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A NOTE ON THE EXISTENCE OF SHAPE-PRESERVING PROJECTIONS

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ABSTRACT. Let X denote a (real) Banach space and V an n-dimensional subspace. We denote by $\mathcal{B} = \mathcal{B}(X, V)$ the space of all bounded linear operators from X into V; let $\mathcal{P}$ be the set of all projections in $\mathcal{B}$. For a given *cone* $S \subset X$, we denote by $\mathcal{P}_S$ the set projections $P \in \mathcal{P}$ such that $PS \subset S$. For a large class of cones S, we characterize when $\mathcal{P}_S \neq \varnothing$.

1. Introduction and preliminaries. The theory of *minimal projections* attempts to describe 'optimal' methods for extending the identity operator I from a Banach space V to an (Banach) overspace X. When V is of finite dimension there is no shortage of possible extensions, and one regards as optimal an extension of smallest possible operator norm. The possibility of extending I, or any linear operator, from V to X changes when we place the additional requirement that the extension leave invariant, or preserve, a particular set. By linearity, it is natural to choose the subset $S \subset X$ to be a *cone*—a convex subset closed under nonnegative scalar multiplication. And, as is often the case, S is chosen so that its elements have in common a particular characteristic, or shape; indeed, we say $f \in X$ *has shape* if $f \in S$ (for example, see [**2–4, 7**]). Thus, if $P : X \to V$ extends I and preserves S, i.e., $PS \subset S$, then we say P is a *shape-preserving projection*. For fixed X, V and S, we denote by $\mathcal{P}_S$ the set of all shape-preserving projections from X onto V. We are interested in characterizing when $\mathcal{P}_S \neq \varnothing$.

The intent of this note is to generalize a characterization of $\mathcal{P}_S$ given in [**3**]. We do so by significantly increasing the cones S for which the characterization is valid. In particular, we include the (rather common) case in which the intersection of the dual cone of S, defined below, and the unit sphere of X^*, the topological dual space of X, contains a weak* null net.

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Throughout this paper, we will denote the ball and sphere of real Banach space X by $B(X)$ and $S(X)$, respectively. For fixed positive integer n, $V \subset X$ will denote an n-dimensional subspace. $\mathcal{B}(X, V)$ will denote the space of linear operators from X into V and $\mathcal{P} \subset \mathcal{B}(X, V)$ will denote the set of all projections. In a (real) topological vector space, a *cone* K is a convex set, closed under nonnegative scalar multiplication. K is *pointed* if it contains no lines. For $\phi \in K$, let $[\phi]^+ := \{\alpha\phi \mid \alpha \geq 0\}$. We say $[\phi]^+$ is an *extreme ray* of K if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of K. When K is a closed, pointed cone of finite dimension we always have $K = \mathrm{co}\,(E(K))$ (this need not be the case when K is infinite dimensional; indeed, we note in [**5**] that it is possible that $E(K) = \varnothing$ despite K being closed and pointed).

Definition 1. Let X be a (fixed) Banach space and $V \subset X$ a (fixed) n-dimensional subspace. Let $S \subset X$ denote a closed cone. We say that $x \in X$ *has shape, in the sense of* S, whenever $x \in S$. If $P \in \mathcal{P}$ and $PS \subset S$, then we say P is a *shape-preserving projection*; we denote the set of all such projections by $\mathcal{P}_{\mathcal{S}}$. For a given cone S, define $S^* = \{\phi \in X^* \mid \langle x, \phi \rangle \geq 0 \text{ for all } x \in S\}$. We will refer to S^* as the *dual cone* of S.

Throughout the remainder of this paper we will consider X^* equipped with the weak* topology. Note that $S^* \subset X^*$ is a (weak*) closed cone; we will assume throughout that S^* is pointed. The following lemma indicates that S^* is in fact "dual" to S.

