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EQUAL SUMS OF SIXTH POWERS AND QUADRATIC LINE COMPLEXES

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1. Introduction. This paper is concerned with the Diophantine equation:

$$x^6 + y^6 + z^6 = u^6 + v^6 + w^6. \tag{1.1}$$

We present a relation between this equation and Kummer's quartic surfaces through the theory of quadratic line complexes.

To this date there have been many numerical solutions to (1.1) discovered by various computer searches. (See Section 7 for more historical details.) A large part of them also satisfy the quadratic equation

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2. \tag{1.2}$$

Furthermore, Bremner [**1**] shows that among those simultaneous solutions, most also satisfy the system of equations:

$$\begin{cases} x^2 + xu - u^2 = w^2 + wz - z^2, \\ y^2 + yv - v^2 = u^2 + ux - x^2, \\ z^2 + zw - w^2 = v^2 + vy - y^2. \end{cases} \tag{1.3}$$

Note that we recover (1.1) by cubing each equation in (1.3) and adding them.

Geometrically, this can be seen as follows. Let V_4 be the fourfold defined by (1.1). Many of the rational points of V_4 are contained in the subthreefold V_3 cut out by the quadric (1.2). The system of equations (1.3) determines a $K3$ surface K_B contained in V_3, and most of the known rational points of V_3 are contained in K_B. Bremner [**1**] investigated this $K3$ surface geometrically in depth. Among others, he gave a theoretical method to find all smooth parametric solutions.

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In this paper we push his investigation one step further in order to understand the geometry of the surface K_B even more. Our results are summarized as follows.

Theorem 1.1. *Bremner's $K3$ surface K_B is the minimal desingularization of Kummer's quartic surface S with 16 double points given by the equation*

$$
\begin{aligned}
(1.4)\quad S : a_0^4 + a_1^4 + a_2^4 + a_3^4 + 3\,(a_0^2a_1^2 + a_1^2a_2^2 + a_2^2a_0^2)\\
- 3\,(a_0^2 + a_1^2 + a_2^2)\,a_3^2 + 4a_0a_1a_2a_3 = 0.
\end{aligned}
$$

The surface S is obtained from the quadratic line complex X defined by the quadric

$$F : x_0^2 + \frac{1}{2}\,x_1^2 - x_2^2 + x_3^2 - \frac{1}{2}\,x_4^2 - x_5^2 - 2x_0x_5 + x_1x_4 = 0,$$

together with the Plücker relation $G : x_0x_5 - x_1x_4 + x_2x_3 = 0$. K_B is the Kummer surface, in the modern sense, of the Jacobian $J(C)$ of the curve C of genus 2 given by

$$C : y^2 = (x^2 + 1)(x^2 + 2x + 5)(x^2 - 2x + 5).$$

The reduced automorphism group $\mathrm{RA}\,(C)$, defined to be $\mathrm{Aut}\,(C)/\langle\iota\rangle$, where ι is the hyperelliptic involution, is isomorphic to the symmetric group S_3. $J(C)$ is isogenous to the product of two copies of the elliptic curve of conductor $320 = 2^6 \cdot 5$ given by

$$E : y^2 = x(x^2 + 4x + 20).$$

In Section 4 we will give explicit formulas for the maps between K_B and S, together with the maps between K_B and the dual Kummer surface S^*. For example, we see that Subba-Rao's first solution to (1.1) (and (1.3)) $(x : y : z : u : v : w) = (3 : 19 : 22 : -23 : 10 : -15)$ corresponds to the solution $(a_0 : a_1 : a_2 : a_3) = (5 : 3 : 4 : 7)$ to S. Also, we study how the lines contained in K_B map to the double points of S and S^*.

In Section 5 we study the curve C of genus 2 in more detail. From the fact that $J(C)$ is isogenous to $E \times E$, and that E does not have complex multiplication, we conclude that the rank of the Néron-Severi group $NS(K_B, \mathbf{C})$ is 19. Bremner [**1**] showed that the rank of the Néron-Severi group $NS(K_B, \mathbf{Q}(i))$ is 19, using a descent argument to the pencil of elliptic curves discovered by Brudno and Kaplansky [**2**]. Bremner left the determination of $NS(K_B, \mathbf{C})$ as a problem, and this provided a first motivation for us to study the surface K_B in more detail.

Also from the fact that $J(C)$ is isogenous to $E \times E$ over $\mathbf{Q}$, we obtain a rational point on K_B from two rational points in the Mordell-Weil group of a quadratic twist E^d. This is because the $E \times E/\{\pm 1\}$ is isomorphic over $\mathbf{Q}$ to $E^d \times E^d/\{\pm 1\}$. (See Remark 5.4.)

In Section 6 we study the symmetries, i.e., the automorphisms induced by a linear map of the ambient space, of K_B, S and S^*. It turns out that K_B has a bigger symmetry than S and S^* do, the difference being accounted for by the isomorphism between S and S^*.

2. Quadratic line complexes. In this section we summarize the results we need concerning quadratic line complexes. We refer to Griffiths-Harris [**5**, Chapter 6] for detail.

Let G be the Grassmannian of all lines in $\mathbf{P}^3$. G may be identified with the Grassmannian $G(2,4)$ of 2-planes in a vector space V of dimension 4. The Plücker embedding

$$G(2,4) \to \mathbf{P}(\textstyle\bigwedge^2 V)$$

is given by sending the 2-plane spanned by v_1, $v_2 \in V$ to the wedge product $v_1 \wedge v_2 \in \bigwedge^2 V$.

Now we choose a basis (e_0, e_1, e_2, e_3) of V. Then

$$(e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$$

is a basis of $\bigwedge^2 V$. We use this basis to identify $\mathbf{P}(\bigwedge^2 V)$ with $\mathbf{P}^5$. With this identification the line in $\mathbf{P}^3$ passing through $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$ is given by

$$(a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2).$$

It is known that an equivalence class of multi-vector $\omega \in \bigwedge^2 V$ is an image of the Plücker embedding if and only if $\omega \wedge \omega = 0$. If we write

$$\omega = x_0\, e_0 \wedge e_1 + x_1\, e_0 \wedge e_2 + x_2\, e_0 \wedge e_3 + x_3\, e_1 \wedge e_2 + x_4\, e_1 \wedge e_3 + x_5\, e_2 \wedge e_3,$$

then the condition $\omega \wedge \omega = 0$ is equivalent to

$$2(x_0 x_5 - x_1 x_4 + x_2 x_3) = 0.$$

Thus the image of the Plücker embedding is the quadric hypersurface given by the above equation. From now on we identify G with this quadric hypersurface in $\mathbf{P}^5$.

