)B(\tau)u(\tau)\,d\tau \Big\|_2 \\ \leq \operatorname*{Sup}_{t\geq 0}\left[\left(\int_0^\infty \|T(t,\tau)\|_2^2\,d\tau\right)^{1/2}\right]\left(\int_0^\infty \|u(\tau)\|_2^2\,d\tau\right)^{1/2} < \infty,\end{aligned}$$

since $B \in L_\infty^{m\times n}$, $u \in L_2^m$ and $T(t,\tau) \in L_2([0,\infty), \mathbf{R}^{m\times n})$. Thus, the system $\mathbf{S}$ is GES for all $u \in L_2^m$.

(xii) It is very similar to that of (xi) by building the solution $z(t^- + T) = T(t^- + T, t)\bar{z}(t)$ where $\bar{z}(t) = z(t^-) + \int_{t^-}^{t^-+T} T^{-1}(\tau', t)z(\tau')\,d\tau$. □

4.3 *Closed-loop stabilization.* Theorem 2 may be easily extended to the case when $u(t)$ is generated from $x(t)$ or $y(t)$, namely, through the state or output measurements, in the context of the so-called closed-loop stabilization problem via a linear feedback regulator. Assume that the state-feedback control law:

(66)

$$\begin{aligned}u(t) = u'(t) + u''(t) = \{(K_0K_c(t))x(t) + K_d(t)x(t-r(t))\} \\ + \Bigg\{ \sum_{t_i\in TN} K_{\text{cimp}}(t_i)\delta(t-t_i)x(t_i^-) \\ + \sum_{t_i\in TN_d} K_{\text{dimp}}(t_i)\delta(t-t_i)x(t_i^- - r(t_i))\Bigg\}\end{aligned}$$

where $K_c(t)$, $K_d(t)$ are bounded matrix functions from $[0,\infty)$ to $\mathbf{R}^{n\times m}$ allowed to have removable and/or jump discontinuities in the sets TD and TD_d, respectively, and $K_{\text{cimp}}(t)$, $K_{\text{dimp}}(t)$ are (impulsive) matrix functions of ranges in $\mathbf{R}^{n\times n}$ with supports of zero Lebesgue measure TN and TN_d, respectively. Thus, the closed-loop system is GE for all $\varphi \in \mathrm{IC}\,([-\bar{r},0],\mathbf{R}^n)$ and the control law is stabilizing if and only if the results of Theorem 2 (ix)–(xii) hold for the subsequent replacements: $A_0 \to A_0 + B(t)K_0$, $A'(t) \to A'(t) + B(t)K_c(t)$, $A_d'(t) \to A_d'(t) + B(t)K_d(t)$ for all $t \in \mathbf{R}_0^+$, $A''(t) = 0$ for all $t \notin TN$; $A_d''(t) = 0$ for all $t \notin TN_d$; and

$$\begin{aligned}A''(t_i) &\longrightarrow A''(t_i) + B(t_i)K_{\text{cimp}}(t_i) \quad \text{for all} \quad t_i \in TN, \\ A_d''(t_i) &\longrightarrow A_d''(t_i) + B(t_i)K_{\text{dimp}}(t_i) \quad \text{for all} \quad t_i \in TN_d.\end{aligned} \tag{67}$$

(67) If output-feedback, instead of state-feedback, is used and $D(t) = 0$ in (1.b), then the control law (66) is substituted with a similar one with the replacements $x(.) \to C(.)x(.)$ everywhere in its righthand side while (67) is reformulated with the controller gain replacements $K_{0,c,d,\mathrm{cimp},\mathrm{dimp}}(t) \to K_{0,c,d,\mathrm{cimp},\mathrm{dimp}}(t)C(t)$, $K_0(t) = K_0$ being constant. If $D(t) \neq 0$, then the control law becomes implicit in $u(t)$ and some rank conditions have to be fulfilled in order to convert it into an explicit form, one of them being, for instance, that the matrix $(I - (K_0 + K_c(t))D(t))$ is nonsingular for all $t \notin (TN \cup TN_d)$.

5. Compactness of some of the input/output, input/state and state/output operators defining the state/output trajectory solutions. Note that the Banach space $U \subset L_2^m$ is also a real Hilbert space endowed with the (semi-)norm of the inner product $\|\cdot\|_{L_2^m}$ defined by $\|u\|_{L_2^m} := \langle u, u\rangle_{L_2^m}^{1/2}$, for all $u \in U$. The subscript for the space L_2^m for inner products and associated endowed norms are omitted in the sequel when no confusion is expected. Let $\{\phi_i^m\}_1^\infty$ and $\{\theta_i^{(m)*}\}_1^\infty = \{\theta_i^{(m)T}\}_1^\infty$ be an orthonormal basis and its reciprocal one for the L_2^m-space (this notation for orthonormal basis is adopted independently of the dimension m which is easily elucidated depending on context) so that:

$$
\begin{aligned}
&\langle \phi_i^{(m)}, \phi_j^{(m)}\rangle = \delta_{ij}; \quad \langle \theta_i^{(m)}, \theta_j^{(m)}\rangle = \delta_{ij};\\
&\langle \phi_i^{(m)}, \phi_j^{(m)}\rangle = \delta_{ij}; \quad i, j = 1, 2, \dots, \infty,
\end{aligned}
\tag{68}
$$

with δ_{ij} being the Kronecker delta, [**24**]. Then, the evolution operator $T(t,\tau)$ has a representation

$$
T(t,\tau) = \sum_{k=1}^{\infty} \psi_l(t)\theta_k^{(m)T}(\tau) = \sum_{k=1}^{\infty} \int_0^t T(t,\tau)\phi_k(\tau')\theta_k^{(m)^T}(\tau)\, d\tau' \tag{69}
$$

where $\Psi_k(t) := \int_{-\infty}^{\infty} T(t,\tau)\phi_k^{(m)}(\tau)\, d\tau = \int_0^t T(t,\tau)\phi_k^{(m)}(\tau)\, d\tau$, since $T(t,\tau) = 0$ for $t \geq \tau$, is an n-matrix function which is the image of ϕ_k via the operator T. From Theorem 1, the state and output trajectories are:

$$
x(t) = (S\varphi)(t) + (S^u u)(t); \qquad y(t) = (M\varphi)(t) + (M^u u)(t) \tag{70}
$$

for all $t \in \mathbf{R}_0^+$ so that for zero initial conditions:

$$
\begin{aligned}
[x(t^-)]_{\varphi\equiv 0} &= (S^u u)(t^-) = \int_0^{t^-} [H_x(t^-,\tau)u(\tau)\,d\tau; y(t^-)]_{\varphi\equiv 0} \\
&= (M^u u)(t^-) = \int_0^{t^-} H(t^-,\tau)u(\tau)\,d\tau
\end{aligned} \tag{71}
$$

