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ITERATIVE APPROXIMATION OF SOLUTIONS TO NONLINEAR EQUATIONS OF ϕ-STRONGLY ACCRETIVE OPERATORS IN BANACH SPACES

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ABSTRACT. Suppose that X is an arbitrary Banach space and $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator. It is proved that, for a given $f \in X$, the Ishikawa iteration method with errors converges strongly to the solutions of the equations $f = Tx$ and $f = x + Tx$ under suitable conditions. Related results deal with the iterative approximation of fixed points of ϕ-strongly pseudocontractive operators. Our results generalize, improve and unify the corresponding results in [**2**]–[**11**], [**13**]–[**16**], [**19**], [**20**], [**23**]–[**26**] and [**28**].

1. Introduction. Let X be an arbitrary Banach space with norm $\|\cdot\|$ and dual X^* and J denote the normalized duality map from X into 2^{X^*} given by

$$J_X = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \operatorname{Re}\langle x, f^*\rangle\},$$

where $\langle \cdot, \cdot \rangle$ stands for the generalized duality pairing between X and X^*. It is well known that, if X^* is convex, then J is single-valued. In the sequel we shall denote the single-valued duality mapping by j.

An operator T with domain $D(T)$ and range $R(T)$ in X is called strongly accretive if there exists a constant $k > 0$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1.1}$$

Without loss of generality we may assume $k \in (0,1)$. If $k = 0$ in (1.1), then T is called accretive. Furthermore, T is called ϕ-strongly accretive

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if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\mathrm{Re}\,\langle Tx - Ty, j(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|. \tag{1.2}$$

Closely related to the class of strongly accretive operators is the class of strongly pseudocontractive operators where an operator T is called a strong pseudocontraction if there exists a constant $t > 1$ such that for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\mathrm{Re}\,\langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{t}\,\|x - y\|^2. \tag{1.3}$$

If $t = 1$ in (1.3), then T is called pseudocontractive. We call T ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\mathrm{Re}\,\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \tag{1.4}$$

If I denotes the identity operator on X, then it follows from inequalities (1.1) to (1.4) that T is pseudocontractive, respectively, strongly pseudocontractive, ϕ-strongly pseudocontractive, if and only if $(I - T)$ is accretive, respectively, strongly accretive, ϕ-strongly accretive. If T is accretive and $(I + rT)(D(T)) = X$ for all $r > 0$, then T is called m-accretive.

The classes of operators introduced above have been studied by several researchers, see, for example, [**1**]–[**28**]. Osilike [**23**] proved that the class of strongly accretive operators, respectively, the class of strongly pseudocontractive operators, is a proper subclass of the class of ϕ-strongly accretive operators, respectively, the class of ϕ-strongly pseudocontractive operators.

We recall the following iterative processes due to Ishikawa [**17**], Mann [**21**] and Liu [**19**], respectively.

(a) Let K be a nonempty convex subset of X, and let $T : K \to K$ be an operator. For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-a_n)x_n+a_nTy_n, \qquad y_n = (1-b_n)x_n+b_nTx_n \quad \text{for all } n \geq 0,$$

is called the Ishikawa iteration sequence, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying appropriate conditions.

(b) In particular, if $b_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, \quad x_{n+1} = (1-a_n)x_n + a_n T x_n \quad \text{for all } n \geq 0,$$

is called the Mann iteration sequence.

(c) Let K be a nonempty convex subset of X, and let $T : K \to X$ be an operator. For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by

$$x_{n+1} = (1-a_n)x_n + a_n T y_n + v_n, \qquad y_n = (1-b_n)x_n + b_n T x_n + u_n \quad \forall\, n \geq 0,$$

where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are in K, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two summable sequences in X, i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$ and $\sum_{n=0}^{\infty} \|v_n\| < \infty$, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying suitable conditions, is called the Ishikawa iteration sequence with errors.

(d) If, with K, T and x_0 as in (c), the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by

$$x_{n+1} = (1-a_n)x_n + a_n T x_n + u_n \quad \text{for all } n \geq 0,$$

where $\{x_n\}_{n=0}^{\infty}$ is in K, $\{u_n\}_{n=0}^{\infty}$ is a summable sequence in X and $\{a_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ satisfying suitable conditions, is called the Mann iteration sequence with errors.

It is clear that the Ishikawa and Mann iterative processes are all special cases of the Ishikawa and Mann iterative processes with errors, respectively.

