THE LENGTH OF DUCCI'S FOUR-NUMBER GAME

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ABSTRACT. The length of a Ducci 4-number game, defined immediately below, is at most six if the initial vector is not cyclically monotone. If it is cyclically monotone, then the length is shown here to be, with error at most 5.4, a linear function of the logarithm of the Euclidean distance from the initial vector (normalized) to the unique normalized vector of infinite length.

1. The main theorem. Define $T : \mathbf{R}^4 \to \mathbf{R}^4$ by the formula

$$T(a, b, c, d) = (|b - a|, |c - b|, |d - c|, |d - a|).$$

For each $v \in \mathbf{R}^4$ the sequence $(T^n(v))_{n \geq 0}$ is called a *Ducci sequence*, or *4-number game*. A minimal $n \geq 0$ with $T^n(v) = 0$ is called the *length* of v, or of the 4-number game; if there is no such n, then v is said to have *infinite length*. Lengths of 4-number games, and more generally of n-number games, have been one of the major topics in the study of Ducci sequences. (See any of the references below, but especially [**1, 4, 6, 7**].) In this paper we give a simple method for estimating with error less than 5.4 the length of any 4-number game.

As in [**6**], a vector $v \in \mathbf{R}^4$ is said to be *equivalent* to $w \in \mathbf{R}^4$ if w is in the orbit of v under the action on $\mathbf{R}^4$ of the group generated by the permutations R, S, A_f (for $f \in \mathbf{R}$) and M_e (for $0 \neq e \in \mathbf{R}$) of $\mathbf{R}^4$ defined by the formulas $R(a, b, c, d) = (d, c, b, a)$, $S(a, b, c, d) = (d, a, b, c)$, $M_e(v) = ev$, and $A_f(v) = v + (f, f, f, f)$. We also call v *normalized* if it has the form $(0, x, y, 1)$ where $0 < x < y \leq 1 - x$.

In Section 3 we will see that the problem of estimating length reduces to the case of normalized vectors. (Briefly, if a vector is cyclically monotone, then its length equals that of an equivalent normalized vector, and otherwise the length is at most 6 and hence is known to within 5.4.) We now state our main theorem and give a simple estimate

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for the length of a normalized vector. This estimate will be illustrated
by examples in Section 2. In Sections 4 and 5 we will prove the theorem
and show that the estimate has error less than 5.4.

In the theorem below and throughout the rest of the paper, we let

$$r = \left(2 - \frac{2\sqrt{33}}{9} \right)^{1/3} + \left(2 + \frac{2\sqrt{33}}{9} \right)^{1/3} \approx 2.38$$

denote the unique real zero of $x^3 - 4x - 4$ and also set

$$\mu = \frac{2}{\ln(r+1)} \approx 1.64$$

and

$$v_\infty = \left(0, \frac{6 - r^2}{2}, \frac{r^2 - 2r}{2}, 1 \right) \approx (0, .16, .46, 1).$$

1.1 Theorem. *There exist constants C and D such that if d is the
Euclidean distance from a normalized vector v to v_∞, where $v \neq v_\infty$,
then the length L of v satisfies*

$$C < L - \mu \ln \frac{1}{d} < D.$$

1.2 *Remark*: *The simple estimate.* The proof of the above theorem
will show that one can choose $C = -2.4$ and $D = 8.4$, so that

$$\left| L - \left(3 + \mu \ln \frac{1}{d} \right) \right| < 5.4.$$

We will refer to $3 + \mu \ln 1/d$ as the *simple estimate* for L. For example,
the vector $(0, \sqrt{2}, e + \sqrt{3}, \pi^2)$ has length 10 and the simple estimate
of its length (rounded) is 9.6. More examples appear in Section 2
including one of a vector in $\mathbf{Z}^4$ of length 3782 (the simple estimate is
about 3782.1). The lower bound $-2.4 + \mu \ln 1/d$ for L is of interest
only if $d < e^{-7.4/\mu} \approx 0.011$, since otherwise $-2.4 + \mu \ln 1/d \leq 5$ and,
by Lemma 3.3 below, normalized vectors always have length at least

5. The above values for C and D can be improved by more careful numerical work; see Remark 5.2.

Note that Theorem 1.1 implies that, if $v \in \mathbf{Z}^4$, then v has finite length. This observation is attributed to Professor Ducci in [2] and the observation is generalized to arbitrary 2^n-number games and proven. Theorem 1.1 of course implies that all vectors $v \in \mathbf{R}^4$ have finite length except for those equivalent to v_∞. The fact that up to equivalence there is exactly one element of $\mathbf{R}^4$ of infinite length was first proved by Lotan [3]. One can give a rather short proof of this fact using ideas in this paper but without using the sledgehammer of Theorem 1.1. Such a proof is sketched in Section 6.

2. Examples. In this section we give some examples based on the "simple estimates" of Remark 1.2.

2.1 Example: Tribonacci numbers. The Tribonacci numbers $t_0, t_1, t_2, \ldots$ are defined by the equations $t_0 = 0$, $t_1 = t_2 = 1$ and for $n \geq 3$, $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. Webb [7] has shown that the "Tribonacci vector" $T_n = (t_n, t_{n-1}, t_{n-2}, t_{n-3})$ has length $3\,[n/2]$ for all $n \geq 3$ and that any vector in $\mathbf{Z}^4$ whose coordinates all are between 0 and t_n (inclusive) has length at most $3\,[n/2]+1$. Each T_n is equivalent to the normalized vector

$$\left(0,\ \frac{t_{n-2} - t_{n-3}}{t_n - t_{n-3}},\ \frac{t_{n-1} - t_{n-3}}{t_n - t_{n-3}},\ 1\right).$$

The following tables record the length of T_n for all n from 5 to 24, together with the simple estimate of its length, rounded to the nearest tenth.

n	5	6	7	8	9	10	11	12	13	14
length	6	9	9	12	12	15	15	18	18	21
estimate	8.1	9.3	10.2	12.3	14.1	14.7	16.5	18.9	19.4	20.8

n	15	16	17	18	19	20	21	22	23	24
length	21	24	24	27	27	30	30	33	33	36
estimate	23.3	24.1	25.2	27.6	28.8	29.7	31.8	33.6	34.3	36.0