(71) with similar expressions for $x(t^+)$ and $y(t^+)$ with the appropriate replacements $t^- \to t^+$. The kernels of the operators $S^u \in \mathbf{L}(U,X)$ and $M^u \in \mathbf{L}(U,Y)$ are, respectively $H_x := [0,\infty) \to \mathbf{L}(U,X)$ and $H : [0,\infty) \to \mathbf{L}(U,Y)$, respectively. Let now $\{\phi_i\}_1^\infty \equiv \{\phi_i^{(1)}\}_1^\infty$ be an orthonormal basis of the real (scalar) space L_2. It follows that $\{\phi_{i_j}^{(l)}\}_1^\infty \equiv \{e_j^{(l)}\phi_i\}_1^\infty$, $i_j, j = 1,2,\dots,l$, is an orthonormal basis of L_2^l (any $l \in \mathbf{Z}^+$) for unity Euclidean vectors $e_j^{(l)} \in R^l$, i.e., its jth component is unity while the remaining ones are zero for $j = 1,2,\dots,l$ so that the inner product satisfies $\langle e_j^{(m)}\phi_i, e_l^{(m)}\theta_k\rangle = \delta_{jl}\delta_{ik}$. Then,

$$\varphi(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj} e_j^{(n)} \phi_k(t)$$

with

$$x_0 = \varphi(0) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj} e_j^{(n)} \phi_k(0)$$

for all $\varphi \in \mathrm{IC}\,([-\bar{r},0],\mathbf{R}^n)$ admitting a component-wise representation

$$\varphi_j(t) = \sum_{k=1}^{\infty} \alpha_{\varphi kj}\phi_k(t),$$

with

$$x_j(0) = \varphi_j(0) = \sum_{k=1}^{\infty} \alpha_{\varphi kj}\phi_k(0),$$

where the $\alpha_{\varphi kj}$, $j = 1,2,\dots,n$, $k = 0,1,\dots$, are the (real constant) components of φ_j in the basis $\{\varphi_k\}_1^\infty$. An artifice is now used to represent (possibly impulsive) inputs $u \in L_2^m$ of the class of the above sections. Such an artifice consist of introducing time-varying components which are essentially discontinuous (and not constant) only at discontinuity points so that $u(t)$ takes the form

$$u(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{kj}(t) e_j^{(m)} \phi_k(t)$$

with time-varying components $\alpha_{kj}(t) = \alpha_{kj}^{-} + \alpha_{kj}^{\delta}\mu_u(t)\delta(0)$ so that:

$$u(t^-) = u'(t^-) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{kj}^{-} e_j^{(m)} \phi_k(t)$$

$$\begin{aligned} u(t^+) = u'(t^+) + u''(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{kj} e_j^{(m)} \phi_k(t) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n} (\alpha_{kj}^{-} + \alpha_{kj}^{\delta}\mu_u(t)\delta(0)) e_j^{(m)} \phi_k(t), \end{aligned}$$

i.e., $\alpha_{kj}(t^-) = \alpha_{kj}^{-}$ and $\alpha_{kj}(t^+) = \alpha_{kj}^{-} + \alpha_{kj}^{\delta}\mu_u(t)\delta(0)$ and having (possibly discontinuous) components to the left and right of any $t \in \mathbf{R}_0^+$ represented by

$$u_j(t^{\mp}) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{kj}(t^{\mp}) \phi_k(t).$$

Note that the above representation is well-posed since $u, u', u'' \in L_2^m$ so that both u' and u'' admit representations with real constant components α_{kj}^{-} and $\alpha_{kj}^{\delta}\delta(\tau - t)$ for input impulses at $\tau = t$. By convenience, the representations of operators for zero initial state and zero input are discussed separately.

5.1 *Representations of the zero-state relevant operators.* Note that the state trajectory solution satisfies

(72)
$$[x(t)]_{\varphi\equiv 0} = (S^u u)(t) = \int_0^t H_x(t,\tau) u(\tau)\, d\tau = \int_0^t T(t,\tau) B(\tau) u(\tau)\, d\tau,$$

with possible eventual discontinuities. Thus, the kernel $H_x(t,\tau) = T(t,\tau)B(\tau)$ of $S^u \in \mathbf{L}(U, X)$ admits the representation:

$$H_x(t,\tau) = \sum_{k=1}^{\infty} \psi_{xk}(t) \theta_k^T(\tau), \tag{73}$$

where

$$\begin{aligned}\psi_{xk}(t) &= \sum_{j=1}^{m} \int_0^t \alpha_{kj}(t) T(t,\tau) B(\tau) (e_j^{(m)} \phi_k(\tau))\, d\tau \\ &= \sum_{k=1}^{\infty} \int_{k=1}^{\infty} \int_0^t T(t,\tau) B(\tau) \underline{\alpha_k}(t) \phi_k(\tau)\, d\tau\end{aligned}$$

is the n-vector state for zero initial conditions with input

$$\sum_{j=1}^{m} \alpha_{kj}(t) e_j^{(m)} \phi_k(t)$$

with

$$\underline{\alpha_k}(t) = \sum_{j=1}^{m} \alpha_{kj}(t) e_j^{(m)}$$

since

$$u_j(t) = \sum_{k=1}^{\infty} \alpha_{kj}(t)\phi_k(t) = \sum_{k=1}^{\infty} \underline{\alpha_k}(t)\phi_k(t),$$