The accretive operators were introduced independently in 1967 by Browder [**1**] and Kato [**18**]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0,$$

is solvable if T is locally Lipschitzian and accretive on X. Martin [**22**] indeed generalized the result of Browder to the continuous accretive operators. That is, he proved that if $T : X \to X$ is strongly accretive and continuous, then T is surjective, so that the equation

$$Tx = f \tag{1.7}$$

has a solution for any given $f \in X$. Meanwhile, he proves also that if $T : X \to X$ is accretive and continuous, then T is m-accretive, so that the equation

$$x + Tx = f \tag{1.8}$$

has a solution for any given $f \in X$.

In [**4**], Chidume proved that if $X = L_p$, or l_p, $p \geq 2$, and K is a nonempty closed convex and bounded subset of X and $T : K \to K$ is a Lipschitz strongly pseudo-contractive operator, then the Mann iteration method converges strongly to the unique fixed point of T. Afterwards, several authors applied the Mann iteration method and the Ishikawa iteration method to approximate fixed points of strong pseudo-contractions and to approximate solutions of equations (1.7) and (1.8) (see, for example, [**2**], [**3**], [**5**]–[**11**], [**13**]–[**16**], [**20**], [**23**], [**25**], [**26**], [**28**]). Using the Mann and Ishikawa iteration methods with errors, Liu [**19**] and Osilike [**24**] also obtained the convergence theorems for strongly accretive operators under certain conditions.

Motivated and inspired by the above works, the purpose of this paper is to study the iterative approximation of solutions to equations (1.7) and (1.8) in the case when T is a uniformly continuous ϕ-strongly accretive operator and X is an arbitrary Banach space. By the way, we also obtain the iterative approximation of fixed points for a uniformly continuous ϕ-strongly pseudo-contractive operator. Our results generalize, improve and unify the corresponding results of Chang [**2**], Chang, Cho, Lee and Kang [**3**], Chidume [**4**]–[**8**], Chidume and Osilike [**9**]–[**11**], Deng [**13**]–[**15**], Deng and Ding [**16**], Liu [**19**], Liu [**20**], Osilike [**23**]–[**25**], Tan and Xu [**26**] and Zeng [**28**] and others.

2. Preliminaries. The following lemmas play a crucial role in the proofs of our main results.

Lemma 2.1 [**19**]. *Let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \beta_n + \gamma_n,$$

for all $n \geq 0$, *where* $\{\omega_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$, $\beta_n = o(\omega_n)$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a nonnegative and bounded sequence and* $\phi : [0,\infty) \to [0,\infty)$ *strictly increasing and* $\phi(0) = 0$. *Assume that* $A(a_n) = (\phi(a_n))/(1 + a_n + \phi(a_n))$ *for all* $n \geq 0$. *Then the following statements are equivalent*:

(i) $\inf\{A(a_n) : n \geq 0\} = 0$;

(ii) $\inf\{\phi(a_n) : n \geq 0\} = 0$;

(iii) *There exists a subsequence* $\{a_{n_k}\}_{k=0}^{\infty}$ *of* $\{a_n\}_{n=0}^{\infty}$ *such that* $a_{n_k} \to 0$ *as* $k \to \infty$.

Proof. Note that $0 \leq A(a_n) \leq \phi(a_n)$ for all $n \geq 0$. This means that (ii) implies (i).

Set $\inf\{\phi(a_n) : n \geq 0\} = r$. Suppose that $r > 0$. Since $\{a_n\}_{n=0}^{\infty}$ is bounded, there exists $d > 0$ satisfying $a_n \leq d$ for all $n \geq 0$. It follows that $A(a_n) \geq r/(1 + d + \phi(d))$ for all $n \geq 0$. It is clear that $\inf\{A(a_n) : n \geq 0\} \geq r/(1 + d + \phi(d)) > 0$. That is, (i) implies (ii).