Lemma 1. *Let $x \in X$. If $\langle x, \phi \rangle \geq 0$ for all $\phi \in S^*$, then $x \in S$.*

Proof. We prove the contrapositive; suppose $x \in X$ such that $x \notin S$. Then, since S is closed and convex, there exists a separating functional $\phi \in X^*$ and $\alpha \in \mathbf{R}$ such that $\langle x, \phi \rangle < \alpha$ and

$$\langle s, \phi \rangle > \alpha, \quad \forall\, s \in S. \tag{1}$$

Note that we must have $\alpha < 0$ because $0 \in S$. In fact, for every $s \in S$, we claim

$$\langle s, \phi \rangle \geq 0 > \alpha. \tag{2}$$

To check this, suppose there exists $s_0 \in S$ such that $\langle s_0, \phi \rangle = \beta < 0$; this would imply

$$\left\langle \frac{\alpha}{\beta} s_0, \phi \right\rangle = \alpha$$

while $(\alpha/\beta)s_0 \in S$. And this is in contradiction to (1). The validity of (2) implies that $\phi \in S^*$ and this completes the proof. □

Lemma 2. *Let $P \in \mathcal{P}$. Then $PS \subset S \iff P^* S^* \subset S^*$.*

Proof. The proof is an immediate consequence of the duality equation $\langle Px, \phi \rangle = \langle x, P^* \phi \rangle$ and Lemma 1. □

2. Main result. Lemma 2 indicates that in the search for shape-preserving projections on X we may work exclusively in X^*. This is attractive since, once we fix a basis $v_1, \dots, v_n$ for V, every element of $P \in \mathcal{B}(X, V)$ is completely determined by n elements $u_1, \dots, u_n$ of X^* by expressing $P = \sum_{i=1}^n u_i \otimes v_i$ where $Px = \sum_{i=1}^n \langle x, u_i \rangle v_i$. In fact, we will be interested in the finite dimensional cone $S^*_{|V}$. Since $\dim(V) = n$ we know $\dim(S^*_{|V}) \le n$. Without loss, we can (and will) assume $\dim(S^*_{|V}) = n$; indeed, suppose $S^*_{|V}$ were k-dimensional where $0 \le k < n$. If $k = 0$, then every projection onto V is shape-preserving and the (following) characterization theorem holds trivially. For $k \ge 1$, choose a basis for V, $v_1, \dots, v_n$ such that, for all $\phi \in S^*$, $\langle v_i, \phi \rangle = 0$ for $i = 1, \dots, n-k$. With this basis, we can express any projection $P \in \mathcal{P}$ as $P = u_1 \otimes v_1 + \cdots + u_n \otimes v_n$ for some choice of u_i's $\in X^*$. And thus we note that projection $P : X \to V$ is shape-preserving if and only if projection $P_1 : X \to V_1$ is shape-preserving where $V_1 := [v_{n-k+1}, \dots, v_n]$ and $P_1 = u_{n-k+1} \otimes v_{n-k+1} + \cdots + u_n \otimes v_n$. Therefore, we might as well assume $S^*_{|V}$ is n-dimensional.

Before going forward it is necessary to place an additional assumption on the cone S^*; we describe this property in the following definition. Note that, in the context of our current considerations, we say a finite (possibly) signed measure μ with support $E \subset X^*$ is a *generalized representing measure* for $\phi \in X^*$ if $\langle x, \phi \rangle = \int_E \langle s, x \rangle \, du(s)$ for all $x \in X$. A nonnegative measure μ satisfying this equality is simply a *representing measure*.

Definition 2. Let X be a Hausdorff topological vector space over $\mathbf{R}$, and let X^* be the topological dual of X. We say that a pointed closed cone $K \subset X^*$ is *simplicial* if K can be recovered from its extreme rays, (i.e., $K = \overline{\text{co}}\,(E(K))$) and the set of extreme rays of K form an independent set (independent in the sense that any generalized representing measure for $x \in K$ supported on $E(K)$ must be a representing measure).

Proposition 1. *A pointed closed cone $K \subset X^*$ of finite dimension n is simplicial if and only if K has exactly n extreme rays.*