2.2 Example: Decimal approximations. Here are lengths and estimated lengths for the vectors $(0, x_n, y_n, 1)$ where x_n and y_n are obtained by truncating to n digits the decimal expansions of the second and third coordinates of v_∞, respectively. Thus, for example, $x_2 = 0.16$ and $y_2 = 0.45$.

n	4	5	6	7	8	9	10	100	1000
length	22	24	27	29	34	38	41	382	3782
estimate	21.0	23.7	25.6	29.5	33.5	37.3	41.0	381.5	3782.1

If a normalized vector has small length, then the simple estimate for its length is both unnecessary and uninformative. Theorem 1.1 shows that the normalized vectors of large length are those close to v_∞. The previous example focused on one way of coming up with vectors close to v_∞; the next gives a more natural way to do this.

2.3 Example: Convergents. In the mth row and nth column of the table below we have listed the length and the simple estimate of length, respectively, of the vector $(0, x_m, y_n, 1)$, where x_m and y_n are the mth and nth convergents to the continued fraction expansions of the second and third coordinates of v_∞, respectively. Because convergents give good approximations relative to the size of their denominators, by multiplying $(0, x_m, y_n, 1)$ by the product of the denominators of x_m and y_n, we can hope to get elements of $\mathbf{Z}^4$ of large length relative to the magnitude of their coordinates.

	1	2	3	4	5	6
1	6, 8.1	11, 11.3	11, 11.4	11, 11.4	11, 11.4	11, 11.4
2	6, 8.1	13, 12.3	15, 14.8	15, 14.9	15, 14.9	15, 14.9
3	6, 8.1	13, 13.4	17, 16.9	25, 25.5	25, 25.6	25, 25.6
4	6, 8.1	13, 13.4	17, 16.9	27, 27.6	35, 35.1	34, 35.2
5	6, 8.1	13, 13.4	17, 16.9	27, 27.6	37, 37.1	39, 38.6
6	6, 8.1	13, 13.4	17, 16.9	27, 27.6	37, 37.2	41, 40.7

One might observe that the fourth row of the above table gives an example of a pair of vectors for which the vector *closer* to v_∞ actually has the smaller length. Note that

$$(0, x_3, y_4, 1) = \left(0, \, \frac{9}{56}, \, \frac{47}{103}, \, 1 \right)$$

is equivalent to

$$(0, 927, 2632, 5768) = (0, t_{13}, t_{13} + t_{14}, t_{16}),$$

which is essentially the vector (in the case $n = 16$) that Webb constructs [**7**, last displayed formula, p. 35] to give a vector of maximal length among those vectors in $\mathbf{Z}^4$ with largest coordinate at most t_{16}. It might be interesting to see if there are other connections between the examples arising from Tribonacci numbers and from convergents.

2.4 Last example. The largest discrepancy we have observed between actual length and the simple estimate of length occurs with vectors of the form $(0, \varepsilon, (1/3) + \varepsilon, 1)$ where ε is chosen very small and positive. Then the lengths are all eight but the estimated lengths are about 5.6. That the simple estimate predicts length as well as it does in the above examples suggests that tighter bounds are possible for length and, further, that the upper and lower bounds for length that we have given are, essentially by accident, about equally far from these hypothesized tighter upper and lower bounds.

3. Normalization. In this section, for which we make no great claim to originality, we prove some basic lemmas, including for the sake of completeness some proofs of known results. We use the maps R, S, A_f and M_e from Section 1.

Let $v \in \mathbf{R}^4$.

3.1 Definition. We say v is *cyclically monotone* if for some $i \geq 0$ we have $S^i v = (e, f, g, h)$ where $e < f < g < h$ or $e > f > g > h$.

The reduction of the problem of estimating lengths to the normalized case is accomplished by the next two lemmas.

3.2 Lemma. *If v is cyclically monotone, then it is equivalent to a normalized vector of $\mathbf{R}^4$, and any normalized vector equivalent to v has the same length as v.*

In Proposition 6.1 below we will show that a cyclically monotone vector is equivalent to only *one* normalized vector. The proof of Lemma 3.2 indicates at least implicitly how to construct this normalized vector.

Proof. By the definition of equivalence we may assume without loss of generality, i.e., after suitable applications of R and S, that $v = (a, b, c, d)$ where $a < b < c < d$. Applying $M_{1/(d-a)}A_{-a}$ we may assume v has the form $(0, b, c, 1)$. If $b+c \le 1$ we are finished, so suppose $b + c > 1$. Then

$$RA_1 M_{-1}(v) = (0, 1 - c, 1 - b, 1),$$

which is clearly normalized, so there is a normalized vector equivalent to v. That equivalent cyclically monotone vectors have the same length follows from the fact that the bijections R, S, M_e and A_f clearly preserve length when applied to nonconstant vectors, i.e., to vectors not of the form (g, g, g, g).   □

The cyclically monotone vector $(0, 1, 2, 3)$ has length 5 and the vector $(0, 2, 5, 1)$ has length 6. The next lemma gives a sense in which these lengths are extremal.

3.3 Lemma. *If v is not cyclically monotone, then its length is at most 6. If it is cyclically monotone, then its length is at least 5.*

Proof. The first assertion of the lemma is a standard result [**5**, Property F] and can be easily proved by considering in the order given below the cases of vectors in $\mathbf{R}^4$ of the following forms: (a, b, a, b); (a, b, a, c); (a, m, b, M) where $M \ge a \ge m$, $M \ge b \ge m$; and, (M, a, b, m) where $m \le a \le b \le M$. Any vector which is not cyclically monotone is equivalent to a vector in one of the last two forms.