For $x \in G$ we denote by l_x the corresponding line in $\mathbf{P}^3$. For a point $p \in \mathbf{P}^3$ we define $\sigma(p)$ to be the set of lines passing through p, and for a hyperplane $h \in (\mathbf{P}^3)^*$ we define $\sigma(h)$ to be the set of lines contained in the plane h:

$$\sigma(p) = \{x \in G \mid p \in l_x\}, \quad \sigma(h) = \{x \in G \mid l_x \subset h\}.$$

These are so-called Schubert cycles. Both $\sigma(p)$ and $\sigma(h)$ are 2-planes contained in the parameter space $G \subset \mathbf{P}^5$. The intersection $\sigma(p) \cap \sigma(h)$, denoted by $\sigma(p, h)$, is a line in $\mathbf{P}^5$, which represents the pencil of lines in $\mathbf{P}^3$ contained in h and passing through the 'focus' p. In fact, it is known that every 2-plane contained in G is either $\sigma(p)$ for some p or $\sigma(h)$ for some h. Also it is known that a line L in G is the intersection of a unique $\sigma(p)$ with a unique $\sigma(h)$:

$$L = \sigma(p, h) = \{x \in G \mid p \in l_x \subset h\}.$$

We sometimes write $h = h_L$, $p = p_L$ and $L = L_{p,h}$. Thus a point on the line L in G corresponds uniquely to a line in the pencil in $h_L \subset \mathbf{P}^3$ with focus p_L.

A quadratic line complex X is defined as the intersection of G with another quadric hypersurface F. We assume $X = G \cap F$ to be smooth.

For any point $p \in \mathbf{P}^3$, the intersection of $\sigma(p)$ with the quadric F can be viewed as a conic in the 2-plane $\sigma(p)$. There are three possibilities:

(1) $\sigma(p)$ meets F transversely, and thus $F \cap \sigma(p)$ is a smooth conic.

(2) $\sigma(p)$ is tangent to F at a point, and thus $F \cap \sigma(p)$ is the union of two distinct lines.

(3) $\sigma(p)$ is tangent to F along a line, and thus $F \cap \sigma(p)$ consists of this line only, which should be viewed as a double line.

If the case (2) occurs and $F \cap \sigma(p) = L \cup L'$, then each of two lines L and L' represents a pencil of lines in G with focus p. These two pencils are called *confocal pencils*, meaning having the same focus p. The intersection $L \cap L'$ represents the line l contained in both of the pencil. The line l is called a *singular line* of the complex X. If the case (3) occurs, then all the lines contained in the pencil represented by $F \cap \sigma(p)$ are considered to be singular.

Dually, for every hyperplane $h \in (\mathbf{P}^3)^*$, the set $F \cap \sigma(h)$ can be regarded as a conic in the 2-plane $\sigma(h)$. Again, there are three possibilities:

$(1)^*$ $\sigma(h)$ meets F transversely, and thus $F \cap \sigma(h)$ is a smooth conic.

$(2)^*$ $\sigma(h)$ is tangent to F at a point, and thus $F \cap \sigma(h)$ is the union of two distinct lines.

$(3)^*$ $\sigma(h)$ is tangent to F along a line, and thus $F \cap \sigma(h)$ consists of this line only, which should be viewed as a double line.

If the case $(2)^*$ occurs, then $F \cap \sigma(h) = L \cup L'$ represents two pencils of lines in h with two distinct focii p_1 and p_2. The line $l = \overline{p_1 p_2}$ turns out to be a singular line. If the case $(3)^*$ occurs, then all the lines contained in the pencil represented by $F \cap \sigma(h)$ are singular.

Define three surfaces:

$$\begin{aligned} \Sigma &= \{x \in X \mid l_x \text{ is a singular line}\}, \\ S &= \{p \in \mathbf{P}^3 \mid \sigma(p) \cap F \text{ is singular}\}, \\ S^* &= \{h \in (\mathbf{P}^3)^* \mid \sigma(h) \cap F \text{ is singular}\}. \end{aligned}$$

If l_x is a singular line, then it is contained in a pencil $\sigma(p, h)$ with $p \in l_x \subset h$. It can be shown that such p and h are unique, and thus we have maps

$$\begin{aligned} \pi &: \Sigma \to S; \quad l_x \mapsto p, \\ \pi^* &: \Sigma \to S^*; \quad l_x \mapsto h. \end{aligned}$$

S is called the associated Kummer surface of the quadratic line complex, and S^* is called the dual Kummer surface. Let R be the set of

p where the case (3) occurs, and let R^* be the set of h where the case $(3)^*$ occurs. We list properties of these surfaces.

• Both S and S^* are quartic surfaces with 16 double points, the singular locus being R and R^*, respectively.

• Σ is a smooth $K3$ surface defined by three quadrics, G, F and another quadric H.

• Σ contains 32 lines, 16 of them are mapped to R by π, and the other 16 are mapped to R^* by π^*.

• Σ is the minimal desingularization of both S and S^*.

• S and S^* are dual to each other. They are isomorphic to each other over some extension of the base field.

In order to write down equations of Σ, the following characterization is essential.

Lemma 2.1 [**5**, p. 767]. *A point $x \in X$ belongs to Σ if and only if the tangent plane $T_x(F)$ to F is also tangent to G at some point x'.*

Suppose that G and F are given respectively by

$$ {}^t x Q_G x = 0, \quad \text{and} \quad {}^t x Q_F x = 0, $$

where Q_G and Q_F are symmetric matrices. The equations of the tangent plane to G and F at a point $a \in X$ is then given respectively by

$$ {}^t a Q_G x = 0, \quad \text{and} \quad {}^t a Q_F x = 0. $$

Conversely, a row vector ${}^t y$ viewed as a point in $(\mathbf{P}^3)^*$ is a tangent plane to G if and only if ${}^t y$ is of the form ${}^t b Q_G$ for some b. Thus, if $T_x(F)$ is tangent to G at x', then we have

$$ {}^t x Q_F = \lambda\, {}^t x' Q_G \qquad \text{for some } \lambda \in k. $$

The condition ${}^t x' Q_G x' = 0$ then gives

$$ {}^t x Q_F Q_G^{-1} Q_F x = 0. \tag{2.5} $$

Let H be the quadric defined by (2.5). Then we have

$$\Sigma = G \cap F \cap H.$$

It is known that the intersection is everywhere transverse, and thus Σ is a nonsingular $K3$ surface [**5**, p. 767].