Assume that (ii) holds. Then there is a subsequence $\{a_{n_k}\}_{k=0}^{\infty}$ of $\{a_n\}_{n=0}^{\infty}$ satisfying $\phi(a_{n_k}) \to 0$ as $k \to \infty$. It follows from boundedness of $\{a_n\}_{n=0}^{\infty}$ that there exists a subsequence $\{a_{n_{k_j}}\}_{j=0}^{\infty}$ of $\{a_{n_k}\}_{k=0}^{\infty}$ such that $a_{n_{k_j}} \to t$ as $j \to \infty$. Clearly, $t \geq 0$. We claim that $t = 0$. If not, then $t > 0$. Therefore, there exists a subsequence $\{a_{n_{k_{j_m}}}\}_{m=0}^{\infty}$ of $\{a_{n_{k_j}}\}_{j=0}^{\infty}$ with $a_{n_{k_{j_m}}} \geq t/2$ for all $m \geq 0$. Note that ϕ is strictly increasing. Thus $0 = \lim_{m\to\infty} \phi(a_{n_{k_{j_m}}}) \geq \phi(t/2) > 0$. This is a contradiction. Hence, (ii) implies (iii).

Assume that (iii) holds. If $\inf\{\phi(a_n) : n \geq 0\} = s > 0$, then $\phi(a_n) \geq s$ for all $n \geq 0$. Since ϕ is strictly increasing, $a_n \geq \phi^{-1}(s) > 0$ for all $n \geq 0$. Therefore, each subsequence of $\{a_n\}_{n=0}^{\infty}$ does not converge to zero. This is a contradiction. This completes the proof. □

Lemma 2.3. *Suppose that* X *is an arbitrary Banach space and* $T : X \to X$ *is a continuous* ϕ*-strongly accretive operator. Then the equation* $Tx = f$ *has a unique solution for any* $f \in X$.

Proof. Given $f \in X$ and $n \geq 1$, define $T_n : X \to X$ by $T_n x = (1/n)x + Tx$ for all $x \in X$. Then, for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\begin{aligned} \langle T_n x - T_n y, j(x-y)\rangle &\geq \|x-y\|^2/n + \phi(\|x-y\|)\|x-y\| \\ &\geq \|x-y\|^2/n. \end{aligned} \tag{2.1}$$

That is, T_n is strongly accretive. Since T_n is continuous, it follows from Deimling [**12**, Theorem 13.1] that the equation $T_n x = f$ has a solution $x_n \in X$. In view of (2.1), we have

$$\begin{aligned} \phi(\|x_n - x_1\|)\|x_n - x_1\| &\leq \langle T_n x_n - T_n x_1, j(x_n - x_1)\rangle \\ &= \langle f - (x_1/n) - Tx_1, j(x_n - x_1)\rangle \\ &= \langle (1-(1/n))x_1, j(x_n - x_1)\rangle \\ &\leq \|x_1\| \cdot \|x_n - x_1\|, \end{aligned}$$

which implies that $\phi(\|x_n - x_1\|) \leq \|x_1\|$. Consequently, $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence. This yields that $Tx_n \to f$ as $n \to \infty$. Note that, for all $n > 0$ and $m > 0$,

$$\|x_n - x_m\| \leq \phi^{-1}(\|Tx_n - Tx_m\|) \leq \phi^{-1}(\|Tx_n - f\| + \|Tx_m - f\|),$$

which means that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Hence, $\{x_n\}_{n=0}^{\infty}$ converges to some $p \in X$. From the continuity of T, we get that $Tx_n \to Tp$ as $n \to \infty$. Therefore, $Tp = f$.

Suppose that the equation $Tx = f$ has another solution $q \in X - \{p\}$. Then there is $j(p-q) \in J(p-q)$ such that

$$0 = \langle Tp - Tq, j(p-q)\rangle \geq \phi(\|p-q\|)\|p-q\|,$$

which implies that $\phi(\|p-q\|) = 0$. Since ϕ is strictly increasing, $\phi(0) = 0$ and $\|p-q\| > 0$ so that $\phi(\|p-q\|) > 0$. This is a contradiction. Therefore, the equation $Tx = f$ has a unique solution in X. This completes the proof.

3. Main results. In the sequel, $F(T)$ denotes the set of fixed points of T. Now we prove the following theorems.

Theorem 3.1. *Suppose that X is an arbitrary Banach space and $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator. Assume that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are sequences in X and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying the conditions*

(3.1)
$$\sum_{n=0}^{\infty} a_n^2 < +\infty, \quad \sum_{n=0}^{\infty} \|v_n\| < +\infty, \quad \|v_n\| = o(a_n), \quad \lim_{n\to\infty} \|u_n\| = 0;$$

$$\lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} a_n = +\infty \quad \textit{and} \quad a_n \neq 0, \quad n \geq 0. \tag{3.2}$$

Then for any given $f \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined for arbitrary $x_0 \in X$ by

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n(f + x_n - Tx_n) + u_n, \quad n \geq 0, \\ x_{n+1} &= (1-a_n)x_n + a_n(f + y_n - Ty_n) + v_n, \quad n \geq 0, \end{aligned} \tag{3.3}$$

converges strongly to the solution of the equation $Tx = f$ provided that the sequences $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{y_n - Ty_n\}_{n=0}^{\infty}$ are bounded.