Now suppose v is cyclically monotone; by the previous lemma we may suppose that v is a normalized vector $(0, x, y, 1)$. Note that $1-x \ge y \ge 2x-y$, and $1-x > y-2x$, so that in any case $1-x \ge |2x-y|$. It follows easily that the third coordinate of $T^4(v)$ is the absolute value of $y - |2x - y|$, which is clearly nonzero. Thus the length of v is at least 5.   □

We end this section with two lemmas in the spirit of the two above. We say a vector $(a, b, c, d) \in \mathbf{R}^4$ is *monotone* if $0 \leq a < b < c < d$ and *subnormal* if it is monotone and in addition $b + c \leq a + d$. Thus normalized vectors are subnormal; subnormal vectors are monotone; and monotone vectors are cyclically monotone.

3.4 Lemma. *If* $v = (a, b, c, d)$ *is subnormal and* $T(v)$ *is cyclically monotone, then* $T(v)$ *is subnormal.*

Proof. Since v is monotone, $T(v) = (b - a, c - b, d - c, d - a)$. If $T(v)$ is not monotone, then since $d - a$ is clearly its largest coordinate, we therefore must have $b - a > c - b > d - c$, which contradicts the hypothesis that $b + c \leq a + d$. Also, because $b > a$, we have $(c - b) + (d - c) \leq (b - a) + (d - a)$, so $T(v)$ is indeed subnormal. □

3.5 Lemma. *Define* $U : \mathbf{R}^4 \to \mathbf{R}^4$ *by the formula* $U(a, b, c, d) = (b - a, c - b, d - c, d - a)$. *If* v *is subnormal and* $U^n(v)$ *is not monotone for some* $n \geq 1$, *then* $T^{n+6}(v) = 0$.

Proof. Note that $T(v) = U(v)$ whenever v is monotone. By Lemma 3.3 it suffices to show that $T^k(v)$ is not cyclically monotone for some $k \leq n$. Suppose otherwise. Then by the previous lemma $T^k(v)$ is monotone for all $k \leq n$, and so $U^n(v) = T^n(v)$. This contradicts the hypothesis that $U^n(v)$ is not monotone. □

4. The main lemma. In this section we give a geometric interpretation of the point v_∞; this leads to our main lemma, Lemma 4.5, which shows how bounds on the distances from $((6 - r^2)/2, (r^2 - 2r)/2)$ to various geometric objects lead to bounds on the lengths of normalized vectors. Related geometric methods are used in [**6**] to study the distribution of lengths.

4.1 Notation. Define the linear map $U : \mathbf{R}^4 \to \mathbf{R}^4$ (cf. Lemma 3.5) by the rule

$$U(a, b, c, d) = (b - a, c - b, d - c, d - a).$$

Let $v = (0, x, y, 1) \in \mathbf{R}^4$ and for each $n \geq 0$ write

$$U^n(v) = (a_n, b_n, c_n, d_n).$$

By inspection,

$$a_0 = 0, \quad a_1 = x, \quad a_2 = y - 2x, \quad a_3 = 3x - 3y + 1.$$

Also for each $n \geq 0$,

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (b_n - a_n, c_n - b_n, d_n - c_n, d_n - a_n)$$

so

$$b_n = a_n + a_{n+1},$$
$$c_n = b_n + b_{n+1} = a_n + 2a_{n+1} + a_{n+2},$$

and

$$d_n = c_n + c_{n+1} = a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3}.$$

But also $d_{n+1} = d_n - a_n$, and hence

$$a_{n+4} + 3a_{n+3} + 3a_{n+2} + a_{n+1} = 3a_{n+1} + 3a_{n+2} + a_{n+3},$$

so the sequence $(a_n)_{n \geq 0}$ satisfies what we will call the *first recursion relation*

(1) $$X_{n+4} = 2(X_{n+1} - X_{n+3}).$$

We now define a sequence of linear polynomials $A_n = A_n(X, Y) \in \mathbf{R}[X, Y]$ by setting

$$A_0 = 0, \quad A_1 = X, \quad A_2 = Y - 2X, \quad A_3 = 3X - 3Y + 1$$

and requiring the sequence $(A_n)_{n \geq 0}$ to satisfy the first recursion relation (1) for all $n \geq 0$. Thus,

$$A_n(x, y) = a_n \quad \text{for all} \quad n \geq 0.$$

Note that $A_4 = 2(A_1 - A_3) = 6Y - 4X - 2$.

4.2 Notation. We let $\bar{z}$ and $\mathrm{Tr}\,(z) = z + \bar{z}$ denote the complex conjugate and trace of a complex number z, respectively. Recall the real zero r of $x^3 - 4x - 4$; write

(2) $$x^3 - 4x - 4 = (x - r)(x - s)(x - \bar{s}),$$

where s denotes the nonreal zero whose coefficient of i is positive, so $s \approx -1.19 + 0.51\,i$. We also let

$$\alpha = \frac{9r^2 - 8r - 24}{44} \quad \text{and} \quad \beta = \frac{9s^2 - 8s - 24}{44},$$

so $\alpha \approx 0.18$ and $\beta \approx -0.09 - 0.34\,i$. For all $n \geq 0$, let

$$M_n = \alpha r^n + \beta s^n + \bar{\beta}\,\bar{s}^n = \alpha r^n + \mathrm{Tr}\,(\beta s^n),$$

$N_n = (1/2)M_{n+2}$ and

$$D_n = N_n + M_n + M_{n+1} = M_n + M_{n+1} + \frac{1}{2}\,M_{n+2}.$$

We also set $P_n = (M_n/D_n, N_n/D_n)$. In the next lemma we examine the values of the polynomials A_m on the points P_n.

4.3 Lemma. *For all $n \geq 0$, $A_{n+1}(P_n) = A_{n+2}(P_n) = A_n(P_{n+1}) = 0$. Also, for all $n > 0$, $A_n(P_{n+j}) > 0$ whenever $j = 0$ or $j \geq 2$. Finally, for all $n \geq 0$, $A_{n+3}(P_n) > 0$.*

The proof of Lemma 4.3 will develop a useful recursion relation satisfied by the M_n, N_n and D_n sequences, and will also give a table (which will be needed below) of values of M_n, N_n, D_n and P_n for small n. Together these will imply that for all n the numbers M_n, N_n and D_n are nonnegative integers with $D_n > 0$.