3. Another basis of Bremner's net of quadrics. Let K_B be Bremner's $K3$ surface defined by (1.3). As Bremner shows, K_B contains 32 lines, which gives us a reason to suspect that K_B is isomorphic to the surface Σ for some quadratic line complex. In order to verify this is actually the case, consider the net (= two-dimensional linear system) of quadrics spanned by the three quadrics (1.3), and find a basis of the form (G, F, H) as described in the previous section. To do so, we write the equations of K_B in terms of matrices. Let Q_1, Q_2 and Q_3 be three symmetric matrices such that the equations (1.3) are written as

$$ {}^t\xi Q_1 \xi = 0, \quad {}^t\xi Q_2 \xi = 0, \quad {}^t\xi Q_3 \xi = 0, \tag{3.6}$$

where ${}^t\xi = (x, y, z, u, v, w)$. We would like to find (r_1, r_2, r_3), (s_1, s_2, s_3) and (t_1, t_2, t_3) such that

$$\begin{aligned} Q'_1 &= r_1 Q_1 + r_2 Q_2 + r_3 Q_3, \\ Q'_2 &= s_1 Q_1 + s_2 Q_2 + s_3 Q_3, \\ Q'_2 {Q'_1}^{-1} Q'_2 &= t_1 Q_1 + t_2 Q_2 + t_3 Q_3. \end{aligned}$$

Straightforward calculations show that every triple $(r_1, r_2, r_3) \neq (0, 0, 0)$ satisfying

$$r_1^2 + r_2^2 + r_3^2 + 3r_1 r_2 + 3r_2 r_3 + 3r_3 r_1 = 0$$

yields solutions (s_1, s_2, s_3) and (t_1, t_2, t_3). Since we only need one solution, we choose r_i's so that the Q_i's are as simple as possible. We set $(r_1, r_2, r_3) = (1/2, -1/2, -1/2)$, and obtain $(s_1, s_2, s_3) = (-1/2, 3/2, -1/2)$ and $(t_1, t_2, t_3) = (-2, -2, 2)$. Then, the quadratic forms corresponding to the Q'_i's are

$$\begin{aligned} q'_1(\xi) &= -y^2 + v^2 + xu - zw, \\ q'_2(\xi) &= x^2 + y^2 - z^2 - u^2 - v^2 + w^2 - 2xu + 2yv, \\ q'_3(\xi) &= -4x^2 + 4u^2 - 4yv + 4zw. \end{aligned}$$

By the change of variables

(3.7)
$$x_0 = x, \quad x_1 = y + v, \quad x_2 = z, \quad x_3 = -w, \quad x_4 = y - v, \quad x_5 = u,$$

these forms are transformed to

$$\begin{aligned} q_1''(x) &= x_0x_5 - x_1x_4 + x_2x_3, \\ q_2''(x) &= x_0^2 + \frac{1}{2}\,x_1^2 - x_2^2 + x_3^2 - \frac{1}{2}\,x_4^2 - x_5^2 - 2x_0x_5 + x_1x_4, \\ q_3''(x) &= -\,4x_0^2 - x_1^2 + x_4^2 + 4x_5^2 - 4x_2x_3, \end{aligned}$$

where $x = {}^t(x_0, x_1, x_2, x_3, x_4, x_5)$. This shows that the surface K_B is the smooth Kummer surface Σ associated to the quadratic line complex defined by $F : q_2''(x) = 0$.

4. The quartic Kummer surface. Now that we showed that K_B is obtained from the quadratic line complex $X = G \cap F$, we would like to describe the quartic Kummer surface S and its dual S^*.

Proposition 4.1. *The Kummer surface S associated to the quadratic line complex $F : q_2''(x) = 0$ is given by the equation*

$$\begin{aligned} (4.8) \quad S : a_0^4 + a_1^4 + a_2^4 + a_3^4 + 3\,(a_0^2a_1^2 + a_1^2a_2^2 + a_2^2a_0^2) \\ - 3\,(a_0^2 + a_1^2 + a_2^2)\,a_3^2 + 4a_0a_1a_2a_3 = 0. \end{aligned}$$

The morphism $\pi : \Sigma \to S$ is given by

$$\begin{aligned} (4.9) \quad \pi : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \longmapsto \\ (x_0x_1 + x_0x_4 + 2\,x_1x_5 : -2\,x_0x_2 + 2\,x_0x_3 + 2\,x_3x_5 : \\ -\,2\,x_1x_2 + x_1x_3 - \,x_3x_4 : -2\,x_2^2 - x_4^2 - 2\,{x_5}^2 - x_0x_5 + x_2x_3), \end{aligned}$$

and the rational map $\pi^{-1} : S \to \Sigma$ is given by

$$\pi^{-1} : (a_0 : a_1 : a_2 : a_3) \longmapsto (x_0 : x_1 : x_2 : x_3 : x_4 : x_5),$$

where

$$\begin{aligned} x_0 &= -\,a_0^4 - a_1^4 - 3\,a_0^2a_1^2 - a_0^2a_2^2 + 2\,a_0^2a_3^2 - 2\,a_1^2a_2^2 + a_1^2a_3^2 - 2\,a_0a_1a_2a_3, \\ x_1 &= a_0^3a_3 - a_1^3a_2 - 2\,a_2^3a_1 - 3\,a_0^2a_1a_2 + a_1^2a_0a_3 - a_2^2a_0a_3 + a_3^2a_1a_2, \\ x_2 &= a_0^3a_2 - a_1^3a_3 + a_3^3a_1 + a_1^2a_0a_2 - 2\,a_2^2a_1a_3 - a_3^2a_0a_2, \\ x_3 &= a_0^3a_2 + a_1^3a_3 + a_2^3a_0 + a_0^2a_1a_3 + a_2^2a_1a_3 - 2\,a_3^2a_0a_2, \\ x_4 &= a_0^3a_3 + a_1^3a_2 - 2\,a_3^3a_0 + a_0^2a_1a_2 + 3\,a_1^2a_0a_3 + a_2^2a_0a_3 + a_3^2a_1a_2, \\ x_5 &= a_0^2a_2^2 - a_0^2a_3^2 + a_1^2a_2^2 - a_1^2a_3^2 + 3\,a_0a_1a_2a_3. \end{aligned}$$

Proof. Let $p = (a_0 : a_1 : a_2 : a_3)$ be a point in $\mathbf{P}^3$. If $a_0 \neq 0$, the Schubert cell $\sigma(p)$ is given by

$$\begin{aligned}(4.10)\quad &(x_0 : x_1 : x_2 : x_3 : x_4 : x_5)\\ &= (a_0 s_0 : a_0 s_1 : a_0 s_2 : -a_2 s_0 + a_1 s_1 : -a_3 s_0 + a_1 s_2 : -a_3 s_1 + a_2 s_2),\end{aligned}$$

with $(s_0 : s_1 : s_2) \in \mathbf{P}^2$. The conic $X \cap \sigma(p)$ is

$$\begin{aligned}-(2a_0^2 + 2a_2^2 - a_3^2)\, s_0^2 - (a_0^2 + 2a_1^2 - 2a_3^2)\, s_1^2 + (2a_0^2 + a_1^2 + 2a_2^2)\, s_2^2\\ - 2(a_0 a_3 - 2a_1 a_2)\, s_0 s_1 - 2(a_0 a_1 + 2a_2 a_3)\, s_1 s_2 s\\ - 2(a_1 a_3 - 2a_0 a_2)\, s_0 s_2 = 0.\end{aligned}$$