Proof. It is widely known that a pointed closed cone K of dimension n has at least n extreme rays; let $[y_1]^+, \dots , [y_n]^+$ be a linearly independent set of extreme rays of K. So to prove the necessity of the condition, it suffices to show that K has at most n extreme rays. To see this suppose K has $n+1$ extreme rays; let $[y_{n+1}]^+$ denote the $(n+1)$st. Because $\dim(K) = n$, there exist scalars $\alpha_1, \dots , \alpha_n$ such that $y_{n+1} = \Sigma_{i=1}^n \alpha_i y_i$, where $\alpha_i \neq 0$ for at least two i's and at least one of these nonzero α_i's is negative (as each y_i belongs to a distinct ray). This gives a generalized representing measure for y_{n+1} supported on $E(K)$ which is not a representing measure. Conversely, suppose K has n extreme rays. Choose linearly independent vectors $y_1, \dots , y_n$, one from each of the distinct n extreme rays. Then for any $x \in K$, $x = \Sigma_{i=1}^n \beta_i y_i$ where the β_i are nonnegative scalars (because K is a cone). The uniqueness of representation with respect to the basis $\{y_1, \dots , y_n\}$ implies that there exists no generalized representing measure for x supported on $E(K)$ which is not a representing measure. □

Throughout the remainder of the paper, we will assume that S^* is simplicial.

The main result of the paper is contained in the following theorem. It says that in order for there to exist a shape-preserving projection, it is necessary and sufficient that the (n-dimensional) cone $S^*_{|V}$ have exactly n extreme rays.

Theorem 1. *$\mathcal{P}_S \neq \varnothing$ if and only if the cone $S^*_{|V}$ is simplicial.*

For convenience, we will refer to the condition "$S^*_{|V}$ is simplicial" as simply *the simplicial condition*.

We prove the sufficiency and necessity of the simplicial condition in Section 4. But before presenting this we include several motivating examples. Example 1 illustrates the primary advantage of working in X^* to determine when $\mathcal{P}_S \neq \varnothing$. Examples 2 and 3 showcase how the necessity of the simplicial condition can fail outside of the projection case. Specifically, despite the existence of a shape-preserving operator, we find, in one instance, that $S^*_{|V}$ is not closed and in another case $S^*_{|V}$ is closed but possesses too many extreme rays. Finally, Example 4 indicates that a seemingly natural generalization of Theorem 1 fails to hold; that is, if $P \in \mathcal{B}$ and $PS \subset S$, it need not be the case that $(P^*S^*)_{|V}$ is contained in a simplicial subcone of $S^*_{|V}$.

3. Examples.

Example 1. What is gained by working in X^* rather than X? For example, suppose in determining if $\mathcal{P}_S \neq \varnothing$, we looked to the cone $D = S \cap V$ for information (note D is the dual cone to $S^*_{|V}$). Let $X = C[0,1]$ with the uniform norm $\|\cdot\|_\infty$, $V = \Pi_2$ (the space of quadratic algebraic polynomials) and S denote the cone of monotone increasing functions. Then $D = S \cap V$ is a cone that looks like a three-dimensional 'wedge' containing the line of constant functions. In fact, D remains unchanged (in shape) if we change the overspace to $X = C^1[0,1]$ (with $\|f\|_X = \max\{\|f\|_\infty, \|f'\|_\infty\}$). However, the cone $S^*_{|V}$ changes significantly with a change of overspace—from not simplicial (see Example 2) to simplicial (see Example 3). This reveals that in the former case no shape-preserving projection exists, i.e., there is no monotonicity-preserving projection from $C[0,1]$ onto the quadratics, while in the latter case we essentially obtain a formula for a shape-preserving projection. In the proof of the sufficiency of the simplicial condition below, we will use the "edges" of $S^*_{|V}$ to construct a shape-preserving projection.