Proof. Note that the last fact of the lemma is implied by the earlier ones, since by the first recursion relation (1) if $n \geq 1$, then

$$A_{n+3}(P_n) = 2(A_n(P_n) - A_{n+2}(P_n)) > 0,$$

and also $A_3(P_0) = 1 > 0$.

Since by definition $r^3 = 4r + 4$ and similarly for s and $\bar{s}$, it follows that the sequence $(M_n)_{n \geq 0}$ satisfies the recursion relation

$$(3) \qquad\qquad X_{n+3} = 4X_{n+1} + 4X_n$$

for all $n \geq 0$. We call (3) the *second recursion relation.* It is of course also satisfied by the sequences $(N_n)_{n \geq 0}$ and $(D_n)_{n \geq 0}$. By equation (2) we have $r + s + \bar{s} = 0$, $rs\bar{s} = 4$, $rs + r\bar{s} + s\bar{s} = -4$, and hence

$$r^2 + s^2 + \bar{s}^2 = (r + s + \bar{s})^2 - 2(rs + r\bar{s} + s\bar{s}) = 8.$$

Using these identities, the definitions of M_n, N_n and D_n, and their recursion relation (3), one easily verifies the values in the following table.

n	0	1	2	3	4
M_n	0	1	0	4	4
N_n	0	2	2	8	16
D_n	1	3	6	16	36
P_n	$(0,0)$	$(1/3, 2/3)$	$(0, 1/3)$	$(1/4, 1/2)$	$(1/9, 4/9)$

For each pair m, n of nonnegative integers we set $K_{m,n} = D_n A_m(P_n)$.

Fix m for the moment and write $A_m = aX + bY + c$. Then

$$K_{m,n} = aM_n + bN_n + cD_n.$$

Therefore, the sequence $(K_{m,n})_{n \geq 0}$ satisfies the second recursion relation (3) since the sequences (M_n), (N_n) and (D_n) all satisfy it.

Also, for fixed n the sequence $(K_{m,n})_{m \geq 0}$ satisfies the first recursion relation (1) since the polynomials A_m satisfy it.

Claim. *For all $n \geq 1$, $K_{n,n} = 2^{n-1}$ and $K_{n,n\pm1} = 0$.*

Proof of Claim. Using the values of D_n, A_n and P_n computed explicitly above, it is routine to verify the claim for all $n \leq 3$. Now suppose inductively that $n > 3$ and that the claim is valid for all smaller

integers. Using this hypothesis and the second recursion relation (3) on sequences of the form $(K_{m,n})_{n\geq 0}$, we deduce in turn that

$$K_{n-3,n-1} = 4(K_{n-3,n-3} + K_{n-3,n-4}) = 4(2^{n-4} + 0) = 2^{n-2}$$

and similarly that $K_{n-3,n} = 2^{n-2}$, $K_{n-3,n+1} = 2^n$, $K_{n-2,n} = 2^{n-1}$, $K_{n-2,n+1} = 2^{n-1}$, and $K_{n-1,n+1} = 2^n$. Now using the fact that sequences of the form $(K_{m,n})_{m\geq 0}$ satisfy the first recursion relation (1), we deduce that

$$K_{n,n-1} = 2(K_{n-3,n-1} - K_{n-1,n-1}) = 2(2^{n-2} - 2^{n-2}) = 0$$

and, similarly, that $K_{n,n} = 2^{n-1}$ and $K_{n,n+1} = 0$. The claim is proved.

We can now verify the assertions of the lemma. Without loss of generality, we may assume that $n \geq 1$ (since $A_0 = 0$). Then by the claim $A_n(P_n) = K_{n,n}/D_n > 0$ and $A_{n+1}(P_n) = K_{n+1,n}/D_n = 0$. Similarly, $A_n(P_{n+1}) = K_{n,n+1}/D_{n+1} = 0$. If $n = 1$, then one checks that $A_{n+2}(P_n) = 0$; if $n > 1$, then the first recursion relation shows that

$$A_{n+2}(P_n) = 2(A_{n-1}(P_n) - A_{n+1}(P_n)) = 0.$$

Finally, an easy induction argument using the above claim and the fact that the sequences $(D_n)_{n\geq 0}$ and $(K_{m,n})_{n\geq 0}$ satisfy the second recursion relation establishes that for $j \geq 2$, $A_n(P_{n+j}) = K_{n,n+j}/D_{n+j} > 0$.
□

Let $P_\infty = ((6 - r^2)/2, (r^2 - 2r)/2)$, so that the coordinates of P_∞ are just the middle coordinates of v_∞.

4.4 Lemma.

$$P_\infty = \lim_{n\to\infty} P_n.$$

Proof. First,

$$\frac{M_n}{D_n} = \frac{M_n}{M_n + M_{n+1} + (1/2)M_{n+2}}$$

$$= \frac{\alpha r^n + \mathrm{Tr}\,(\beta s^n)}{\alpha(1 + r + r^2/2)r^n + \mathrm{Tr}\,((1 + s + s^2/2)s^n\beta)}$$

$$= \frac{\alpha + \mathrm{Tr}\,((s/r)^n\beta)}{\alpha(1 + r + r^2/2) + \mathrm{Tr}\,((s/r)^n(1 + s + s^2/2)\beta)}$$

$$\longrightarrow \frac{1}{1 + r + r^2/2} = \frac{6 - r^2}{2}$$

as $n \to \infty$ since $|s/r| < 1$. Similarly,

$$\frac{N_n}{D_n} = \frac{1}{2}\frac{M_{n+2}}{D_n}$$

$$= \frac{\alpha r^{n+2} + \mathrm{Tr}\,(\beta s^{n+2})}{\alpha(2 + 2r + r^2)r^n + \mathrm{Tr}\,(\beta(2 + 2s + s^2)s^n)}$$

$$\longrightarrow \frac{r^2}{r^2 + 2r + 2} = \frac{2}{r + 2} = \frac{r^2 - 2r}{2}.$$

This completes the proof that $P_n \to P_\infty$ as $n \to \infty$. $\qquad\square$

For any line L in $\mathbf{R}^2$ and any points P, Q in $\mathbf{R}^2$, we let $\mathrm{dist}\,(P, Q)$ and $\mathrm{dist}\,(P, L)$ denote the Euclidean distance from P to Q and to L, respectively. We also write

$$\omega = (r + 1)^{-1/2} = \left|\frac{s}{r}\right| \approx 0.544.$$

Finally, L_n will denote the line whose equation is $A_n = 0$.