The symmetric matrix corresponding to this quadratic form in (s_0, s_1, s_2) is given by

$$Q_p = \begin{pmatrix} a_3^2 - 2\,a_2^2 - 2\,a_0^2 & 2\,a_2 a_1 - a_0 a_3 & 2\,a_2 a_0 - a_1 a_3 \\ 2\,a_2 a_1 - a_0 a_3 & -a_0^2 - 2\,a_1^2 + 2\,a_3^2 & -2\,a_2 a_3 - a_1 a_0 \\ 2\,a_2 a_0 - a_1 a_3 & -2\,a_2 a_3 - a_1 a_0 & 2\,a_2^2 + 2\,a_0^2 + a_1^2 \end{pmatrix}.$$

The determinant of Q_p is $4a_0^2$ times the left-hand side of (1.4). Since we know that S is an irreducible hypersurface in $\mathbf{P}^3$, we conclude that the equation of S is given by (1.4).

Assuming $p \in \Sigma$, the rank of the matrix Q_p is less than or equal to 2. In case $\operatorname{rank} Q_p = 2$, the intersection point of the two lines that form the degenerated conic can be obtained by solving the first two equation of $Q_p{}^t(s_0, s_1, s_2) = 0$:

$$\begin{cases} (a_3^2 - 2\,a_2^2 - 2\,a_0^2)\, s_0 + (2\,a_2 a_1 - a_0 a_3)\, s_1 + (2\,a_2 a_0 - a_1 a_3)\, s_2 = 0, \\ (2\,a_2 a_1 - a_0 a_3)\, s_0 + (-a_0^2 - 2\,a_1^2 + 2\,a_3^2)\, s_1 + (-2\,a_2 a_3 - a_1 a_0)\, s_2 = 0. \end{cases}$$

Plugging the solution (s_0, s_1, s_2) into the formula (4.10), we obtain the formula for the map π^{-1}.

Let $x = (x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ be a point in $K_B = \Sigma$. If $x_0 \neq 0$, then a point p on the line l_x representing x is expressed by the formula

$$\begin{aligned}(4.11)\qquad p &= (a_0 : a_1 : a_2 : a_3)\\ &= (x_0 t_0 : x_0 t_1 : -x_3 t_0 + x_1 t_1 : -x_4 t_0 + x_2 t_1),\end{aligned}$$

where $(t_0 : t_1) \in \mathbf{P}^1$. Plugging this into the formula (4.10), we can parametrize $G \cap T_x(G) = \cup_{p \in l_x} \sigma(p)$ in terms of x:

$$\begin{aligned}(x_0 t_0 s_0 : x_0 t_0 s_1 : x_0 t_0 s_2 : (x_3 t_0 - x_1 t_1)\, s_0 + x_0 t_1 s_1 :\\ (x_4 t_0 - x_2 t_1)\, s_0 + x_0 t_1 s_2 : (x_4 t_0 - x_2 t_1)\, s_1 - (x_3 t_0 - x_1 t_1)\, s_2).\end{aligned}$$

The condition $x \in \Sigma$ is equivalent to the condition that $\sigma(p)$ is tangent to F at x, for some point $p \in l_x$. In terms of the matrix Q_F, this condition is expressed by

$$(x_0, x_1, x_2, x_3, x_4, x_5) Q_F \begin{pmatrix} x_0 t_0 s_0 \\ x_0 t_0 s_1 \\ x_0 t_0 s_2 \\ (x_3 t_0 - x_1 t_1)\, s_0 + x_0 t_1 s_1 \\ (x_4 t_0 - x_2 t_1)\, s_0 + x_0 t_1 s_2 \\ (x_4 t_0 - x_2 t_1)\, s_1 - (x_3 t_0 - x_1 t_1)\, s_2 \end{pmatrix} = 0.$$

This equation is linear with respect to (s_0, s_1, s_2). Under the condition $x \in \Sigma$, the above equation holds for any (s_0, s_1, s_2). Thus, we obtain the condition for (t_0, t_1) by equating the coefficients of s_i to 0. Plugging the solution for (t_0, t_1) into (4.11), we obtain the formula for the map π. □

Proposition 4.2. *The dual Kummer surface S^* associated to the quadratic line complex F is given by the equation*

$$\begin{aligned}(4.12)\quad S^* : \alpha_0^4 + \alpha_1^4 + \alpha_2^4 + \alpha_3^4 + 3\,(\alpha_0^2\alpha_1^2 + \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_0^2)\\ - 3\,(\alpha_0^2 + \alpha_1^2 + \alpha_2^2)\,\alpha_3^2 - 4\alpha_0\alpha_1\alpha_2\alpha_3 = 0.\end{aligned}$$

The morphism $\pi^* : \Sigma \to S^*$ is given by

$$\begin{aligned}\pi^* : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \longmapsto\\ (-x_3 x_4 - x_3 x_1 + 2\, x_4 x_2, {x_1}^2 - x_0 x_5 - 2\, x_2^2 + x_2 x_3 - 2\, x_5^2,\\ x_4 x_0 - x_1 x_0 + 2\, x_5 x_4, -2\, x_5 x_3 + 2\, x_2 x_0 - 2\, x_0 x_3),\end{aligned}$$

and the rational map $(\pi^*)^{-1} : S^* \to \Sigma$ is given by

$$(\pi^*)^{-1} : (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3) \longmapsto (x_0 : x_1 : x_2 : x_3 : x_4 : x_5),$$