Proof. It follows from Lemma 2.3 that the equation $Tx = f$ has a unique solution $q \in X$. Define $S : X \to X$ by $Sx = f + (I-T)x$ for all $x \in X$. Obviously, S is uniformly continuous and q is a unique fixed point of S. Thus, for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\begin{aligned} \operatorname{Re}\langle (I-S)x - (I-S)y, j(x-y)\rangle &= \operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \\ &= \phi(\|x-y\|)\|x-y\| \\ &\geq A(x,y)\|x-y\|^2, \end{aligned}$$

where
$$A(x,y) = \frac{\phi(\|x-y\|)}{1 + \|x-y\| + \phi(\|x-y\|)} \in [0,1)$$

for all $x, y \in X$. This implies that

$$\operatorname{Re}\langle (I-S-A(x,y))x - (I-S-A(x,y))y, j(x-y)\rangle \geq 0,$$

and it follows from Lemma 1.1 of Kato [**18**] that

$$\|x-y\| \le \|x-y+r[(I-S-A(x,y))\,x-(I-S-A(x,y))\,y]\|, \tag{3.4}$$

for all $x, y \in X$ and $r > 0$. In view of (3.3), we have

$$\begin{aligned} x_n &= x_{n+1}+a_nx_n-a_nSy_n-v_n \\ &= (1+a_n)x_{n+1}+a_n(I-S-A(x_{n+1},q))x_{n+1} \\ &\quad -(1-A(x_{n+1},q))a_nx_n+(2-A(x_{n+1},q))a_n^2(x_n-Sy_n) \\ &\quad +a_n(Sx_{n+1}-Sy_n)-[1+(2-A(x_{n+1},q))a_n]v_n. \end{aligned} \tag{3.5}$$

Note that

$$q=(1+a_n)q+a_n(I-S-A(x_{n+1},q))q-(1-A(x_{n+1},q))a_nq. \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned} \|x_n-q\| &\ge (1+a_n)\,\Big\|x_{n+1}-q \\ &\qquad\qquad + \frac{a_n}{1+a_n}\,[(I-S-A(x_{n+1},q))x_{n+1} \\ &\qquad\qquad - (I-S-A(x_{n+1},q))q)]\,\Big\| \\ &\quad - a_n(1-A(x_{n+1},q))\|x_n-q\| \\ &\quad - (2-A(x_{n+1},q))a_n^2\|x_n-Sy_n\|-a_n\|Sx_{n+1}-Sy_n\| \\ &\quad - [1+(2-A(x_{n+1},q))a_n]\|v_n\| \\ &\ge (1+a_n)\|x_{n+1}-q\|-a_n(1-A(x_{n+1},q))\|x_n-q\| \\ &\quad - (2-A(x_{n+1},q))a_n^2\|x_n-Sy_n\| \\ &\quad - a_n\|Sx_{n+1}-Sy_n\|-[1+(2-A(x_{n+1},q))a_n]\|v_n\|, \end{aligned}$$

which implies that

(3.7)

$$\begin{aligned} \|x_{n+1}-q\| &\le \frac{1+(1-A(x_{n+1},q))a_n}{1+a_n}\|x_n-q\| \\ &\quad + (2-A(x_{n+1},q))a_n^2\|x_n-Sy_n\| \\ &\quad + a_n\|Sx_{n+1}-Sy_n\|+[1+(2-A(x_{n+1},q))a_n]\|v_n\| \\ &\le (1-A(x_{n+1},q)a_n+a_n^2)\|x_n-q\| \\ &\quad + M_1a_n^2\|x_n-Sy_n\|+a_n\|Sx_{n+1}-Sy_n\|+M_2\|v_n\|, \end{aligned}$$

for some constants $M_1 \geq 0$, $M_2 \geq 0$. Put

$$d = \sup\{\|Sx_n - q\| + \|Sy_n - q\| : n \geq 0\} + \|x_0 - q\|, \quad D = d + \sum_{n=0}^{\infty} \|v_n\|.$$