4.5 Lemma. *Suppose B and G are positive numbers such that for all $n \geq 0$,*

$$\mathrm{dist}\,(P_n, P_\infty) \leq B\omega^n \quad \text{and} \quad B \geq \omega\,\mathrm{dist}\,(P_\infty, (1, 0)),$$

and for all $n \geq 1$,

$$\mathrm{dist}\,(P_\infty, L_n) \geq G\omega^n.$$

Then the length L of any normalized vector $v = (0, x, y, 1) \neq v_\infty$ satisfies

$$\frac{\ln d}{\ln \omega} - \frac{\ln G}{\ln \omega} + 1 \leq L < \frac{\ln d}{\ln \omega} - \frac{\ln B}{\ln \omega} + 9,$$

where $d = \text{dist}((x, y), P_\infty)$.

Proof. We will use the notation of Notation 4.1 so that for all $n \geq 0$ we write $a_n = A_n(x, y)$ and $U^n(v)$ equals

$$(a_n, a_n + a_{n+1}, a_n + 2a_{n+1} + a_{n+2}, a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3}).$$

Since v is normalized, (x, y) lies in the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$. The point of this triangular region farthest from P_∞ is $(1, 0)$, so

$$B \geq \omega \, \text{dist}(P_\infty, (1, 0)) > \omega \, d,$$

where the first inequality is a hypothesis of the lemma. Since $\omega < 1$, there exists a least $n \geq 0$ with $d \geq B\omega^n$. If $n > 0$, then of course $d < B\omega^{n-1}$; this is also true if $n = 0$ by the remarks above.

If $k \geq 0$, then

$$\text{dist}(P_\infty, P_{n+k}) \leq B\omega^{n+k} \leq B\omega^n \leq d.$$

Hence (x, y) is not inside the triangle $P_n P_{n+1} P_{n+2}$. (The closed disk with center P_∞ and radius $d = \text{dist}(P_\infty, (x, y))$ contains this triangle.) The sides of this triangle are the lines L_{n+1}, L_{n+2}, L_{n+3} by Lemma 4.3. Thus, for some $k \in \{1, 2, 3\}$, the point (x, y) cannot be on the same side of L_{n+k} as P_{n+k} (if $k \neq 3$) or P_n (if $k = 3$). (The interior of the triangle is the intersection of the open half planes $A_{n+j} > 0$ for $j = 1, 2, 3$.) Hence, $A_{n+k}(x, y) \leq 0$. Thus, there is a smallest positive $t \leq n + 3$ with $A_t(x, y) \leq 0$. Therefore, the first coordinate of $U^t(v)$ is not positive, so $U^{t-1}(v)$ cannot be monotone. Thus, there exists a least $j \leq t - 1 \leq n + 2$ with $U^j(v)$ not monotone. Hence by Lemma 3.5, $(0, x, y, 1)$ has length at most $j + 6 \leq n + 8$. Now, by the choice of n,

$$\ln B + (n - 1) \ln \omega > \ln d$$

so, keeping in mind that $\ln \omega < 0$,

$$L \le 8 + n < 8 + 1 + \frac{\ln d}{\ln \omega} - \frac{\ln B}{\ln \omega},$$

proving the upper bound for L in Lemma 4.5.

We may assume that there exists a unique integer $n \ge 1$ with

(4) $$G \omega^{n+1} \le d < G \omega^n.$$

(The use of the symbol n here has no connection with its use in the previous paragraph.) After all, if this is not true, then $d \ge G\omega$, so that

$$1 + \frac{\ln d}{\ln \omega} - \frac{\ln G}{\ln \omega} \le 2 \le L$$

(Lemma 3.3), proving the lemma in this case. But then (4) implies that

(5) $$\text{dist}\,(P_\infty, (x, y)) < G\omega^n \le \text{dist}\,(P_\infty, L_n)$$

so (x, y) lies on the same side of L_n as P_∞. Now, for large N, $A_n(P_N) > 0$, so by Lemma 4.4 we have $A_n(P_\infty) \ge 0$. But (5) then implies that $A_n(P_\infty) > 0$, so $A_n(x, y) > 0$. Similarly, if $1 \le k \le n$, then

$$\text{dist}\,(P_\infty, (x, y)) < G\omega^n \le G\omega^k \le \text{dist}\,(P_\infty, L_k),$$

so $a_k = A_k(x, y) > 0$. Therefore, if $1 \le k \le n - 2$, then

$$U^k(v) = (a_k, a_k + a_{k+1}, a_k + 2a_{k+1} + a_{k+2}, a_k + 3a_{k+1} + 3a_{k+2} + a_{k+3})$$

has its first three terms positive. Hence, for all $k \le n - 3$, $U^k(v)$ is monotone and hence $T^k(v) = U^k(v)$. Thus, by Lemma 3.3, $T^{n-3}(v)$ has length at least 5, so v has length at least $(n - 3) + 5 = n + 2$. But, from (4) we have $(\ln G/\ln \omega) + n + 1 \ge (\ln d/\ln \omega)$, so

$$L \ge n + 2 \ge 1 + \frac{\ln d}{\ln \omega} - \frac{\ln G}{\ln \omega}. \qquad \square$$

5. Proof of Theorem 1.1. In this section we will find numbers B and G satisfying the conditions of Lemma 4.5, thereby proving