and

$$u(t) = \sum_{\substack{k_j=1 \\ 1\le j\le m}}^{\infty} \underline{\alpha_k}_j(t)(e_j^{(m)}\phi_k(t)) = \sum_{k=1}^{\infty}\sum_{j=1}^{m} \alpha_{k_j}(t) e_j^{(m)} \phi_k(t)$$

where $\underline{\alpha_k}(t) = (\alpha_{k1}(t), \alpha_{k2}(t), \dots, \alpha_{km}(t))^T \in \mathbf{R}^m$. In the same way,

$$\begin{aligned}y(t) &= (M^u u)(t) = \int_0^t H(t,\tau) u(\tau)\, d\tau \\ &= \int_0^t (C(t)T(t,\tau)B(\tau) + D(t)\delta(t-\tau)) u(\tau)\, d\tau\end{aligned} \tag{74}$$

so that

$$y(t^-) = \int_0^{t^-} H(t^-,\tau)u(\tau)\, d\tau = \int_0^{t^-} C(t)T(t^-,\tau)B(\tau)u(\tau)\, d\tau$$

$$\begin{aligned}y(t^+) &= \int_0^{t^+} H(t^+,\tau)u(\tau)\, d\tau \\ &= \int_0^{t^+} C(t)T(t^+,\tau)B(\tau)u'(\tau)\, d\tau + D(t)\delta(0)\mu_u(t)u''(t)\end{aligned} \tag{75}$$

with the operator $H(.,.)$ admitting a representation, [**24**]:

$$H(t^-,\tau) = \sum_{k=1}^{\infty} \Psi_k(t^-)\theta_k^T(\tau); \qquad H(t^+,\tau) = \sum_{k=1}^{\infty} \Psi_k(t^+)\theta_k^T(\tau) \tag{76}$$

$$\begin{aligned} \Psi_k(t^-) &= \sum_{j=1}^{m} \int_0^{t^-} \alpha_{kj}(t^-)C(t)T(t^-,\tau)B(\tau)e_j^{(m)}\phi_k(\tau)\,d\tau \\ \Psi_k(t^+) &= \sum_{j=1}^{m} \int_0^{t^+} \alpha_{kj}(t^+)(C(t)T(t^+,\tau)B(\tau) \\ &\qquad + D(\tau)\delta(\tau-t)\mu_u(\tau))e_j^{(m)}\phi_k(\tau)\,d\tau \\ &= \sum_{j=1}^{m} \alpha_{kj}(t^+)\bigg(\int_0^{t^+} C(t)T(t^+,\tau)B(\tau)\phi_k(\tau)\,d\tau \\ &\qquad + D(t)\delta(0)\mu_u(t)\phi_k(t)\bigg)e_j^{(m)}. \end{aligned} \tag{77}$$

A quite similar representation may be obtained for the operator

$$T(t^{\mp},\tau) = \sum_{k=1}^{\infty} \Psi_{Tk}(t^{\mp})\theta_k^T(\tau),$$

provided that $Bu \in L_2^n$, by replacing the real vector functions $\Psi_k(t^{\mp})$ by vector functions $\Psi_{Tk}(t^{\mp})$ obtained by fixing $C(t) = I_n$ in the right-hand sides of (77) and the use of the representation:

$$\begin{aligned} B(t)u(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{m} \bar{\alpha}_{kj}(t)e_j^{(n)}\phi_k(t); \\ \Psi_{TK}(t^{\mp}) &= \sum_{j=1}^{m} \int_0^{t^{mp}} \bar{\alpha}_{kj}(t^{\mp})T(t^{\mp},\tau)e_j^{(n)}\phi_k(\tau)\,d\tau. \end{aligned}$$

As a result,

$$\begin{aligned} x(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n} \beta_{xkj}(t)e_j^{(n)}\phi_k(t) \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{n} \alpha_{lj}(t)\Psi_{tk}(t)\int_0^t \theta_k^T(\tau)e_j^{(n)}\phi_l(\tau)\,d\tau, \end{aligned}$$

$$
\begin{aligned}
y(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{p}\beta_{kj}(t)e_j^{(p)}\phi_k(t)\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{n}\alpha_{lj}(t)C(t)\Psi_{Tk}(t)\int_0^t\theta_k^T(\tau)e_j^{(n)}\phi_l(\tau)\,d\tau,
\end{aligned}
$$

where $\beta_{xki}(t)$ and $\beta_{xk}(t)$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, p$, $k = 0, 1, \dots,$ are the components of $x : [0,\infty) \to \mathbf{R}^n$ and $y : [0,\infty) \to \mathbf{R}^p$ with respect to the two respective orthonormal bases $\{e_j^{(n)}\phi_k\}_1^\infty$ and $\{e_j^{(p)}\phi_k\}_1^\infty$.

5.2 *Representations of the zero-input relevant operators.* Since the function of initial conditions $\varphi \in \mathrm{IC}\,([-\bar{r},0],\mathbf{R}^n)$ has the form $\varphi(t) = \varphi_0(t) + (\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ with domain of finite measure, $\varphi_0(t)$ being uniformly bounded and $(\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ possessing a support of zero measure and a domain of finite measure. Then, φ, φ_0 and $(\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ are in L_2^m while $\varphi(t^+) \neq \varphi(t^-)$ if $\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t) \neq 0$. Then it is possible to represent $\varphi(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}(t)e_j^{(n)}\phi_k(t)$ with left and right limits:

$$
\begin{aligned}
\varphi(t^-) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}^{-}e_j^{(n)}\phi_k(t),\\
\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}(\alpha_{\varphi kj}^{+} - \alpha_{\varphi kj}^{-})e_j^{(n)}\phi_k(t)\\
\varphi(t^+) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}^{+}e_j^{(n)}\phi_k(t)\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}^{-}e_j^{(n)}\phi_k(t) + \sum_{k=1}^{\infty}\sum_{j=1}^{n}(\alpha_{\varphi kj}^{+} - \alpha_{\varphi kj}^{-})e_j^{(n)}\phi_k(t)
\end{aligned}
$$

with real components $\alpha_{\varphi kj}(t^-) = \alpha_{kj}^{-}$ and $\alpha_{\varphi kj}(t^+) = \alpha_{\varphi kj}^{+} = \alpha_{\varphi kj}^{+} = \alpha_{kj}^{-} + \alpha_{kj}^{\delta}\mu_\varphi(t)\,\delta(0)$ in a similar way as argued for the input representation. The zero-input trajectory of $\mathbf{S}$ is given by:

$$x(t) = (S\varphi)(t) = \sum_{k=1}^{\infty} \Psi_{Tk}(t) \Bigg[\sum_{l=1}^{\infty} \sum_{j=1}^{n} \alpha_{\varphi lj} \Bigg(\theta_k^T(0) e_j^{(n)} \phi_l(0) + \int_{-\bar{r}}^{0} \theta_k^T(\tau+\bar{r}) e_j^{(n)} A_d(\tau+\bar{r}) \phi_l(\tau+\bar{r}-r(\tau)) \Bigg) \Bigg], \tag{78}$$

while its zero-input output state trajectory follows directly from the above expression by pre-multiplying its righthand side by $C(t)$ to obtain a point-wise representation of the operator M via $y(t) = (M\varphi)(t) = C(t)x(t) = (C(t)S\varphi)(t)$.

5.3 *Representations of the zero-state relevant operators associated with gate operators.* In some applications of electronics, the use of so-called gate operators is relevant where the input is impulsive of the form

$$u(t) = u''(t) = \sum_{t_i \in TU} \omega(t_i)\delta(t - t_i) u_0(t)$$

with $u_0 \in L_2^m$ being the reference input and $\omega : [0, \infty) \to \mathbf{R}$ being the piecewise continuous weighting function. Then, one has the point-wise operator definitions:

$$\begin{aligned} (S^u u)(t) &= \int_0^t H_x(t,\tau) u(\tau)\, d\tau \\ &= \int_0^t \sum_{t_i \in TU} T(t,\tau) B(\tau) \omega(\tau) \delta(\tau - t_i) u_0(\tau)\, d\tau, \end{aligned} \tag{79}$$

$$(S^{u_0} u_0)(t) = \sum_{t_i \in TU} T(t,t_i) B(t_i) \omega(t_i) u_0(t_i) \tag{80}$$

$$\begin{aligned} (M^u u)(t) &= \int_0^t H(t,\tau) u(\tau)\, d\tau \\ &= \int_0^t \sum_{t_i \in TU} C(t) T(t,\tau) B(\tau) \omega(\tau) \delta(\tau - t_i) u_0(\tau)\, d\tau \\ &\quad + D(t) \mu_u(t) \omega(t) \delta(0) u_0(t) \end{aligned} \tag{81}$$

$$\begin{aligned} (M^{u_0} u_0)(t) &= \sum_{t_i \in TU} C(t) T(t,t_i) B(t_i) \omega(t_i) u_0(t_i) \\ &\quad + D(t) \mu_u(t) \omega(t) \delta(0) u_0(t). \end{aligned} \tag{82}$$

The following result holds.

Theorem 3. *The operator* $(S^{u_0}u_0)(t)$*, defined in* (80)*, is a Hilbert-Schmidt operator if* $B : [0, \infty) \to R^{m \times n}$ *is in* $L_\infty^{m\times n}$*,* $T(t,\tau) \in L_2^{n\times n}$ *as a function of* τ *over* $[0, \infty)$ *for all* $t \in \mathbf{R}_0^+$ *and* $\sum_{t_i \in TU} \omega^2(t_i) < \infty$ *(which holds if* $\omega \in L_2$*).*

The operator $(M^{u_0}u_0)(t)$*, defined in* (82)*, is a Hilbert-Schmidt operator if* $C : [0, \infty) \to \mathbf{R}^{n\times p}$ *is in* $L_\infty^{n\times p}$*,* $T(t,\tau) \in L_2^{n\times n}$ *as a function of* τ *over* $[0, \infty)$ *for all* $t \in \mathbf{R}_0^+$*,* $\sum_{t_i \in TU} \omega^2(t_i) < \infty$ *and* $D(t) \equiv 0$ *for all* $t \in \mathbf{R}_0^+$*.*

Proof. The operators $(S^{u_0}u_0)(t)$ and $(M^{u_0}u_0)(t)$ are Hilbert-Schmidt if their kernels are Hilbert-Schmidt, i.e., square-integrable on $(-\infty, \infty)$, [**19, 22**], which holds under the given respective conditions. □

5.4 *Compactness of the relevant input-state, state-output and input-output operators.* The main well-known properties of compact operators, which are useful to approximate infinite dimensional spaces by finite dimensional ones; those used are the following [**19, 22**]:

(i) A Hilbert-Schmidt operator is compact.

(ii) An operator of finite dimensional range is compact.

(iii) A degenerate operator from a Banach space to a Banach space which is the limit of operators of finite dimensional ranges is compact.

(iv) An operator of closed range which is compact has a finite-dimensional range.