It is easy to verify that $\|x_n - q\| \leq D$ and $\|x_n - Sy_n\| \leq 2D$ for all $n \geq 0$. Using (3.1), (3.2) and (3.3), we have

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq (1 - a_n)\|x_n - y_n\| + a_n\|Sy_n - y_n\| + \|v_n\| \\ &\leq \|x_n - y_n\| + a_n\|Sy_n - x_n\| + \|v_n\| \\ &\leq b_n\|x_n - Sx_n\| + \|u_n\| + 2Da_n + \|v_n\| \\ &\leq 2D(a_n + b_n) + \|u_n\| + \|v_n\| \to 0, \end{aligned}$$

as $n \to \infty$. It follows from the uniform continuity of S that $\|Sx_{n+1} - Sy_n\| \to 0$ as $n \to \infty$. From (3.7) and (3.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq [1 - A(x_{n+1}, q)a_n + a_n^2]\|x_n - q\| \\ &\quad + 2DM_1a_n^2 + a_n\|Sx_{n+1} - Sy_n\| + M_2\|v_n\| \\ &\leq [1 - A(x_{n+1}, q)a_n]\|x_{n-q}\| + (1 + 2M_1)Da_n^2 \\ &\quad + a_n\|Sx_{n+1} - Sy_n\| + M_2\|v_n\| \\ &\leq [1 - A(x_{n+1}, q)a_n]\|x_n - q\| \\ &\quad + (1 + 2M_1)Da_n^2 + o(a_n), \end{aligned} \tag{3.8}$$

for all $n \geq 0$. Assume that $\inf\{A(x_{n+1}, q) : n \geq 0\} = r$. We assert that $r = 0$. Otherwise $r > 0$. Using (3.8), we obtain that

$$\|x_{n+1} - q\| \leq (1 - ra_n)\|x_n - q\| + (1 + 2M_1)Da_n^2 + o(a_n),$$

for all $n \geq 0$. By virtue of (3.1), (3.2) and Lemma 2.1, we infer immediately that $\|x_n - q\| \to 0$ as $n \to \infty$. Lemma 2.2 implies that $r = 0$. This is a contradiction. Hence $r = 0$. It follows from Lemma 2.2 that there exists a subsequence $\{x_{n_i+1}\}_{i=0}^{\infty}$ of $\{x_{n+1}\}_{n=0}^{\infty}$ such that $x_{n_i+1} \to q$ as $i \to \infty$. Using (3.1) and (3.2), we conclude that, given $\varepsilon > 0$, there exists a positive integer m such that, for all $n \geq m$,

$$\|x_{n_m+1} - q\| < \varepsilon, \tag{3.9}$$

(3.10)

$$(1+2M_1)Da_n + \|Sx_{n+1} - Sy_n\| + M_2\frac{\|v_n\|}{a_n} < \min\left\{\frac{\varepsilon}{2}, \frac{\phi(\varepsilon)\varepsilon}{1+2\varepsilon+\phi(2\varepsilon)}\right\}.$$

Now we claim that

$$\|x_{n_m+j} - q\| \le \varepsilon, \tag{3.11}$$

for all $j \geq 1$. Obviously, (3.9) ensures that (3.11) holds for $j = 1$. Assume that (3.11) holds for $j = k$. If $\|x_{n_m+k+1} - q\| > \varepsilon$, we conclude that by (3.8) and (3.10)

(3.12)

$$\begin{aligned}\|x_{n_m+k+1} - q\| &\le \|x_{n_m+k} - q\| + a_{n_m+k}[(1+2M_1)Da_{n_m+k} \\ &\quad + \|Sx_{n_m+k+1} - Sy_{n_m+k}\| + M_2\|v_{n_m+k}\|(a_{n_m+k})^{-1}] \\ &< \varepsilon + \min\left\{\frac{\varepsilon}{2}, \frac{\phi(\varepsilon)\varepsilon}{1+2\varepsilon+\phi(2\varepsilon)}\right\}a_{n_m+k} < 2\varepsilon.\end{aligned}$$