Theorem 1.1. The bounds found will be sharp enough to imply the assertion of Remark 1.2 that the simple estimate of length differs from true length by less than 5.4. In particular, we will show that for all $n \geq 0$,

$$\text{(6)} \qquad\qquad \text{dist}\,(P_n, P_\infty) < 0.685\,\omega^n$$

and that for all $n \geq 1$ we have

$$\text{(7)} \qquad\qquad \text{dist}\,(P_\infty, L_n) > 0.135\,\omega^n.$$

The inequality (6) is easily checked when $n \leq 2$ using the values of P_n in the table in the proof of Lemma 4.3. (The value 0.685 was obtained by rounding upward the exact value of $\text{dist}\,(P_2, P_\infty)/\omega^2$, namely, $\sqrt{7r^2 - 4r - 26}/3$.) Now suppose $n \geq 3$. Then $\omega^3 \geq \omega^n$ and $1.116r^n < D_n$, as is easily checked using the fact that both the sequences (D_n) and (r^n) satisfy the second recursion relation. It is convenient to write

$$\delta_r = r^2/2 + r + 1 \quad \text{and} \quad \delta_s = s^2/2 + s + 1,$$

so that $P_\infty = (1/\delta_r, r^2/2\delta_r)$, as was noted in the proof of Lemma 4.4.

The triangle inequality implies the useful formula

$$(\text{Tr}\,(z))^2 + (\text{Tr}\,(w))^2 \leq 2\big(|z|^2 + |w|^2 + |z^2 + w^2|\big).$$

Combining these observations we can calculate that

$$\text{dist}\,(P_\infty, P_n)^2$$
$$= \left(\frac{M_n}{D_n} - \frac{1}{\delta_r}\right)^2 + \left(\frac{1}{2}\frac{M_{n+2}}{D_n} - \frac{r^2}{2\delta_r}\right)^2$$
$$= \frac{1}{(2\delta_r D_n)^2}\big[4(\delta_r M_n - D_n)^2 + (\delta_r M_{n+2} - r^2 D_n)^2\big]$$
$$= \frac{1}{(2\delta_r D_n)^2}\big[(\text{Tr}\,(2s^n\beta(\delta_r - \delta_s)))^2 + (\text{Tr}\,(s^n\beta(\delta_r s^2 - \delta_s r^2)))^2\big]$$
$$\leq \frac{1}{(2.232\delta_r r^n)^2}\,|s^n\beta|^2 E$$
$$= \left(\frac{|\beta|\sqrt{E}}{2.232\delta_r}\omega^n\right)^2 < (0.663\omega^n)^2,$$

where $E = 8|\delta_r - \delta_s|^2 + 2|\delta_r s^2 - \delta_s r^2|^2 + 2|4(\delta_r - \delta_s)^2 + (\delta_r s^2 - \delta_s r^2)^2|$.
This proves the inequality (6).

We begin the proof of formula (7) by finding the distance from P_∞
to L_n for any $n \geq 2$. It is convenient to set

$$\rho_n = D_{n-1}D_{n-2} \text{ dist}(P_{n-1}, P_{n-2})$$
$$= \sqrt{(M_{n-1}D_{n-2} - D_{n-1}M_{n-2})^2 + (N_{n-1}D_{n-2} - D_{n-1}N_{n-2})^2}.$$

Suppose that $n \geq 2$ and $k \geq 1$. Now by Lemma 4.3 L_n is the line
through P_{n-1} and P_{n-2}, so

$$\text{dist}(P_{n+k}, L_n) = \frac{1}{\text{dist}(P_{n-1}, P_{n-2})} \left\| \begin{matrix} M_{n+k}/D_{n+k} & N_{n+k}/D_{n+k} & 1 \\ M_{n-1}/D_{n-1} & N_{n-1}/D_{n-1} & 1 \\ M_{n-2}/D_{n-2} & N_{n-2}/D_{n-2} & 1 \end{matrix} \right\|$$
$$= \frac{|\Phi_{n,k}|}{\rho_n D_{n+k}},$$

where on the first line of the above display the double vertical lines
indicate the absolute value of the determinant of the matrix and where

$$\Phi_{n,k} = \begin{vmatrix} M_{n+k} & N_{n+k} & D_{n+k} \\ M_{n-1} & N_{n-1} & D_{n-1} \\ M_{n-2} & N_{n-2} & D_{n-2} \end{vmatrix}.$$

Since the sequences (M_n), (N_n) and (D_n) all satisfy the second recur-
sion relation (3), we have

$$(8) \qquad\qquad \Phi_{n,1} = 0 \quad \text{and} \quad \Phi_{n,2} = 4\Phi_{n,0}.$$

Also $\Phi_{n+1,0} = 4\Phi_{n,0}$ and hence by induction

$$(9) \qquad\qquad \Phi_{n,0} = 4^{n-2}\Phi_{2,0} = 4^{n-2} \begin{vmatrix} 0 & 2 & 6 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{4^n}{8}.$$

We next claim that for all $n \geq 2$ and $k \geq 0$,

$$(10) \qquad\qquad\qquad \Phi_{n,k} = -\frac{4^n M_{k+1}}{8}.$$

This is obvious from formula (9) if $k = 0$ since $M_1 = 1$. It is true for $k = 1$ and $k = 2$ by the formulas in (8) since $M_2 = 0$ and $M_3 = 4$. That (10) holds for all $k \geq 0$ now follows from the fact that the sequences $(M_k)_{k \geq 0}$ and $(\Phi_{n,k})_{k \geq 0}$ both satisfy the second recursion relation. Thus,

$$\begin{aligned}
\text{dist}\,(P_{n+k}, L_n) &= \frac{4^n M_{k+1}}{8\rho_n D_{n+k}} \\
&= \frac{r^{k+1}\alpha + \text{Tr}\,(s^{k+1}\beta)}{r^{n+k}\alpha\delta_r + \text{Tr}\,(s^{n+k}\delta_s\beta)}\,\frac{4^n}{8\rho_n} \longrightarrow \frac{4^n r}{8r^n \delta_r \rho_n}
\end{aligned}$$

as $k \to \infty$. Hence by Lemma 4.4

$$(11) \qquad\qquad \text{dist}\,(P_\infty, L_n) = \frac{4^n r}{8r^n \delta_r \rho_n}.$$