where

$$
\begin{aligned}
x_0 &= \alpha_0^2\alpha_2^2 - \alpha_0^2\alpha_3^2 + \alpha_1^2\alpha_2^2 - \alpha_1^2\alpha_3^2 - 3\,\alpha_0\alpha_1\alpha_2\alpha_3,\\
x_1 &= \alpha_0^3\alpha_3 - 2\,\alpha_0\alpha_3^3 - \alpha_1^3\alpha_2 - \alpha_0^2\alpha_1\alpha_2 + 3\,\alpha_1^2\alpha_0\alpha_3 + \alpha_2^2\alpha_0\alpha_3 - \alpha_3^2\alpha_1\alpha_2,\\
x_2 &= -\alpha_0^3\alpha_2 + \alpha_1^3\alpha_3 - \alpha_2^3\alpha_0 + \alpha_0^2\alpha_1\alpha_3 + \alpha_2^2\alpha_1\alpha_3 + 2\,\alpha_3^2\alpha_0\alpha_2,\\
x_3 &= \alpha_0^3\alpha_2 + \alpha_1^3\alpha_3 - \alpha_3^3\alpha_1 + \alpha_1^2\alpha_0\alpha_2 + 2\,\alpha_2^2\alpha_1\alpha_3 - \alpha_3^2\alpha_0\alpha_2,\\
x_4 &= -\alpha_0^3\alpha_3 - \alpha_1^3\alpha_2 - 2\,\alpha_2^3\alpha_1 - 3\,\alpha_0^2\alpha_1\alpha_2 - \alpha_1^2\alpha_0\alpha_3 + \alpha_2^2\alpha_0\alpha_3 + \alpha_3^2\alpha_1\alpha_2,\\
x_5 &= \alpha_0^4 + \alpha_1^4 + 3\,\alpha_0^2\alpha_1^2 + \alpha_0^2\alpha_2^2 - 2\,\alpha_0^2\alpha_3^2 + 2\,\alpha_1^2\alpha_2^2 - \alpha_1^2\alpha_3^2 - 2\,\alpha_0\alpha_1\alpha_2\alpha_3.
\end{aligned}
$$

Proof. Let $h = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3)$ be a point in $(\mathbf{P}^3)^*$. If $\alpha_0 \neq 0$, the Schubert cell $\sigma(h)$ is given by

$$
\begin{aligned}
&(4.13)\quad (x_0 : x_1 : x_2 : x_3 : x_4 : x_5)\\
&\quad = (\alpha_2 s_0 + \alpha_3 s_1 : \alpha_3 s_2 - \alpha_1 s_0 : -\alpha_1 s_1 - \alpha_2 s_2 : \alpha_0 s_0 : \alpha_0 s_1 : \alpha_0 s_2),
\end{aligned}
$$

Let $x = (x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ be a point in $K_B = \Sigma$. If $x_0 \neq 0$, then a hyperplane h containing the line l_x representing x is expressed by the formula

(4.14)

$$h = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3) = (x_3 t_0 + x_4 t_1 : -x_1 t_0 - x_2 t_1 : x_0 t_0 : x_0 t_1),$$

where $(t_0 : t_1) \in \mathbf{P}^1$. Using these formulas, we obtain the equation of S^* and the maps between Σ and S^* in a similar manner as in the proof of Proposition 4.1. □

Remark 4.3. Abstractly, S and S^* are isomorphic. For example, the map

$$(a_0 : a_1 : a_2 : a_3) \longmapsto (\alpha_0 : \alpha_1 : \alpha_2 : -\alpha_3) \tag{4.15}$$

gives an isomorphism from S to S^*. However, the map $\pi^* \circ \pi^{-1}$ does not coincide with this map. In fact, $\pi^* \circ \pi^{-1}$ is not a map induced by a linear isomorphism from $\mathbf{P}^3$ to $(\mathbf{P}^3)^*$. See Remark 6.3.

Bremner showed that $K_B = \Sigma$ contains 32 lines. From the theory of quadratic line complexes [**5**, p. 776] it is known that these lines form two families of 16 disjoint lines. The lines in one family are sent to the 16 double points of S, and the others are mapped to the 16 double points of S^*.

The 16 lines in Table 1, expressed using Bremner's notation, are mapped to the double points of S by π after the change of variables (3.7).

The 16 lines in Table 2 are mapped to the double points of S^* by π^* after the change of variables (3.7).

5. The Abelian variety associated to the quadric line complex. As is well known, the Kummer surface S is the quotient of the Jacobian of a curve of genus 2. We now identify this curve of genus 2 and describe its Jacobian.

The Jacobian, or a torsor of it to be precise, is realized as the variety of lines lying on the three-dimensional quadratic line complex $X = G \cap F$ [**5**, p. 778]:

$$J = \{L \mid L \text{ is a line contained in } X\} \subset G(2,6).$$

A line contained in G is nothing but a pencil of lines $\sigma(p,h)$. Thus, we have

$$J = \{\sigma(p,h) \mid p \in S, h \in S^*\}.$$

We thus have two maps

$$\begin{aligned} j\ &: J \to S\ ; \quad L = \sigma(p,h) \longmapsto p, \\ j^* &: J \to S^*; \quad L = \sigma(p,h) \longmapsto h. \end{aligned}$$

These are generically two-to-one maps.

Let $L = \sigma(p,h)$ be a point in J. Define C_L to be the hyperplane section $h \cap S$. Suppose that L is not mapped to a singular point by neither j nor j^*. Then the curve C_L is a curve of genus 2, and J is (a torsor of) the Jacobian of C_L [**5**, p. 782].

To write down an equation of C_L, we begin with the point $p = (3 : 4 : 5 : 7) \in S$. This point corresponds to Subba-Rao's first solution to (1.1). In general, even if p is defined over the base field k, the two $\sigma(p,h)$'s contained in X may not be defined over k. In this particular case, however, the $\sigma(p,h)$'s are defined over $\mathbf{Q}$.

TABLE 1.