In view of (3.9), (3.10) and (3.12), we obtain the following estimates:

$$\begin{aligned}\|x_{n_m+k+1} - q\| &\le \left(1 - \frac{\phi(\varepsilon)}{1+2\varepsilon+\phi(2\varepsilon)}a_{n_m+k}\right)\|x_{n_m+k} - q\| \\ &\quad + a_{n_m+k}[(1+2M_1)a_{n_m+k} + \|Sx_{n_m+k+1} - Sy_{n_m+k}\| \\ &\quad + M_2\|v_{n_m+k}\|(a_{n_m+k})^{-1}] \\ &\le \left(1 - \frac{\phi(\varepsilon)}{1+2\varepsilon+\phi(2\varepsilon)}a_{n_m+k}\right)\varepsilon \\ &\quad + \min\left\{\frac{\varepsilon}{2}, \frac{\phi(\varepsilon)\varepsilon}{1+2\varepsilon+\phi(2\varepsilon)}\right\}a_{n_m+k} \\ &\le \varepsilon.\end{aligned}$$

That is,

$$\varepsilon < \|x_{n_m+k+1} - q\| \le \varepsilon,$$

which is impossible. Hence, $\|x_{n_m+k+1} - q\| \le \varepsilon$. By induction, (3.11) holds for all $j \geq 1$. It follows from (3.11) that $x_n \to q$ as $n \to \infty$. This completes the proof. □

From Theorem 3.1 we have the following results.

Theorem 3.2. *Suppose that X is an arbitrary Banach space and $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator. Assume that $\{u_n\}_{n=0}^\infty, \{v_n\}_{n=0}^\infty$ are sequences in X and $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$*

are sequences in $[0,1]$ *satisfying* (3.1) *and* (3.2). *Suppose that the range of* $I-T$ *is bounded. Then, for any given* $f \in X$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined as in (3.3) converges strongly to the solution of the equation* $Tx = f$.

Corollary 3.1. *Suppose that* X *is an arbitrary Banach space and* $T : X \to X$ *is a uniformly continuous* ϕ*-strongly accretive operator. Assume that* $\{v_n\}_{n=0}^{\infty}$ *is a sequence in* X *and* $\{a_n\}_{n=0}^{\infty}$ *is a sequence in* $(0,1]$ *satisfying*

$$\sum_{n=0}^{\infty} a_n = +\infty, \quad \sum_{n=0}^{\infty} a_n^2 < +\infty, \quad \sum_{n=0}^{\infty} \|v_n\| < +\infty, \quad \|v_n\| = o(a_n). \tag{3.13}$$

Then for any given $f \in X$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined for arbitrary* $x_0 \in X$ *by*

$$x_{n+1} = (1-a_n)x_n + a_n(f + x_n - Tx_n) + v_n, \quad n \geq 0, \tag{3.14}$$

converges strongly to the solution of the equation $Tx = f$ *provided that the sequence* $\{x_n - Tx_n\}_{n=0}^{\infty}$ *is bounded.*

Theorem 3.3. *Suppose that* X *is an arbitrary Banach space and* $T : X \to X$ *is a uniformly continuous* ϕ*-strongly accretive operator. Assume that* $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ *are sequences in* X *and* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *satisfying* (3.1) *and* (3.2). *Then, for any given* $f \in X$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined for arbitrary* $x_0 \in X$ *by*

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n(f - Tx_n) + u_n, \quad n \geq 0, \\ x_{n+1} &= (1-a_n)x_n + a_n(f - Ty_n) + v_n, \quad n \geq 0, \end{aligned} \tag{3.15}$$

converges strongly to the solution of the equation $x + Tx = f$ *provided that the sequences* $\{Tx_n\}_{n=0}^{\infty}$ *and* $\{Ty_n\}_{n=0}^{\infty}$ *are bounded.*

Proof. Put $S = I+T$. It is easy to see that S is a uniformly continuous ϕ-strongly accretive operator and

$$f - Tx = f - (S-I)x = f + x - Sx,$$

for all $x \in X$. It follows from Theorem 3.1 that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of the equation $Sx = f$. This completes the proof. $\square$

From Theorem 3.3 we have

Theorem 3.4. *Suppose that X is an arbitrary Banach space and $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator. Assume that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are sequences in X and $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying (3.1) and (3.2). If the range of T is bounded, then for any given $f \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined as in (3.15) converges strongly to the solution of the equation $x + Tx = f$.*

Corollary 3.2. *Suppose that X is an arbitrary Banach space and $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator. Assume that $\{v_n\}_{n=0}^{\infty}$ is a sequence in X and $\{a_n\}_{n=0}^{\infty}$ is a sequence in $(0,1]$ satisfying (3.13). Then, for any given $f \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined for arbitrary $x_0 \in X$ by*

$$x_{n+1} = (1-a_n)x_n + a_n(f - Tx_n) + v_n, \quad n \geq 0 \tag{3.16}$$

converges strongly to the solution of the equation $x + Tx = f$ provided that the sequence $\{Tx_n\}_{n=0}^{\infty}$ is bounded.