We now give an upper bound for ρ_n. We will use the identity

$$\text{Tr}\,(a\Delta)\text{Tr}\,(b\Delta) - \text{Tr}\,(ab\Delta)\text{Tr}\,(\Delta) = \Delta\overline{\Delta}(\overline{a} - a)(b - \overline{b}).$$

For any nonnegative i, j, we have

$$\begin{aligned}
&\left|M_{i+1}D_j - D_{j+1}M_i\right| \\
&= \left|\left(r^{i+1}\alpha + \text{Tr}\,(s^{i+1}\beta)\right)\left(r^j \delta_r \alpha + \text{Tr}\,(s^j \delta_s \beta)\right) \right. \\
&\qquad\qquad\qquad \left. - \left(r^{j+1}\delta_r \alpha + \text{Tr}\,(s^{j+1}\delta_s \beta)\right)\left(r^i \alpha + \text{Tr}\,(s^i \beta)\right)\right| \\
&= \left|\text{Tr}\,\left(r^{i+1}s^j \alpha\beta\delta_s + r^j s^{i+1}\alpha\beta\delta_r - r^{j+1}s^i \alpha\beta\delta_r - r^i s^{j+1}\alpha\beta\delta_s\right) \right. \\
&\qquad\qquad\qquad \left. + \text{Tr}\,(s^{i+1}\beta)\text{Tr}\,(s^j \delta_s \beta) - \text{Tr}\,(s^{j+1}\delta_s \beta)\text{Tr}\,(s^i \beta)\right| \\
&= \left|\text{Tr}\,\left(\alpha\beta(rs)^j\left(r^{i+1-j}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s\right)\right) \right. \\
&\qquad\qquad\qquad \left. + \beta s^i \overline{\beta}\,\overline{s}^i\left(\overline{s} - s\right)\left(\delta_s\,s^{j-i} - \overline{\delta}_s \overline{s}^{j-i}\right)\right|
\end{aligned}$$

$$= |rs|^j \left| \mathrm{Tr} \left(\alpha\beta \left(\frac{s}{|\bar{s}|} \right)^j \left(r^{i+1-j}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s \right) \right. \right.$$

$$\left. \left. + \beta\bar{\beta}(\bar{s}-s) \left(\frac{\bar{s}}{|s|} \right)^j \left(\frac{s}{r} \right)^j \left(\delta_s\bar{s}^{i-j} - \bar{\delta}_s s^{i-j} \right) \right) \right|$$

$$\le |rs|^j \left(2|\alpha\beta| \left| r^{i-j+1}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s \right| \right.$$

$$\left. + |\beta|^2 |\bar{s}-s| \left| \frac{s}{r} \right|^j \left| \delta_s\bar{s}^{i-j} - \bar{\delta}_s s^{i-j} \right| \right).$$

We apply the last inequality with $i = j = n-2$ and with $i-2 = j = n-2$ to see that for any $k \le n-2$

$$\rho_n^2 = (M_{n-1}D_{n-2} - D_{n-1}M_{n-2})^2 + \frac{1}{4}(M_{n+1}D_{n-2} - D_{n-1}M_n)^2$$

$$\le |rs|^{2(n-2)} \mathcal{E}_k$$

where

$$\mathcal{E}_k = \left(2|\alpha\beta| \left| r\delta_s + s\delta_r - r\delta_r - s\delta_s \right| + |\beta^2||\bar{s}-s| \left| \delta_s - \bar{\delta}_s|\omega^k \right) \right)^2$$

$$+ \frac{1}{4} \left(2|\alpha\beta| \left| (4r+4)\delta_s + (4s+4)\delta_r - rs^2\delta_r - r^2 s\delta_s \right| \right.$$

$$\left. + |\beta|^2|\bar{s}-s| \left| \delta_s\bar{s}^2 - \bar{\delta}_s s^2 \right| \omega^k \right)^2.$$

Thus, for any $k \le n-2$

$$\mathrm{dist}\,(P_\infty, L_n)^2 \ge \left(\frac{|rs|^n r}{8\delta_r} \right)^2 \frac{\omega^{2n}}{|rs|^{2(n-2)}\mathcal{E}_k},$$

so

$$\mathrm{dist}\,(P_\infty, L_n) \ge \frac{2(r+1)\omega^n}{r\delta_r\sqrt{\mathcal{E}_k}} = \frac{2}{r+2}\,\frac{\omega^n}{\sqrt{\mathcal{E}_k}}.$$

Hence, if $n \ge 7$ we have (taking $k=5$)

$$\mathrm{dist}\,(P_\infty, L_n) \ge \frac{2}{r+2}\,\frac{\omega^n}{\sqrt{\mathcal{E}_5}} > 0.135\,\omega^n.$$

The validity of formula (7) now follows by directly checking all cases in which $n \le 6$ using formula (11).

We now turn very directly to Theorem 1.1 (and Remark 1.2). If the normalized vector $v = (0, x, y, 1)$ has distance d from v_∞, then the vector (x, y) also has distance d from P_∞. Using the bounds $B = 0.685$ and $G = 0.135$ from formulas (6) and (7) we can apply Lemma 4.5 to conclude that v has length L satisfying

$$-2.30 < L - \mu \ln \frac{1}{d} < -0.62 + 9,$$

so that

(12) $$-5.30 < L - \left(3 + \mu \ln \frac{1}{d}\right) < 5.38.$$

Theorem 1.1 and Remark 1.2 follow immediately.