Example 2. Let $X = C[0,1]$ with the uniform norm $\|\cdot\|_\infty$ and $S \subset X$ denote the cone of monotone increasing functions. An n-dimensional subspace V of X is said to be *monotonically complemented* if there exists a projection $P : X \to V$ that leaves S invariant. This class of subspaces is studied in [**4**], where it is also shown that, for every positive integer $k \geq 2$ the space of k-degree algebraic polynomials Π_k is not monotonically complemented. In fact, with $V = \Pi_2$ we will now show that the cone $S^*_{|_V}$ fails to be closed. This happens despite the existence of the monotonicity-preserving (linear) operator $B_2 : X \to \Pi_2$ which maps a continuous function to its second degree Bernstein polynomial (note the relative "closeness" of B_2 to a projection: for $i = 0, 1$, $B_2 x^i = x^i$ and $B_2 x^2 = (x^2 + x)/2$). Consider the cone $S^*_{|_V}$; since every element of this cone vanishes on the identically 1 function, we can regard $S^*_{|_V}$ as a subset of $\mathbf{R}^2$ by associating each $\phi_{|_V} \in S^*_{|_V}$ with the 2-tuple $\big(\langle x, \phi\rangle, \langle x^2, \phi\rangle\big)$ We claim that the ray determined by $e_1 := (1, 0)$ does not belong to the cone. Suppose, to the contrary, that there exists $\phi \in S^*$ such that $\phi_{|_V} = (1, 0)$. Let m be an arbitrary positive integer and consider the function $F(t) := mt^2 - G(t)$ where $G(t)$ is any C^1 function such that $0 \leq G'(t) \leq 2mt$ for all $t \in [0, 1]$. F is monotone so $\langle F, \phi\rangle \geq 0$; but G is also monotone and ϕ vanishes on t^2. The only possibility then is that ϕ vanishes on G. However, vanishing on all such G leads quickly to the conclusion that ϕ is unbounded. Therefore, the ray determined by e_1 does not belong to the cone and, moreover, the cone is not closed.

Example 3. Here we give an example in which S is preserved by an operator and $S^*_{|_V}$ is closed. However, $S^*_{|_V}$ will fail to be simplicial because the number of extreme rays of $S^*_{|_V}$ exceeds the dimension of $S^*_{|_V}$. At the end of this example, we fulfill a promise of Example 1 and verify that $S^*_{|_V}$ is simplicial when $V = \Pi_2$. Let $X = C^1[0, 1]$ with $\|f\|_X = \max\{\|f\|_\infty, \|f'\|_\infty\}$ and $V = \Pi_3 \subset X$. Let $S \subset X$ denote the cone of monotone increasing functions. Note that the third-degree Bernstein operator leaves S invariant. From the definition of X, we see that, for each $t \in [0, 1]$, derivative evaluation at t is a bounded linear functional; denote this functional by δ'_t and thus $\delta'_t \in S^* \subset X^*$. In fact, for each t, $[\delta'_t]^+$ defines an extreme ray of S^* and moreover $E(S^*) = \cup_{t\in[0,1]}[\delta'_t]^+$. Now, as done in Example 2, we can associate $S^*_{|V}$ with a cone in $\mathbf{R}^3$ via $\phi_{|V} \leftrightarrow \big(\langle x,\phi\rangle, \langle x^2,\phi\rangle, \langle x^3,\phi\rangle\big)$. Consider the restriction of $E(S^*)$ to V: in general, we always have $E(S^*_{|V}) \subset E(S^*)_{|V}$; however, in our current setting, we have that $E(S^*_{|V}) = E(S^*)_{|V}$. Thus, $E(S^*_{|V}) = \cup_{t\in[0,1]}[(\delta'_t)_{|V}]^+$ and so $S^*_{|V}$ has infinitely many extreme rays. That $S^*_{|V}$ is closed follows from the observation that the convex hull of $\{(\delta'_t)_{|V}\}_{t\in[0,1]}$ is a compact set that misses the origin. Notice the change in $S^*_{|V}$ if we replace $V = \Pi_3$ with $V = \Pi_2$; in this case $S^*_{|V}$ becomes a closed two-dimensional cone with

$$(\delta'_t)_{|V} = t(\delta'_1)_{|V} + (1-t)(\delta'_0)_{|V}$$

and thus it is simplicial.

Example 4. If $P \in \mathcal{P}_S$, then, as shown in the proof of Theorem 1, the cone P^*S^* must be simplicial. Suppose $A \in \mathcal{B}$ and $AS \subset S$; then $A^*S^* \subset S^*$ and so $(A^*S^*)_{|V} \subset S^*_{|V}$. While neither $(A^*S^*)_{|V}$ nor $S^*_{|V}$ need be simplicial, one might hope that $(A^*S^*)_{|V}$ must belong to a simplicial *subcone* of $S^*_{|V}$. We now show this is not the case. Let X be a Banach space with three-dimensional subspace $V = [v_1, v_2, v_3]$ and dual space X^*. We define the shape using four dual elements. Choose $\phi_1, \phi_2, \phi_3 \in X^*$ so that $\langle v_i, \phi_j\rangle = \delta_{ij}$. Choose a fourth element ϕ_4 so that

$$\langle v_1, \phi_4\rangle = -1 \quad \text{and} \quad \langle v_2, \phi_4\rangle = \langle v_3, \phi_4\rangle = 1$$