Theorem 3.5. *Suppose that X is an arbitrary Banach space and K is a nonempty closed subset of X. Assume that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are sequences in X and $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying (3.1) and (3.2). Suppose that $T : K \to X$ is a uniformly continuous ϕ-strongly pseudocontractive operator with $F(T) \neq \varnothing$. If the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by*

$$\begin{aligned} y_n &= (1-b_n)x_n + b_nTx_n + u_n, \\ x_{n+1} &= (1-a_n)x_n + a_nTy_n + v_n, \quad n \geq 0, \end{aligned} \tag{3.17}$$

satisfies that $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are in K and $\{Tx_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$ are bounded, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. We claim that $F(T)$ is a singleton. Suppose that T has two fixed points p, q in K and $p \neq q$. Note that T is ϕ-strongly pseudocontractive. Then

$$\|p-q\|^2 = \operatorname{Re}\langle Tp - Tq, j(p-q)\rangle \leq \|p-q\|^2 - \phi(\|p-q\|)\|p-q\|,$$

which implies that $\phi(\|p-q\|) = 0$. Since ϕ is strictly increasing and $\phi(0) = 0$, so that $p = q$. This is a contradiction. That is, $F(T) = \{q\}$ for some $q \in K$.

Since T is ϕ-strongly pseudocontractive, $I - T$ is ϕ-strongly accretive. Thus, for any $x, y \in K$, we have

$$\operatorname{Re}\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\| \geq A(x,y)\|x-y\|^2,$$

where $A(x,y) = (\phi(\|x-y\|)/1 + \|x-y\| + \phi(\|x-y\|)) \in [0,1)$ for all $x, y \in K$. That is,

$$\operatorname{Re}\langle (I-T-A(x,y))x - (I-T-A(x,y))y, j(x-y)\rangle \geq 0,$$

and it follows from Lemma 1.1 of Kato [**18**] that

$$\|x-y\| \leq \|x-y+r[(I-T-A(x,y))x-(I-T-A(x,y))y]\|,$$

for all $x, y \in K$ and $r > 0$. The rest of the proof of this theorem is similar to that of our Theorem 3.1 and is therefore omitted. □

Corollary 3.3. *Suppose that X is an arbitrary Banach space and K is a nonempty closed subset of X. Assume that $\{v_n\}_{n=0}^{\infty}$ is a sequence in X and $\{a_n\}_{n=0}^{\infty}$ is a sequence in $(0,1]$ satisfying* (3.13). *Suppose that $T : K \to X$ is a uniformly continuous ϕ-strongly pseudocontractive operator with $F(T) \neq \varnothing$. If the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by*

$$x_{n+1} = (1-a_n)x_n + a_n T x_n + v_n, \quad n \geq 0, \tag{3.18}$$

is contained in K and the sequence $\{Tx_n\}_{n=0}^{\infty}$ is bounded, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Remark 3.1. The following example shows that the condition that T has a fixed point in K is necessary in Theorem 3.5 and Corollary 3.3.

Example 3.1. Let R denote the reals with the usual norm and $K = [0, 1]$. Define $T : K \to R$ by $Tx = rx + 2$ for all $x \in K$ and some $r \in (0, 1)$. Set $\phi(x) = (1 - r)x$ for all $x \geq 0$. It follows that

$$\langle Tx - Ty, j(x - y)\rangle = \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|,$$

for all $x, y \in K$. Hence, T is a uniformly continuous ϕ-strongly pseudocontractive operator. However, T has no fixed point in K.

It follows from Theorem 3.5 and Corollary 3.3 that

Theorem 3.6. *Suppose that X is an arbitrary Banach space and K is a nonempty closed convex subset of X. Assume that $T : K \to K$ is a uniformly continuous ϕ-strongly pseudocontractive operator with $F(T) \neq \varnothing$, and $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ satisfying*

$$\lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} a_n = +\infty, \quad \sum_{n=0}^{\infty} a_n^2 < +\infty. \tag{3.19}$$

If the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1-b_n)x_n+b_nTx_n, \quad x_{n+1} = (1-a_n)x_n+a_nTy_n, \quad n \geq 0, \tag{3.20}$$

satisfies that $\{Tx_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$ are bounded, then it converges strongly to the fixed point of T.