From the developments of subsections 5.1–5.3, Theorem 3, and the above list of properties, the subsequent result holds for relevant operators defined in (78)–(82) and some related operators:

Theorem 4. (i) *The operators* $S \in \mathbf{L}(L_2^n, L_2^n)$*,* $M \in \mathbf{L}(L_2^n, L_2^p)$*,* $SS^* \in \mathbf{L}(L_2^n, L_2^n)$*,* $MM^* \in \mathbf{L}(L_2^n, L_2^n)$ *are compact.*

(ii) *Let the input* $u \in U \subset L_2^m$ *be impulsive and generated from a mapping* $U_0 \times W \to U$ *(gate operator) via a reference input* $u_0 \in U_0 \subset L_2^m$ *and* $\omega \in W \subset L_2$ *is a modulating weighting function that generates* *an impulsive input from* u_0 *defined by*

$$u(t) = \sum_{t_i \in TU} \omega(t)\delta(t - t_i)uu_0(t) = \sum_{t_i \in TU} \omega(t_i)\delta(0)\omega(t_i)\mu_u(t).$$

Assume that $\int_0^\infty \int_0^\infty \|T(t,\tau)\|^2 \, d\tau \, dt < \infty$*, or in particular that the unforced* $\mathbf{S}$ *is* GES (*see Theorem* 2 *for sufficiency-type conditions*). *Then, the operators* S^{u_0}, $(S^{u_0})^*$ *and* $S^{u_0}(S^{u_0})^*$*, from the appropriate Banach spaces to appropriate Banach spaces, are compact if all the entries of* $B(t)$ *are bounded and* $\sum_{t_i \in TU} \omega^2(t_i) < \infty$*. If, in addition, all the outputs of* $C(t)$ *are uniformly bounded and* $D(t) \equiv 0$ *for all* $t \in \mathbf{R}_0^+$*, then the operators* M^{u_0}, $(M^{u_0})^*$ *and* $M^{u_0}(M^{u_0})^*$ *are Hilbert-Schmidt operators and then compact.*

(iii) *Assume that the input satisfies the constraints in* (ii). *Then the operator* $S^{u_0(k)}$ *has a decomposition* $S^{u_0(k)} = \sum_{i=1}^n \chi_i \Delta_i$ *where* $(\Delta_i x)(t) = \langle x, \xi_i \rangle_{L_2^n} \xi_i(t)$, $i = 1, 2, \ldots, n$, *while*

$$T^{(k)}(t,\tau) = \sum_{i=1}^k \chi_i \xi_i(t) \xi_i^T(\tau) = \sum_{i=1}^k \chi_i \psi_i(t) \theta_i^T(\tau)$$

where $\xi_i(t)$ *is a set of orthonormal eigenvectors of respective eigenvalues* χ_i, $i = 1, 2, \ldots, n$, *of* $T^{(k)}$ *satisfying* $\sum_{i=1}^\infty |\chi_i|^2 < \infty$*. Furthermore,* $S^{u_0} - S^{u_0(k)} = \sum_{i=k+1}^\infty \xi_i \Delta_i$,

$$\begin{aligned} \|S^{u_0} u_0\|_{L_2^n} &= \langle S^{u_0} u_0, S^{u_0} u_0 \rangle_{L_2^n} = \sum_{j=1}^m \sum_{k=1}^\infty |\alpha'_{kj}|^2 |\chi_k|^2 \\ &\le \operatorname*{Sup}_{t \in S^{u_0} u_0} (\|B(t)\|_2^2) \left(\sum_{j=1}^m \sum_{k=1}^\infty |\alpha_{kj}|^2 |\chi_k|^2 \right), \end{aligned}$$

provided that $u_0(t) = \sum_{j=1}^m \sum_{k=1}^\infty \alpha_{kj} e_j^{(m)} \phi_k(t) \in L_2^m$ *and* $B(t)u_0(t) = \sum_{j=1}^m \sum_{k=1}^\infty \alpha'_{kj} e_j^{(m)} \phi_k(t) \in L_2^m$ *since* $B(t)$ *is bounded on* $[0, \infty)$*. If* $\|u_0\| \in L_2^m$*, then* $\sum_{k=1}^\infty |\alpha_{kj}|^2 = 1$*, any integer* $j = 1, 2, \ldots, m$*, so that*

$$\|S^{u_0} u_0\|_{L_2^n} \le \operatorname*{Sup}_{t \in S^{u_0} u_0} (\|B(t)\|_2^2) \left(\sum_{j=1}^m |\chi_j|^2 \right).$$

Proof. (i) Define the sequences of operators

$$\Psi_{A0}^{(k)}(t) := \sum_{l=0}^{k} \frac{A_0^l\, t^l}{l!}$$

and

$$\begin{aligned} T^{(k)}(t^-,0) := \Psi_{A0}^{(k)}(t)\Big[I + \int_{0^-}^{t^-} \Psi_{A0}^{(k)}(\tau)\tilde{A}(\tau)T^{(k)}(\tau,0)\,d\tau \\ + \int_{\bar{r}}^{t^-} \Psi_{A0}^{(k)}(\tau)A_d(\tau)T^{(k)}(\tau-r(\tau),0)\mathbf{1}(t-\tau)\,d\tau\Big]; \\ \text{for all} \quad t\in\mathbf{R}_0^+;\ k=0,1,\dots\,. \end{aligned}$$

with $T^{(0)}(0,0) = T(0,0) = I$, and $T^{(k)}(t,\tau) = T(t,\tau) = 0$, for all $\tau > t$. Assume that for $-\bar{r} \le \tau < t$ and any given finite real $t > 0$, $\|T^{(k)} - T\|_{\mathbf{L}(L_2[0,t],L_2[0,t])} \to 0$. Since $\lim_{k\to\infty}(\|\Psi_{A0}^{(k)}(t) - \Psi_{A0}(t)\|_2) = 0$ for all $t \in \mathbf{R}_0^+$, it follows by using l_2 matrix norms that:

$$\begin{aligned} &\|T^{(k)}(t^-+\varepsilon,0) - T(t^-+\varepsilon,0)\|_2 \\ &\quad \le 2\Big\| \sum_{j=k+1}^{\infty} \Psi_{A0}^{(k)}(t+\varepsilon) - \Psi_{A0}(t+\varepsilon)\Big\|_2 \\ &\qquad \times \left[\int_0^{t^-+\varepsilon} \Big\|\sum_{j=k+1}^{\infty}\Psi_{A0}^{(k)}(\tau) - \Psi_{A0}(\tau)\Big\|_2^2 (\|\tilde{A}(\tau)\|_2 + \|\tilde{A}_d(\tau)\|_2)^2\,d\tau\right]^{1/2} \\ &\qquad \times \Bigg[\operatorname*{Sup}_{0\le\tau'\le\tau-r(\tau)}\left(\int_{0^-}^{t^-}\|T^{(k)}(\tau-\tau',0) - T(\tau-\tau',0)\|_2^2\,d\tau\right) \\ &\qquad\qquad + \operatorname*{Sup}_{\tau-r(\tau)\le\tau'\le\tau}\left(\int_{t^-}^{t^-+\varepsilon}\|T^{(k)}(\tau-\tau',0) - T(\tau-\tau',0)\|_2^2\,d\tau\right)\Bigg] \\ &\quad \longrightarrow 0 \quad \text{as} \quad k\to\infty \end{aligned}$$

and

$$\begin{aligned} &\|T^{(k)}(t^++\varepsilon,0) - T(t^++\varepsilon,0)\|_2 \\ &\quad \le \Bigg[\sum_{j=k+1}^{\infty}\ \sum_{t_l\in TN(t,t+\varepsilon)\cup TN_d(t,t+\varepsilon)}^{\infty} \|\Psi_{A0}^{(k)}(t_l) - \Psi_{A0}(t_l)\|_2 \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad \times \left(\|\tilde{A}(t_l)\|_2 + \|\tilde{A}_d(t_l)\|_2\right) d\tau \Bigg] \\
&\times \Bigg[\operatorname*{Sup}_{t_l \in TN(0,t)\cup TN_d(0,t)} \ \operatorname*{Sup}_{0\le\tau\le t_l - r(t_l)} \left(\|T^{(k)}(t_l-\tau,0) - T(t_l-\tau,0)\|_2\right) \\
&\quad + \operatorname*{Sup}_{t_l \in TN(0,t)\cup TN_d(0,t)} \ \operatorname*{Sup}_{t_l - r(t_l)\le\tau\le t_l} \left(\|T^{(k)}(t_l-\tau,0) - T(t_l-\tau,0)\|_2\right) \Bigg] \\
&\longrightarrow 0 \quad \text{as} \quad k \to \infty
\end{aligned}
$$

since $\|\Psi_{A0}^{(k)}(t) - \Psi_{A0}(t)\|_2 \to 0$ as $k \to \infty$, for all $t \in \mathbf{R}_0^+$. Furthermore, $T^{(k)}(t,\tau) = \sum_{l=1}^{k} \psi_l^{(k)}(t)\theta_l^{(k)T}(\tau)$ is of finite range for all t, τ for all finite integers k. Since $\|\psi^{(k)}(t^+)\|_2 \le k_1^{(k)} \|\psi^{(k)}(t^-)\|_2$ then $\|\psi(t^+)\|_2 \le k_1 \|\psi(t^-)\|_2$ from Theorem 1 (ii) for some nonnegative real constants $k_1^{(k)}$, $k = 0, 1, \dots$, and k_1, then

$$
\begin{aligned}
\|T^{(k)}(t^+ + \varepsilon, 0) - T(t^+ + \varepsilon, 0)\|_2 &\le \|T^{(k)}(t^- + \varepsilon, 0) - T(t^- + \varepsilon, 0)\|_2 \\
&\longrightarrow 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad t \in \mathbf{R}_0^+
\end{aligned}
$$

and $\|T^{(k)} - T\|_{L_{2e}^n} \to 0$, i.e. $\int_{0^-}^{t} \|T^{(k)}(\tau, 0) - T(\tau', 0)\| \, d\tau < \infty$, as $k \to \infty$, for all $t \in \mathbf{R}_0^+$ and $T^{(k)}(\tau, 0)$ is of finite range on $[0, t + \varepsilon)$, provided it is compact on $[0, t)$ for any finite integer $k \in \mathbf{Z}_0^+$. Since $\|T^{(k)}(t,0) - T(t,0)\|_{L_2^n} \to 0$ and $T^{(k)}(t,0)$ is of finite dimensional range as $k \to \infty$, for all $t \in \mathbf{R}_0^+$, $T(t,0)$ is a compact operator. It is now proved that T^*, (T^*T) and (TT^*) are also compact. Note that taking norms in L_2^n:

$$
\begin{aligned}
\|T^*T(f_n - f_m)\|_{L_2^n}^2 &= \langle T^*T(f_n - f_m), T^*T(f_n - f_m)\rangle_{L_2^n} \\
&= \langle TT^*T(f_n - f_m), T(f_n - f_m)\rangle_{L_2^n} \\
&\le \|T(T^*T)(f_n - f_m)\|_{L_2^n} \cdot \|T(f_n - f_m)\|_{L_2^n} \longrightarrow 0
\end{aligned}
$$

as $\mathbf{Z}_0^+ \ni n, m \to \infty$, for any bounded sequence $\{f_n\}_1^\infty$ and its endowed norm from the inner product in the Hilbert space of square-integrable real vector functions of dimension n. Then (T^*T) is a compact operator and, therefore, (TT^*) is compact as well, [**5**]. As a result, weak convergence, i.e., convergence of the inner products, implies strong convergence, i.e., convergence of the sequences, so that $f_k \to f$ as

$k \to \infty$ implying $\langle T^* T f_k, f_k \rangle_{L_2^n} \to 0$ as $k \to \infty$ implies that $T^* T f_k$ as $k \to \infty$. Also,

$$\begin{aligned}
\|T^*(f_n - f_m)\|_{L_2^n}^2 &= \langle T^*(f_n - f_m), T^*(f_n - f_m) \rangle_{L_2^n} \\
&= \langle TT^*(f_n - f_m), (f_n - f_m) \rangle_{L_2^n} \\
&\le \|TT^*(f_n - f_m)\|_{L_2^n} \cdot \|f_n - f_m\|_{L_2^n} \\
&\le 2C\|TT^*(f_n - f_m)\|_{L_2^n}
\end{aligned}$$

since $\|f_n - f_m\|_{L_2^n} \le 2C\|TT^*(f_n - f_m)\|_{L_2^n} \to 0$ for some finite positive real constant C as $\mathbf{Z}_0^+ \ni n,\ m \to \infty$, so that $\{T^* f_n\}_1^\infty$ and T^* is compact. By construction via T, the operators $\mathbf{S}$ (SS^*), M^* and (MM^*) are compact as well and (i) has been proved.