(thus $S^* = \text{cone}(\{\phi_\text{i}\}_{\text{i}=1}^4)$). Let $A = \sum_{i=1}^3 u_i \otimes v_i \in \mathcal{B}$ where

$$u_1 = \phi_1 + \phi_2, \quad u_2 = \phi_1 + \phi_3, \quad \text{and} \quad u_3 = \phi_2 + \phi_4.$$

To show $AS \subset S$, we need only establish $A^*S^* \subset S^*$; thus, with $A^*\phi_j = \sum_{i=1}^3 u_i \langle v_i, \phi_j\rangle$, we note

$$\begin{aligned} A^*\phi_1 &= u_1 = \phi_1 + \phi_2 \\ A^*\phi_2 &= u_2 = \phi_1 + \phi_3 \\ A^*\phi_3 &= u_3 = \phi_2 + \phi_4 \\ A^*\phi_4 &= -u_1 + u_2 + u_3 = \phi_3 + \phi_4. \end{aligned} \tag{3}$$

Therefore, $A^*S^* \subset S^*$. However, we claim that every subcone of $S^*_{|V}$ possessing exactly three extreme rays fails to contain $(A^*S^*)_{|V}$. Now the extreme rays of $A^*_{|V}$ are precisely $[\phi_{i|V}]^+$, for $i = 1, \ldots, 4$; and thus the extreme rays of $(A^*S^*)_{|V}$ are $[A^*\phi_{i|V}]^+$, for $i = 1, \ldots, 4$. From (3) we see that each of these extreme rays belongs to a distinct two-dimensional face of $S^*_{|V}$. Therefore, no simplicial (3-edged) subcone of $S^*_{|V}$ can contain $(A^*S^*)_{|V}$.

4. Lemmas and proofs. The following lemma establishes the sufficiency of the simplicial condition. While Theorem 1 is proven under the assumption that S^* is simplicial, we note that the proof of Lemma 3 does not require this assumption.

Lemma 3. *If $S^*_{|V}$ is simplicial, then $\mathcal{P}_S \neq \varnothing$.*

Proof. Suppose the number of extreme rays of $S^*_{|V}$ equals n. Choose one (nonzero) point from each ray and label the points as $u_{1|V}, \ldots, u_{n|V}$. Thus, we have

$$S^*_{|V} = \operatorname{cone}(u_{1|V}, \ldots, u_{n|V}). \tag{4}$$

Let $\mathbf{u} = (u_1, \ldots, u_n) \in (S^*)^n$ and $\mathbf{v} = (v_1, \ldots, v_n)^T$ be a basis for V; note that we may then write $\langle \mathbf{v}, u\rangle = \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{c}_u$ where $\mathbf{c}_u$ is the vector of nonnegative coefficients guaranteed by (4). Since $S^*_{|V}$ has n independent elements, the matrix $M = \langle \mathbf{v}, \mathbf{u}\rangle$ is nonsingular. Thus we may solve for $\mathbf{c}_u$ and write $\mathbf{c}_u = M^{-1}\langle \mathbf{v}, u\rangle$. Let $P := \mathbf{u}M^{-1} \otimes \mathbf{v}$; obviously, P is a projection from X into V. Moreover, for any $u \in S^*$, we have $P^*u = \mathbf{u}M^{-1}\langle \mathbf{v}, u\rangle \in S^*$ since $M^{-1}\langle \mathbf{v}, u\rangle$ has nonnegative entries. By Lemma 2 the proof is complete. □