$$\begin{array}{cccc}
\Gamma_1 : \begin{cases} x = v \\ y = w \\ z = u \\ x + y + z = 0 \end{cases} & \Gamma_2 : \begin{cases} x = -v \\ y = w \\ z = -u \\ x - y - z = 0 \end{cases} & \Gamma_3 : \begin{cases} x = -v \\ y = -w \\ z = u \\ x - y + z = 0 \end{cases} & \Gamma_4 : \begin{cases} x = v \\ y = -w \\ z = -u \\ x + y - z = 0 \end{cases} \\
\longmapsto (1 : -1 : -1 : 2) & \longmapsto (-1 : 1 : -1 : 2) & \longmapsto (-1 : -1 : 1 : 2) & \longmapsto (1 : 1 : 1 : 2) \\
\Gamma_9 : \begin{cases} x = i \\ v = -z \\ w = iu \\ ix + v + w = 0 \end{cases} & \Gamma_{10} : \begin{cases} x = -iy \\ v = -z \\ w = -iu \\ ix - v - w = 0 \end{cases} & . \quad \Gamma_{11} : \begin{cases} x = -iy \\ v = z \\ w = iu \\ ix - v + w = 0 \end{cases} & \Gamma_{12} : \begin{cases} x = iy \\ v = z \\ w = -iu \\ ix + v - w = 0 \end{cases} \\
\longmapsto (1 : 1 : -2i : i) & \longmapsto (1 : 1 : 2i : -i) & \longmapsto (-1 : 1 : 2i : i) & \mapsto (1 : -1 : 2i : i) \\
\Gamma_{17} : \begin{cases} u = iv \\ z = iy \\ w = x \\ u + z - x = 0 \end{cases} & \Gamma_{18} : \begin{cases} u = -iv \\ z = -iy \\ w = x \\ u + z - x = 0 \end{cases} & \Gamma_{19} : \begin{cases} u = -iv \\ z = iy \\ w = -x \\ u - z + x = 0 \end{cases} & \Gamma_{20} : \begin{cases} u = iv \\ z = -iy \\ w = -x \\ u - z + x = 0 \end{cases} \\
\longmapsto (-1 : 2i : 1 : i) & \longmapsto (1 : 2i : -1 : i) & \longmapsto (1 : -2i : 1 : i) & \longmapsto (1 : 2i : 1 : -i) \\
\Gamma_{25} : \begin{cases} u = y \\ v = -iw \\ x = -iz \\ u - v - x = 0 \end{cases} & \Gamma_{26} : \begin{cases} u = y \\ v = iw \\ x = iz \\ u - v - x = 0 \end{cases} & \Gamma_{27} : \begin{cases} u = -y \\ v = iw \\ x = -iz \\ u + v - x = 0 \end{cases} & \Gamma_{28} : \begin{cases} u = -y \\ v = -iw \\ x = iz \\ u + v - x = 0 \end{cases} \\
\longmapsto (-2 : i : i : 1) & \longmapsto (2 : i : i : -1) & \longmapsto (2 : i : -i : 1) & \longmapsto (2 : -i : i : 1)
\end{array}$$

TABLE 2.

$$\Gamma_5 : \begin{cases} x = -w \\ y = -u \\ z = -v \\ x + y + z = 0 \end{cases} \longmapsto (-1 : 1 : 1 : 2) \qquad \Gamma_6 : \begin{cases} x = w \\ y = u \\ z = -v \\ x - y - z = 0 \end{cases} \longmapsto (1 : -1 : 1 : 2) \quad . \qquad \Gamma_7 : \begin{cases} x = w \\ y = -u \\ z = v \\ x + y - z = 0 \end{cases} \longmapsto (1 : 1 : 1 : -2) \qquad \Gamma_8 : \begin{cases} x = -w \\ y = u \\ z = v \\ x - y + z = 0 \end{cases} \longmapsto (1 : 1 : -1 : 2)$$

$$\Gamma_{13} : \begin{cases} x = -iz \\ v = -iu \\ w = y \\ ix + v + w = 0 \end{cases} \longmapsto (1 : 1 : -2i : -i) \qquad \Gamma_{14} : \begin{cases} x = iz \\ v = iu \\ w = y \\ ix - v - w = 0 \end{cases} \longmapsto (1 : 1 : 2i : i) \qquad \Gamma_{15} : \begin{cases} x = iz \\ v = -iu \\ w = -y \\ ix + v - w = 0 \end{cases} \longmapsto (1 : -1 : 2i : -i) \qquad G_{16} : \begin{cases} x = -iz \\ v = iu \\ w = -y \\ ix - v + w = 0 \end{cases} \longmapsto (1 : -1 : -2i : i)$$

$$\Gamma_{21} : \begin{cases} u = -z \\ x = iy \\ w = iv \\ u + x - w = 0 \end{cases} \longmapsto (1 : -2i : -1 : i) \qquad \Gamma_{22} : \begin{cases} u = -z \\ x = -iy \\ w = -iv \\ u + x - w = 0 \end{cases} \longmapsto (1 : 2i : -1 : -i) \qquad \Gamma_{23} : \begin{cases} u = z \\ x = -iy \\ w = iv \\ u + x + w = 0 \end{cases} \longmapsto (1 : -2i : 1 : -i) \qquad \Gamma_{24} : \begin{cases} u = z \\ x = iy \\ w = -iv \\ u + x + w = 0 \end{cases} \longmapsto (1 : 2i : 1 : i)$$

$$\Gamma_{29} : \begin{cases} u = -iw \\ v = -x \\ y = -iz \\ u - v - y = 0 \end{cases} \longmapsto (2 : -i : -i : 1) \qquad \Gamma_{30} : \begin{cases} u = iw \\ v = -x \\ y = iz \\ u - v - y = 0 \end{cases} \longmapsto (2 : i : i : 1) \qquad \Gamma_{31} : \begin{cases} u = iw \\ v = x \\ y = -iz \\ u + v + y = 0 \end{cases} \longmapsto (2 : -i : i : -1) \qquad \Gamma_{32} : \begin{cases} u = -iw \\ v = x \\ y = iz \\ u + v + y = 0 \end{cases} \longmapsto (2 : i : -i : -1)$$

Parametrizing $\sigma(p)$ as in (4.10), we see that the conic $X \cap \sigma(p)$ splits into two factors over $\mathbf{Q}$:

$$(s_1 - 3s_2 + 2s_3)(19s_1 + 19s_2 - 42s_3) = 0.$$

We choose as L the line defined by the equation $s_1 - 3s_2 + 2s_3 = 0$ and see that the plane h in $\mathbf{P}^3$ corresponding to L in X is given by

$$a_0 - a_1 + 3a_2 - 2a_3 = 0.$$

The intersection of this plane and the surface S (projected down to $\mathbf{P}^2$ defined by $a_3 = 0$) is given by

$$\begin{aligned} &5a_0^4 + 5a_1^4 - 11a_2^4 + 20a_0^3a_1 - 60a_0^3a_2 + 20a_0a_1^3 + 60a_1^3a_2 \\ &\quad + 36a_0a_2^3 - 36a_1a_2^3 + 30a_0^2a_1^2 - 18a_0^2a_2^2 - 18a_1^2a_2^2 \\ &\qquad + 68a_0^2a_2a_1 - 68a_0a_1^2a_2 + 12a_0a_1a_2^2 = 0. \end{aligned} \tag{5.16}$$

This curve in $\mathbf{P}^2$ has an ordinary double point at $(a_0 : a_1 : a_2) = (-1 : 4 : 1)$. Blowing up at this point, we obtain a smooth curve C of genus 2. In fact, by using the change of coordinates

$$\begin{aligned} a_1 &= \frac{(x-1)y + 2x^4 - 4x^3 - 3x^2 - 34x - 5}{4x(x^3 + 2x^2 + 6x + 2)}\, a_0, \\ a_2 &= -\frac{(3x+1)y + 2x^4 + 4x^3 + 7x^2 + 14x + 5}{4x(x^3 + 2x^2 + 6x + 2)}\, a_0, \end{aligned}$$

the equation (5.16) is transformed to

$$y^2 = (x^2+1)(x^2+2x+5)(x^2-2x+5).$$

Let C be the curve defined by the above equation. The hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$ is in the center of the automorphism group $\mathrm{Aut}\,(C)$. Define the reduced automorphism group $\mathrm{RA}\,(C)$ to be the quotient group $\mathrm{Aut}\,(C)/\langle\iota\rangle$.