(ii) Since $u(t) = u''(t) = \sum_{t_i \in TU} \omega(t_i)\delta(t - t_i)u_0(t)$ and $(S^{u_0}u_0)(t)$ and $(M^{u_0}u_0)(t)$ are defined via (80), (81), it follows that their kernels are square-integrable if $\sum_{t_i \in TU} \omega^2(t_i) < \infty$ and S^{u_0} and M^{u_0} are Hilbert-Schmidt and then compact operators. In the same way as in (i), $(S^{u_0}S^{u_0^*})$, $(M^{u_0}M^{u_0^*})$, $S^{u_0^*}$ and $M^{u_0^*}$ are compact since S^{u^0} and M^{u^0} are compact.

(iii) It follows directly from the definitions of the various operators and their spectral decompositions since they are compact since the reference input is square integrable. □

Using similar reasoning as that used in Theorem 4 (i), it may be proved that the operators $\Psi_{A0} \in \mathbf{L}(\mathrm{IC}, L_2^n)$, $S_{A0e} \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$, $M_{A0e} \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$, $S_{A0} \in \mathbf{L}(\mathrm{IC}, L_2^n)$ and $M_{A0} \in \mathbf{L}(\mathrm{IC}, L_2^n)$ are compact operators. The evolution operator $\Psi_A \in \mathbf{L}(\mathrm{IC}, L_2^n)$ as well as $S_{Ae} \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$, $M_{Ae} \in \mathbf{L}(\mathrm{IC}, L_{2e}^p)$, $S_A \in \mathbf{L}(\mathrm{IC}, L_2^n)$ and $M_A \in \mathbf{L}(\mathrm{IC}, L_2^p)$ are proved to be compact as well under similar guidelines for the proofs. Those properties are useful to approximate the zero-input responses of the auxiliary systems **S1** and **S2** through finite dimensional real vector functions what holds irrespective of the stability or not of the infinitesimal generator A_0 or that of the matrix function $A(t)$. □

Remarks 1. Theorem 4 (ii) also holds under similar conditions if $U_0 \in L_2^{m'}$, $W \in L_2^{m \times m'}$ and $\sum_{t_i \in TU} \|\omega(t_i)\|^2 < \infty$, which implies that $U \subset L_2^m$ although the proof is omitted.

2. sp (T), the spectrum of T. is close, belongs to $[-\|T\|, \|T\|]$ and includes zero. The remaining values of the spectrum excluding zero are eigenvalues of T. Such a spectrum is either zero, finite (when excluding zero) or a sequence which converges to zero. All the points of sp $(T)/\{0\}$ are isolated.

3. $\Psi_{A0}(t) = \sum_{k=0}^{\bar{\mu}-1} \alpha_k(t) A_0^k$ for any integer $\bar{\mu} \geq \mu$, μ being the degree of the minimal polynomial of A_0 and $\alpha_k : \mathbf{R}_0^+ \to \mathbf{R}$, $k = 0, 1, \dots, \bar{\mu}-1)$ are linearly independent functions calculated from the algebraic system of linear equations:

$$\left(\frac{d^j}{d\lambda^j}\right)[\mathbf{1}\lambda\lambda^2 \cdots \lambda^{\bar{v}_i}] \cdot [\alpha_0(t)\alpha_1(t) \cdots \alpha_{\bar{\mu}-1}(t)]^T$$
$$= [e^{\lambda_1 t} \cdots t^{\bar{\nu}_1 - 1} e^{\lambda_1 t} \cdots e^{\lambda_\sigma t} \cdots t^{\bar{\nu}_1 - 1} e^{\lambda_\sigma t}]^T,$$
$$j = 0, 1, \dots, \bar{\nu}_i - 1;\ i = 1, 2, \dots \sigma,$$

where λ_i, of respective multiplicities ν_i, $i = 1, 2, \dots, \sigma$, are the distinct eigenvalues of A_0 and $\bar{\nu}_i$, $i = 1, 2, \dots, \sigma$, are (nonuniquely) chosen satisfying the constraints $\mathbf{Z}^+ \ni \bar{\nu}_i \geq \nu_i$ and $\bar{\mu} = \sum_{i=1}^{\sigma} \bar{\nu}_i \geq \mu = \sum_{i=1}^{\sigma} \nu_i$. The set $\{\alpha_i(t), i = 1, 2, \dots, \bar{\mu}\}$ is not unique since it depends on the set $\bar{\nu}_i$, $i = 1, 2, \dots, \sigma$, (the minimum set $\{\alpha_i(t), i = 1, 2, \dots, \mu\}$ is unique). As a result, each unique real n-dimensional vector trajectory solution of **S1** is represented in any of the forms

$$z_{A0}(t) = \sum_{k=0}^{\bar{\mu}-1} \alpha_k(t) A_0^k x_0 = \sum_{k=0}^{\bar{\mu}-1} \alpha_k(t) \left(\sum_{i=1}^{n} c_i A_0^k e_i^{(n)} \right)$$

for any initial condition

$$x_0 = \sum_{i=1}^{n} c_i e_i^{(n)}, \quad c_i \in \mathbf{R};\ i = 1, 2, \dots, n,$$

so that it is of finite dimension as it is the real p-vector function $C(t) z_{A0}(t)$. As a result, the operators $S_{A0} \in \mathbf{L}(\mathrm{IC}, L_2^n)$, $S_{A0e} \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$, $M_{A0} \in \mathbf{L}(\mathrm{IC}, L_2^p)$, $M_{A0e} \in \mathbf{L}(\mathrm{IC}, L_{2e}^p)$. If, furthermore, $\|A_d'\| \in L_2$, then $\overline{S}_{A0e} \in \mathbf{L}(L_{2e}^n, L_{2e}^n)$, $\overline{M}_{A0e} \in \mathbf{L}(L_{2e}^n, L_{2e}^p)$, $\overline{S}_{Ae} \in \mathbf{L}(L_{2e}^n, L_{2e}^n)$ and $\overline{M}_{Ae} \in \mathbf{L}(L_{2e}^n, L_{2e}^p)$ are also compact, see (4.c), (4.d), (4.g) and (4.h) in Theorem 1.