To establish the necessity of the simplicial condition will require more work. The approach we take is to attempt to represent, in a useful way, elements of S^* using only points that belong to extreme rays of S^*. To facilitate this, we define $E_1 := E(S^*) \cap S(X^*)$. One might hope that every element of S^* can be written as a positive scalar multiple of an element from $\overline{\text{co}}\,(E_1)$ (where the closure is taken with respect to the weak* topology). However this is not always possible. For example, consider $X = l_2$ and $S^* \subset X^* = l_2$ consisting of all nonnegative sequences. S^* is clearly a simplicial cone and $E(S^*) = \cup_{i \in \mathbf{N}}[e_i]^+$

where $e_i(j) = \delta_{ij}$. Note that $\overline{\text{co}}\,(E_1)$ contains only summable sequences ($x \in l_2$ is summable if $\sum_i x(i)$ is finite valued). But of course S^* contains sequences which are *strictly* square summable, i.e., sequences $x(i)$ which are not summable but for which $(x(i)^2)$ is summable, and thus it is exactly these elements that cannot be expressed as positive scalar multiples of elements from $\overline{\text{co}}\,(E_1)$. The following proposition gives a condition which will allow (a set homeomorphic to) $\overline{\text{co}}\,(E_1)$ to 'reach' every element of S^*. Note in the following that all closures are taken with respect to the weak* topology.

Proposition 2. *Let $E_1 = E(S^*) \cap S(X^*)$. If $0 \notin \overline{E}_1$, then there exists a compact convex set $C \subset S^*$ such that every element of S^* is a positive scalar multiple of an element from C. Moreover, distinct extreme points of C belong to distinct extreme rays of S^*.*

Proof. We construct the set C in two steps. First we define the cone $K := \{\rho e \mid \rho \geq 0,\ e \in \overline{\text{co}}\,(E_1)\}$. Note that $K \subset S^*$; we claim $K = S^*$. From the definitions of K and $S \subset X$, it is clear that $f \in S$ if and only if $\langle f, \phi \rangle \geq 0$ for all $\phi \in K$. Therefore, if K is closed then, by an argument identical to that in the proof of Lemma 1, we will have $K = S^*$. We now verify that K is closed. To do this, we first establish that $0 \notin \overline{\text{co}}\,(E_1)$. From our assumption, $0 \notin \overline{E}_1$ and therefore, by the Krein-Milman theorem, 0 is not an extreme point of $\overline{\text{co}}\,(E_1)$. Suppose $0 \in \overline{\text{co}}\,(E_1)$; since 0 is not extreme there exists nonzero $x, y \in \overline{\text{co}}\,(E_1)$ such that $0 = x + y$; but this would imply that $-x \in S^*$ and this contradicts the fact that S^* is pointed. Thus, $0 \notin \overline{\text{co}}\,(E_1)$. Now let $\{y_\alpha\} \subset K$ be a net that converges to y; we may write $y_\alpha = \rho_\alpha e_\alpha$, where $e_\alpha \in \overline{\text{co}}\,(E_1)$. By compactness, there exists a convergent subnet $\{e_{\alpha_\beta}\}$ of $\{e_\alpha\}$ possessing a nonzero limit point, call it e, contained in $\overline{\text{co}}\,(E_1)$. The (real) net $\{\rho_\alpha\}$ is bounded and thus, passing to subnets if necessary, we have $\rho_{\alpha_\beta} \to \rho$ for some $\rho \in \mathbf{R}^+$. Therefore

$$y = \lim y_\alpha = \lim y_{\alpha_\beta} = \lim \rho_{\alpha_\beta} e_{\alpha_\beta} = \rho e$$

and hence K is closed which implies $K = S^*$.

We begin the second step by noting that 0 and $\overline{\text{co}}\,(E_1)$ can be strictly separated with a hyperplane H, i.e., there exists $\alpha > 0$ and $x \in X$ such that $\langle x, \phi \rangle \geq \alpha$ for all $\phi \in \overline{\text{co}}\,(E_1)$. So $H = x^{-1}(\{\alpha\})$. Let

$$C := \{\alpha\phi / \langle x, \phi \rangle \mid \phi \in \overline{\text{co}}\,(E_1)\};$$