5.1 Last example. Using the second recursion relation (3) it is easy to verify that the vectors $V_n = (0, M_n, N_n, D_n)$ have length $L = n + 4$ for all $n \geq 0$. Thus, we have a family of vectors whose lengths are unbounded and easy to compute (the Tribonacci vectors give another such family). For these vectors the error in estimating length by the simple estimate is less than 1.3. Indeed, suppose $n > 1$. The simple estimate of length for V_n, applied to the normalization of V_n, is $L* = 3 + \mu \ln(1/d)$ where $d = \text{dist}(P_\infty, P_n)$. By Lemma 4.3 P_n is on L_{n-1} so $d \geq \text{dist}(P_\infty, L_{n-1})$. Hence by the inequalities (6) and (7),

$$0.135\omega^{n-1} < d < 0.685\omega^n.$$

It follows easily that

$$-1.29 < L - L* = n + 1 + \mu \ln d < 0.38.$$

For $n \leq 30$ we found no examples in which $|L - L*|$ was more than 0.9.

5.2 *Remark.* Formula (12) implies but is slightly stronger than the result in Remark 1.2. One could improve the inequality (12) of course by less crude estimates for the parameters B and G in Lemma 4.5.

6. Uniqueness proofs. In this section we give two arguments which were referred to earlier in the paper, but which were not actually

needed in the proof of Theorem 1.1. Combined with Lemma 3.2, the first shows that each cyclically monotone vector is equivalent to exactly one normalized vector.

6.1 Proposition. *If v, $w \in \mathbf{R}^4$ are normalized and equivalent, then $v = w$.*

Proof. For any cyclically monotone vector u, denote the entries of u by m_u, a_u, b_u, n_u where $m_u < a_u < b_u < n_u$. Call u *smooth* if $a_u + b_u < m_u + n_u$ and *rough* if $a_u + b_u > m_u + n_u$. (A normalized vector may not be smooth, but it is never rough.)

That v is equivalent to w implies that w can be obtained from v by applying a sequence of operators of the types R, S, M_e ($e > 0$), A_f, and M_{-1}. Let I denote the identity map on $\mathbf{R}^4$. R and S generate a dihedral group $\mathcal{G}$ whose elements commute with all other generators above. Using this fact and the additional relations

$$M_{-1}A_f = A_{-f}M_{-1}, \ M_{-1}M_e = M_eM_{-1}, \ M_{-1}^2 = I$$

and

$$M_eA_f = A_{fe}M_e,$$

we can write $w = M_eA_f\Delta M_{-1}^i v$ where $e > 0$, $\Delta \in \mathcal{G}$, and $i \in \{0,1\}$. If v is smooth, then we must have $i = 0$ since M_{-1} interchanges smoothness and roughness, while M_e, A_f and Δ preserve them, and since w cannot be rough. On the other hand, if v is not smooth, then we have $a_v + b_v = 1$ and hence $M_{-1}v = A_{-1}Rv$, so without loss of generality in this case we may also suppose $i = 0$. Also $\Delta = I$ since M_e^{-1} and A_f^{-1} preserve monotonicity and all elements of $\mathcal{G}$ except I destroy it. Then $f = 0$ (otherwise $m_w = ef \neq 0$) and finally $e = 1$ (otherwise $n_w = e \neq 1$). Therefore, $v = w$. $\square$

We end this section by sketching a short proof (promised at the end of Section 1) of Lotan's theorem [**3**] that *up to equivalence there is only one vector of infinite length*. This result and the above proposition together show that v_∞ *is the unique normalized vector of infinite length*. (That v_∞ has infinite length follows from the easily verified identity

$$T(v_\infty) = (\lambda, \lambda, \lambda, \lambda) + (1 - \lambda)v_\infty$$

where $\lambda = (6 - r^2)/2$, since this identity implies that for all $n \geq 1$,

$$T^n(v_\infty) = (1 - \lambda)^{n-1}(\lambda, \lambda, \lambda, \lambda) + (1 - \lambda)^n v_\infty \neq 0.)$$

Now suppose that v is any normalized vector of infinite length. Then for all $k \geq 0$ the vector $U^k(v)$ is monotone (Lemma 3.5), so $U^k(v) = T^k(v)$ for all $k \geq 0$. The complex eigenvalues of U have the form 0, t, u, $\bar{u}$ where t is real and strictly between 0 and 1 and u is nonreal with modulus greater than 1. Let corresponding eigenvectors be v_1, v_2 (both in $\mathbf{R}^4$), v_3, and $v_4 = \overline{v_3}$, respectively. Since the eigenvalues are distinct, the v_i are a basis for $\mathbf{C}^4$, so we can write $v = av_1 + bv_2 + cv_3 + \bar{c}\,\overline{v_3}$ for some a, b (both in $\mathbf{R}$), and $c \in \mathbf{C}$. For all $n \geq 1$ we have $T^n v = b\,t^n v_2 + cu^n v_3 + \bar{c}\;\bar{u}^n\overline{v_3}$. Now on the one hand $(T^n(v))_{n\geq 0}$ is bounded (no coordinate of any $T^n(v)$ is larger than the maximal coordinate of v), and, on the other hand, $t^n \to 0$ and $|u|^n = |\bar{u}|^n \to \infty$ as $n \to \infty$. It follows that $c = 0$ and so $v = av_1 + bv_2$. (The basic observation here is that if $e \in \mathbf{C}$, then $\mathrm{Tr}\,(eu^n)$ can be bounded only if $e = 0$.) Since clearly $b \neq 0$, v is equivalent to v_2. Thus, all vectors of infinite length are equivalent.

REFERENCES

1. E.R. Berlekamp, *The design of slowly shrinking squares*, Math. Comp. **29** (1975), 25–27.

2. C. Ciamberlini and A. Marengoni, *Su una interessante curiosità numerica*, Period Mat. **7** (1937), 25–30.

3. M. Lotan, *A problem in difference sets*, Amer. Math. Monthly **56** (1949), 535–541.

4. A. Ludington-Young, *Length of the n-number game*, Fibonacci Quart. **28** (1990), 259–265.

5. L. Meyers, *Ducci's four-number problem*: *A short bibliography*, Crux Math. **8** (1982), 262–266.

6. D. Ullman, *More on the four-numbers game*, Math. Mag. **65** (1992), 170–174.

7. W. Webb, *The length of the 4-number game*, Fibonacci Quart. **20** (1982), 33–35.

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