Theorem 5.1. *The quartic surface S is the Kummer surface* (*in the modern sense*) *obtained from the Jacobian $J(C)$ of the curve C of genus 2 given by*

$$C : y^2 = (x^2+1)(x^2+2x+5)(x^2-2x+5).$$

The reduced automorphism group $\mathrm{RA}\,(C)$ *is isomorphic to the symmetric group* S_3 *generated by*

$$\tau : (x, y) \longmapsto (-x, y), \quad \sigma : (x, y) \longmapsto \left(-\frac{x+3}{x-1}, \frac{8y}{(x-1)^3} \right).$$

In particular, τ*,* $\sigma\tau$ *and* $\sigma^2\tau$ *are three involutions of* C*. The quotients* $C/\langle\tau\rangle$*,* $C/\langle\sigma\tau\rangle$*, and* $C/\langle\sigma^2\tau\rangle$ *are all birationally equivalent to the same elliptic curve given by*

$$E : y^2 = x(x^2 + 4x + 20).$$

The conductor of E *is* $320 = 2^6 \cdot 5$*, and* E *has no complex multiplication. The natural maps from* C *to the quotient* E *are given by*

$$\begin{aligned}
p_1 &: C \to C/\langle\tau\rangle; & (x, y) &\longmapsto (x^2 + 1, y), \\
p_2 &: C \to C/\langle\sigma\tau\rangle; & (x, y) &\longmapsto \left(\frac{2(x^2 + 2x + 5)}{(x-1)^2}, \frac{8y}{(x-1)^3} \right). \\
p_3 &: C \to C/\langle\sigma^2\tau\rangle; & (x, y) &\longmapsto \left(\frac{2(x^2 - 2x + 5)}{(x+1)^2}, \frac{-8y}{(x+1)^3} \right),
\end{aligned}$$

$J(C)$ *is isogenous to* $E \times E$ *with an isogeny of degree* 4.

Proof. We have already shown the equation of C. It is obvious that C admits the involutions τ, and it is easy to check that σ is an automorphism of order 3. Furthermore, σ and τ satisfy $\tau^{-1}\sigma\tau = \sigma^{-1}$. This shows that the group $\langle\sigma, \tau\rangle$ is isomorphic to S_3. According to the classification of automorphisms of curves of genus 2 (see Igusa [**8**] or Ibukiyama-Katsura-Oort [**7**]), there are only three isomorphism classes over $\overline{\mathbf{Q}}$ of curves whose reduced automorphism group is strictly larger than S_3, and our C is not one of them. Thus, we conclude $RA(C) \simeq S_3$.

The function field of $C/\langle\tau\rangle$ is the subfield of the function field $k(C) = k(x, y)$ generated by x^2 and y. If we let $X = x^2 + 1$ and $Y = y$, then (X, Y) satisfies $Y^2 = X(X^2 + 4X + 20)$. For the quotient $C/\langle\sigma\tau\rangle$ we see that $p_1 \circ \sigma$ is the quotient map. Indeed, if $P \in C$, then we have

$$(p_1 \circ \sigma)(\sigma\tau(P)) = p_1(\sigma^2\tau(P)) = p_1(\tau\sigma(P)) = p_1(\sigma(P)) = (p_1 \circ \sigma)(P).$$

Similarly, the quotient map p_3 equals $p_1 \circ \sigma^2$.

Calculations using standard formulas show the conductor of E is 320. This implies that E has semi-stable reduction at $p = 5$. Since a curve with complex multiplication cannot have semi-stable reduction, we see that E has no complex multiplication.

The map $p_1 \times p_2 : C \to E \times E$ induces the map $(p_1 \times p_2)^* : E \times E \to J(C)$, is an isogeny of degree 4. $\square$

Corollary 5.2. *The Picard number $\rho(K_B)$, that is, the rank of the Néron-Severi group $NS(K_B, \mathbf{C})$, equals* 19.

Proof. Since $J(C)$ is isogenous to the self-product of an elliptic curve without complex multiplication, the Picard number of $J(C)$ is 3. In general, the Picard number of the minimal desingularization of $J(C)/\{\pm 1\}$ equals $16 + \rho(J(C))$, 16 being accounted for by the exceptional curves arising from the blow-ups. Thus, we see that $\rho(K_B)$ equals 19. $\square$

Remark 5.3. C admits three other involutions: $\iota\tau$, $\iota\sigma\tau$ and $\iota\sigma^2\tau$. The quotients $C/\langle\iota\tau\rangle$, $C/\langle\iota\sigma\tau\rangle$, and $C/\langle\iota\sigma^2\tau\rangle$ are all birationally equivalent to the elliptic curve given by

$$E' : y^2 = x(x^2 - 44x + 500).$$

There is an isogeny of degree 3 from E to E' is given by

$$(x, y) \longmapsto \left(\frac{x(x+10)^2}{(x+2)^2}, \frac{y(x+10)(x^2-4x+20)}{(x+2)^3}\right).$$

Remark 5.4. Let $p_i^* : \mathrm{Pic}^0(E) \to \mathrm{Pic}^0(C)$ be the map induced from p_i for $i = 1, 2$. Identifying $\mathrm{Pic}^0(E)$ with E, and $\mathrm{Pic}^0(C)$ with $J(C)$, we obtain a map $E \times E \to J(C)$. Taking quotients, we have

$$E \times E/\{\pm 1\} \longrightarrow J(C)/\{\pm 1\}.$$

If P_1 and P_2 are points in the Mordell-Weil group $E^d(\mathbf{Q})$ of a quadratic twist of E by some d, then we obtain a rational point of $E \times E/\{\pm 1\}$.

Thus, using the above map, we obtain a rational point in Σ. If P_1 and P_2 are linearly dependent, then we can show that the point obtained is on the curve Bremner constructed. Thus, to obtain a rational solution of (1.1), we look for d's such that $\operatorname{rank} E^d(\mathbf{Q}) \geq 2$. In order to write down the formula for the above map, however, we need to compute the addition map of $J(C)$.