thus, C is the intersection of H and S^* and, as such, is convex and compact. Clearly every element of S^* can be (positively) scaled into C. Let T denote the set of extreme points of C. Since $T \subset H \cap S^*$, it is clear that distinct points of T belong to distinct rays of S^*. To see that the elements of T belong to extreme rays, i.e., $[T]^+ \subset E(S^*)$, consider the set $C_1 := \overline{\text{co}}\,(C \cup 0)$. It follows from the definition of C that C_1 is convex and compact, that $S^* \backslash C_1$ is convex and that the set of nonzero extreme points of C_1 is T. We show $[T]^+ \subset E(S^*)$ by contradiction; let $x \in T$ and assume $[x]^+ \not\subset E(S^*)$. Then $x \in \text{co}\,([\phi]^+, [\psi]^+)$ for some $\phi, \psi \in S^* \backslash [x]^+$. The properties of C_1 guarantee the existence of positive constants $s, t \in \mathbf{R}$ such that $s\phi, t\psi \in C_1$ and

$$s = \sup\{c \in \mathbf{R} \mid c\phi \in C_1\} \quad \text{and} \quad t = \sup\{c \in \mathbf{R} \mid c\psi \in C_1\}. \tag{5}$$

Finally consider the triangle formed by vertices 0, $s\phi$ and $t\psi$. If x belongs to this triangle then x is not an extreme point of C_1 and we have a contradiction; if x fails to belong to the triangle, then there exist $\hat{s} > s$ and $\hat{t} > t$ such that $x = \hat{s}\phi/2 + \hat{t}\psi/2$ which, by (5) and the convexity of $S^* \backslash C_1$, would imply $x \notin C_1$—again a contradiction. Therefore $[T]^+ \subset E(S^*)$. □

Note 1. Using the language of convex analysis, Proposition 2 verifies that S^* possesses a compact *base* whenever $0 \notin \overline{E}_1$. The discussion prior to the proposition illustrates that not every closed, pointed cone contains a base. A base is generalized by the notion of a *cap*: a compact, convex subset of a cone such that the cone, take away the subset, is still convex. An introduction to bases and caps can be found in [**6**] and a more definitive treatment in [**1**].

Lemma 4. *If $\mathcal{P}_S \neq \varnothing$, then the cone $S^*_{|_V}$ is closed.*

Proof. Let $P \in \mathcal{P}_S$, and let $\mathbf{v} = [v_1, \dots, v_n]^T$ denote a fixed basis for V. Let $\overline{P^*S^*}$ denote the closure of P^*S^*, and let $P^*\phi \in \overline{P^*S^*} \subset P^*X^*$. Choose a sequence $\{P^*\phi_k\}_{k=1}^{\infty} \subset P^*S^*$ such that $P^*\phi_k \to P^*\phi$. Notice, by Lemma 2, $\{P^*\phi_k\}_{k=1}^{\infty} \subset S^*$. S^* is weak*-closed and therefore $P^*\phi \in S^*$; this implies $P^*\phi \in P^*S^*$ since $(P^*)^2 = P^*$. Thus P^*S^* is closed. Note that P^*S^* is homeomorphic to $(P^*S^*)_{|_V}$ and thus $(P^*S^*)_{|_V}$ is closed. Finally, we claim $(P^*S^*)_{|_V} = S^*_{|_V}$. To verify this, choose $\phi \in S^*$, $v \in V$ and consider

$$\langle v, P^*\phi\rangle = \langle Pv, \phi\rangle = \langle v, \phi\rangle,$$

where the last equality follows from the fact that P is a projection. But this equation simply says that $P^*\phi$ and ϕ agree on V, thus establishing the claim. From here we can conclude that $S^*_{|V}$ is closed. □

Lemma 5. *If $\mathcal{P}_S \neq \varnothing$, then $S^*_{|V}$ is simplicial.*

Proof. From Lemma 4 we have $E(S^*_{|V}) \neq \varnothing$. We will show that the number of extreme rays of $S^*_{|V}$ is exactly n. Let $P = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{P}_S$ and, from Lemma 2, we have $P^*S^* \subset S^*$. There is an obvious bijection between P^*S^* and $(P^*S^*)_{|V}$; and from our work above in Lemma 4, we have $(P^*S^*)_{|V} = S^*_{|V}$. This implies that the number of extreme rays of $S^*_{|V}$ is equal to the number of extreme rays of P^*S^*, which we now show must be n. Since P^*S^* is n-dimensional, there exists a linearly independent subset $\{P^*w_1, \dots, P^*w_n\}$ such that $[P^*w_i]^+ \in E(P^*S^*)$ for each i. We will now show that it is impossible for there to be any other extreme rays. Consider first the case that $0 \notin \overline{E}_1$. From Proposition 2 (and the positive scaling of each w_i), there exists a compact set C such that $P^*w_i \in C \subset S^*$ for each i. This implies that, for each P^*w_i, we have a representing (probability) measure μ_i on C (in the sense of Choquet; see [**6**]) supported on a subset S_i containing extreme points of C such that