4. If $\varphi_{\mathrm{imp}} \equiv 0$ for $t \in [-\bar{r}, 0]$, then $S_e \in \mathbf{L}(\mathrm{IC}, L_{2e}^n)$ and $M_e \in \mathbf{L}(\mathrm{IC}, L_{2e}^p)$ are compact if the auxiliary system **S3**, or equivalently the unforced **S**, equation (1), is GAS.

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REFERENCES

1. R.P. Agarwal, *Difference equations and inequalities: Theory, methods and applications*, Marcel Dekker, New York, 1992.

2. A.K. Agrawall and J.N. Yang, *Compensation of time-delay for control of civil engineering structures*, Earthquake Engrg. Struct. Dynam. **29** (2000), 37–62.

3. N.I. Akhiezer and I.M. Glazman, *Theory of linear operators in Hilbert space*, Frederick Ungar Pub. Co., New York, 1963; reprinted by Dover Inc., New York, 1993.

4. B. Aziz, *Nonlinear robust control problems of parabolic type equations with time-varying delays given in the integral form*, J. Dynam. Control Systems **9** (2003), 489–512.

5. H. Brézis, *Analyse fonctionelle*, Masson, Paris, 1983.

6. T.A. Burton, *Stability and periodic solutions of ordinary and differential equations*, Elsevier, New York, 1985.

7. T. Caraballo, J.A. Langa and J.C. Robinson, *Attractors for differential equations with variable delays*, J. Math. Anal. Appl. **260** (2001), 421–438.

8. M. De la Sen, *The use of Gronwall's lemma for robust compensation of time-varying linear systems*, Internat. J. Systems Sci. **22** (1991), 885–903.

9. M. De la Sen and V. Etxebarria, *Discretized models and use of multirate sampling for finite spectrum assignment in linear systems with commensurate time-delays*, Nonlinear Anal. Appl. **38** (1999), 193–228.

10. M. De la Sen and Ningsu Luo, *A note on the stability of linear time-delay systems with impulsive inputs*, IEEE Trans. Circuits Systems I Fund. Theory Appl. **50** (2003), 149–152.

11. ———, *On the uniform exponential stability of a wide class of linear time-delay systems*, J. Math. Anal. Appl. **289** (2004), 456–476.

12. T. Faria, W. Huang and J. Wu, *Smoothness of center manifolds for maps and formal adjoints for semilinear FDES in generic Banach spaces*, SIAM J. Math. Anal. **34** (2002), 173–203.

13. L.E. Franks, *Signal theory*, Prentice Hall, Englewood Cliffs, 1975.

14. Z. Gang Zeng, J. Wang and X. Liao, *Global exponential stability of neural networks with time-varying delays*, IEEE Trans. Circuits Systems I Fund. Theory Appl. **50** (2003), 1353–1358.

15. Z. Gang Zeng, J. Wang and X. Liao, *Stability of neural networks with time-varying delays*, Dyn. Contin. Discrete Impuls. Syst. **10** (2003), 340–345.

16. P. Ioannou and A. Datta, *Robust adaptive control*: *Design, analysis and robustness bounds*, in *Foundations of adaptive control* (M. Thoma and A. Wyner, eds.), Lecture Notes in Control and Inform. Sci. (P.V. Kokotovic, ed.), No. 160, Springer-Verlag, Berlin, 1991.

17. J. Jugo and M. De la Sen, *Input-output based pole-placement controller for a class of time-delay systems*, IEEE Proc. Control Theory Appl. **149** (2002), 323–330.

18. S. Larsson, V. Thomée and L.B. Wahlbin, *Numerical solutions of parabolic integro-differential equations by the discontinuous Galerkin method*, Math. Comp. **67** (1998), 45–71.

19. B. Lehman and S.P. Weibel, *Averaging theory for delay difference equations with time-varying delays*, SIAM J. Appl. Math., **5** (1999), 1487–1506.

20. Ningsu Luo and M. De la Sen, *State feedback sliding mode control of a class of uncertain time-delay systems*, IEEE Proc. Control Theory Appl. **140** (1993), 456–476.

21. M.C. Memory, *Stable and unstable manifolds for practical functional differential equations*, Nonlinear Anal. **16** (1991), 131–142.

22. S.I. Nakagiri, *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. **25** (1988), 353–398.

23. V.A. Oliveira, M.C.M. Teixeira and L. Cossi, *Stabilizing a class of time-delay systems using the Hermite-Biehler theorem*, Linear Algebra Appl. **369** (2003), 203–216.

24. S. Oucheriah, *Exponential stabilization of linear delayed systems*, IEEE Trans. Circuits Systems I Fund. Theory Appl. **50** (2003), 826–830.

25. V. Ngoc Phat, *Stabilization of linear continuous time-varying systems with state delays in Hilbert spaces*, Electron. J. Differential Equations **2001** (2001), 1–13.

26. J.P. Richard, *Time-delay systems*: *An overview of some recent advances and open problems*, Automatica **39** (2003), 1667–1694.

27. F. Zheng and P.M. Frank, *Finite dimensional variable structure control design for distributed delay systems*, Internat. J. Control **74** (2001), 398–408.

Department of Systems Engineering and Automatic Control, University of Basque Country, Campus of Leioa (Bizkaia), Aptdo. 644–Bilbao, 48080 Bilbao
E-mail address: wepdepam@lg.ehu.es