6. Symmetries of Σ and S. Bremner showed that the group of rational symmetries, i.e., the automorphisms defined over $\mathbf{Q}$ induced by the linear map of the ambient space, of the surface K_B is a group of order 48 generated by

$$\begin{aligned}
\alpha &: (x:y:z:u:v:w) \longmapsto (-x:y:z:-u:v:w),\\
\beta &: (x:y:z:u:v:w) \longmapsto (x:-y:z:u:-v:w),\\
\gamma &: (x:y:z:u:v:w) \longmapsto (w:v:u:z:y:x),\\
\delta &: (x:y:z:u:v:w) \longmapsto (y:z:x:v:w:u),\\
\varepsilon &: (x:y:z:u:v:w) \longmapsto (-u:-v:-w:x:y:z).
\end{aligned}$$

Proposition 6.1. *The symmetries α, β, γ and δ induce symmetries of S given by*

$$\begin{aligned}
\bar\alpha &: (a_0:a_1:a_2:a_3) \longmapsto (-a_0:-a_1:a_2:a_3),\\
\bar\beta &: (a_0:a_1:a_2:a_3) \longmapsto (-a_0:a_1:-a_2:a_3),\\
\bar\gamma &: (a_0:a_1:a_2:a_3) \longmapsto (-a_2:a_1:-a_0:a_3),\\
\bar\delta &: (a_0:a_1:a_2:a_3) \longmapsto (-a_2:-a_0:a_1:a_3).
\end{aligned}$$

The group $\langle \bar\alpha, \bar\beta, \bar\gamma, \bar\delta \rangle$ is isomorphic to the symmetric group S_4, identified with the permutation group of the set $R_{\mathbf{Q}} = \{(1:1:1:2), (1:-1:-1:2), (-1:1:-1:2), (-1:-1:1:2)\}$ of rational double points of S.

Proof. Using the formula (4.9) for the map $\Sigma \to S$, it is easy to verify that α, β, γ and δ induce $\bar\alpha$, $\bar\beta$, $\bar\gamma$ and $\bar\delta$, respectively. These symmetries fixes the set $R_{\mathbf{Q}}$. The symmetry $\bar\alpha\bar\gamma$ acts as the 4-cycle

$$\begin{aligned}
(1:1:1:2) \longmapsto (1:-1:-1:2) \longmapsto (-1:1:-1:2)&\\
\longmapsto (-1:-1:1:2)&,
\end{aligned}$$

whereas $\bar{\beta}\bar{\delta}\bar{\gamma}\bar{\delta}$ acts as the transposition between $(1 : 1 : 1 : 2)$ and $(1 : -1 : -1 : 2)$. These two generate the full symmetric group acting on $R_{\mathbf{Q}}$. It is easy to check that the only element of $\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \rangle$ that fixes each element of $R_{\mathbf{Q}}$ is the identity. $\square$

Remark 6.2. Over the field $\mathbf{Q}(i)$, S has more symmetries, such as $(a_0 : a_1 : a_2 : a_3) = (a_1 : a_2 : ia_3 : -ia_0)$. The order of the group of symmetries over $\mathbf{Q}(i)$ is 96.

Remark 6.3. The symmetry ε does not induce a symmetry on S. As a matter of fact, the map $\pi^* \circ \varepsilon \circ \pi^{-1} : S \to S^*$ coincides with the map (4.15). As a consequence, from one rational point of K_B modulo symmetry, we obtain a pair of two rational points that do not coincide by any symmetry. Table 3 is the list of the nine smallest solutions of (1.3) in terms of the size $x^6 + y^6 + z^6$, together with their images by π (first row) and by $\pi \circ \varepsilon$ (second row). It corresponds to the list of [**9**] minus $(x : y : z : u : v : w) = (25 : 62 : 138 : 82 : 92 : 135)$, which does not satisfy (1.2).

7. Historical remarks. The first solution $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$ to the equation (1.1) was found by Rao [**10**] in 1934. In 1967, Lander, Parkin and Selfridge published a survey [**9**] of results concerning equal sums of like Powers obtained by computer searches. At that time ten solutions to (1.1) were known. The search was conducted up to $x^6 + y^6 + z^6 \leq 2.5 \times 10^{14}$. It was noted that all but one of the ten solutions satisfied (1.2).

Bremner [**1**], after a suggestion of Swinnerton-Dyer, found that the nine simultaneous solutions to (1.1) and (1.2) satisfy the equations (1.3) after suitable relabeling and change of sign. In the second edition of his book [**6**] published in 1994, Guy remarks that Peter Montgomery found 18 solutions to (1.1) that do not satisfy (1.2). In 1998 Ekl published a first systematic survey [**4**] on equal sums of like powers since [**9**]. Ekl extended the search of Lander, Parkin and Selfridge [**9**] up to $x^6 + y^6 + z^6 \leq 1.3 \times 10^{19}$, and found 87 solutions to (1.1), though he did not present the list of solutions in [**4**]. In his master's thesis [**11**], Womack found 207 solutions to (1.1) up to $x^6 + y^6 + z^6 \leq 3.4 \times 10^{22}$. He used elaborated searching techniques and distributed computations.

TABLE 3. Rational points of K_B and S.

x	y	z	u	v	w	a_0	a_1	a_2	a_3
3	19	22	−23	10	−15	5	3	4	7
						29	131	79	122
36	37	67	−65	−15	−52	1	40	23	73
						39	−139	−71	−98
33	47	74	−73	54	23	71	41	76	−107
						39	89	31	82
32	43	81	−3	80	55	19	−79	−271	−178
						8	15	31	53
37	50	81	−78	−11	−65	4	19	13	35
						89	−411	−229	−298
51	113	136	−125	−40	−129	41	201	184	413
						31	−97	−85	−82
71	92	147	−132	133	−1	64	29	91	97
						211	309	139	362
111	121	230	26	225	169	41	−89	−215	−146
						39	164	311	547
75	142	245	14	243	163	20	−41	−167	−107
						39	161	479	802

He also noted that only 22 out of the 207 solutions he found do not satisfy (1.2). Since he did not publish the list of solutions, we do not know how many of 185 simultaneous solutions to (1.1) and (1.2) satisfy Bremner's equations (1.3).

In 2000 Choudhry [**3**] found infinitely many solutions to (1.1) and (1.2) that do not satisfy Bremner's equations (1.3). He studied the simultaneous equations (1.1) and (1.2) together with $x + y + z = u + v + w$. Geometrically, these three equations determine another $K3$ surface. Choudhry found an elliptic pencil on this $K3$ surface and found some fibers with positive Mordell-Weil rank. Of the 26 solutions he listed, 20 fall in the range of Womack's search. In other words at least 20 out of 185 simultaneous solutions to (1.1) and (1.2) do not satisfy Bremner's equations (1.3).

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