$$\begin{aligned} P^*w_i = P^*(P^*w_i) &= \sum_{j=1}^n \langle P^*w_i, v_j\rangle u_j \\ &= \sum_{j=1}^n \int_{S_i} \langle v_j, s\rangle \, d\mu_i \; u_j \\ &= \int_{S_i} \sum_{j=1}^n \langle v_j, s\rangle u_j \, d\mu_i \\ &= \int_{S_i} P^*s \, d\mu_i. \end{aligned} \tag{6}$$

But each P^*w_i belongs to an extreme ray of P^*S^* and thus for μ_i almost everywhere $s \in S_i$ we must have $P^*s = c_s P^*w_i$, where $c_s \geq 0$

(note that if $c_s = 0$ then $P^*s = 0$ and, consequently, we may remove such s from S_i and not affect (6)). Therefore, we may conclude that, for each i, there exists $\widehat{S}_i \subset S_i$ such that

$$\mu_i(\widehat{S}_i) > 0 \quad \text{and} \quad \mu_j(\widehat{S}_i) = 0 \quad \text{whenever} \quad j \neq i. \tag{7}$$

Now suppose there exists $P^*w_{n+1} \in E(P^*S^*)$ such that $[P^*w_{n+1}]^+ \neq [P^*w_i]^+$, $i = 1, \ldots, n$. Then the n-dimensionality of P^*S^* implies the existence of constants c_i, $i = 1, \ldots, n$ such that, for all $x \in X$,

$$\langle P^*w_{n+1}, x\rangle = \langle c_1 P^*w_1 + \cdots + c_n P^*w_n, x\rangle. \tag{8}$$

Since each ray $[P^*w_i]^+$ is extreme, it follows that there exists $i \in \{1, \ldots, n\}$ such that $c_i < 0$. Let $\mu = \sum_{i=1}^n c_i\mu_i$, where each μ_i is the representing measure from (6). Note from (7) that μ is necessarily a signed measure. And finally, by rewriting (8) as

$$\langle P^*w_{n+1}, x\rangle = \int_{S_1\cup\cdots\cup S_n} \langle s, x\rangle \, d\mu, \tag{9}$$

we obtain a contradiction to the fact that S^* is simplicial, since μ is a signed measure with support on $E(S^*)$. Thus we must have $|E(P^*S^*)| = n$.

In the case that $0 \in \overline{E}_1$, begin by writing

$$P = \sum_{i=1}^n u_i \otimes v_i \quad \text{for} \quad P \in \mathcal{P}_S. \tag{10}$$

Via a change basis, we may assume $u_i \in S^*$ for each i and recall, for each i, $P^*u_i = u_i \in S^*$ since $P \in \mathcal{P}_S$. Consider the simplicial cone

$$Q^* := \overline{\text{co}}\left(\bigcup\{[e_s + u_1]^+ \mid e_s \in E_1 := E(S^*) \cap S(X^*)\}\right)$$

and note that $P^*Q^* \subset Q^*$. By construction we have $E(Q^*) = \cup\{[e_s + u_1]^+\}$. Since S^* is pointed, and thus $-u_1 \notin S^*$, it follows that there exists $\lambda > 0$ such that $\lambda < \|e_s + u_1\| \leq 1 + \|u_1\|$ for every $e_s \in E_1$. Let $\widehat{E_1} := \{e_s + u_1 \mid e_s \in E_1\}$ (we regard $\widehat{E_1}$ as the set of "normalized" extreme rays of Q^*, as we do for E_1 relative to S^*).

Now 0 is not in the weak* closure of $\widehat{E_1}$, and thus, as in the previous case, we must conclude that $Q^*_{|V}$ has exactly n extreme rays. And, since $u_1 \in S^*$, the cones $Q^*_{|V}$ and $S^*_{|V}$ must have the same number of extreme rays, which completes the proof. $\square$

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