Corollary 3.4. *Suppose that X is an arbitrary Banach space and K is a nonempty closed convex subset of X. Assume that $T : K \to K$ is a uniformly continuous ϕ-strongly pseudocontractive operator with $F(T) \neq \varnothing$ and $\{a_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ satisfying*

$$\sum_{n=0}^{\infty} a_n^2 < +\infty, \quad \sum_{n=0}^{\infty} a_n = +\infty. \tag{3.21}$$

If the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = (1 - a_n)x_n + a_nTx_n, \quad n \geq 0, \tag{3.22}$$

satisfies that $\{Tx_n\}_{n=0}^{\infty}$ *is bounded, then it converges strongly to the fixed point of* T.

Remark 3.2. Theorems 3.1–3.6 extend, improve and unify Theorems 3.3, 3.4 and 5.2 of [**2**], Theorems 3.2, 3.4, 4.2 and 5.2 of [**3**], the theorem of [**4**], Theorems 1 and 2 of [**5**], Theorem 2 of [**6**], Theorems 2 and 4 of [**7**], Theorems 4, 5, 6, 9, 10 and 13 of [**8**], Theorems 1, 2, 3 and 4 of [**9**], Theorem 1 of [**10**], Theorems 1, 2 and 3 of [**11**], Theorems 1 and 2 of [**13**], Theorems 1, 2, 3 and 4 of [**14**], Theorems 1 and 2 of [**15**], Theorems 1 and 2 of [**16**], Theorem 1 of [**19**], Theorem 1 of [**20**], Theorem 1 of [**23**], Theorems 1 and 3 of [**24**], Theorem 1 of [**25**], Theorems 4.1 and 4.2 of [**26**] and Theorems 1.2 and 3.4 of [**28**] in the following sense:

1. The Mann iteration method in [**3**]–[**5**], [**8**], [**9**], [**20**] and the Ishikawa iteration method in [**2**], [**3**], [**6**]–[**10**], [**13**]–[**16**], [**23**], [**25**], [**26**], [**28**] are replaced by the more general Ishikawa iteration method with errors introduced by Liu [**19**].

2. The assumptions that the equation $f = Tx$ has solutions in [**2**], [**3**], [**23**], [**25**] and $a_n \le b_n$ in [**6**]–[**9**] are omitted.

3. Theorems 3.1–3.6 hold in arbitrary Banach spaces whereas the results of [**2**]–[**11**], [**13**]–[**16**], [**19**], [**23**], [**26**], [**28**] have been proved in the restricted real uniformly smooth Banach spaces, L_p, or l_p, spaces, p-uniformly convex Banach spaces, smooth real Banach spaces and p-uniformly smooth real Banach spaces, respectively.

4. The Lipschitz strongly pseudocontractive operators in [**2**]–[**10**], [**13**]–[**16**], [**20**], [**26**], [**28**], the uniformly continuous strong pseudocontractive operators in [**2**], [**3**], [**8**], [**11**], the Lipschitz strongly accretive operators in [**2**], [**3**], [**5**], [**7**], [**9**], [**13**]–[**16**], [**19**], [**24**], [**26**], [**28**], the uniformly continuous strong accretive operators in [**8**], [**11**], [**24**] and the Lipschitz ϕ-strongly accretive operators in [**23**], [**25**] are replaced by the more general uniformly continuous ϕ-strongly accretive operators and uniformly continuous ϕ-strongly pseudocontractive operators, respectively.

Remark 3.3. The iteration parameters $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ in Theorems 3.1–3.6 and Corollaries 3.1–3.4 do not depend on any geometric structure of the underlying Banach space X or on any property of the operator T. A prototype for $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ in our results is $a_n = b_n = 1/(n+2)$ for all $n \geq 0$.

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REFERENCES

1. F.E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.

2. S.S. Chang, *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. **30** (1997), 4197–4208.

3. S.S. Chang, Y.J. Cho, B.S. Lee and S.M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl. **224** (1998), 149–165.

4. C.E. Chidume, *Iterative approximation of fixed points of Lipschitz strictly pseudo-contractive mappings*, Proc. Amer. Math. Soc. **99** (1987), 283–288.

5. ———, *An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces*, J. Math. Anal. Appl. **151** (1990), 453–461.

6. ———, *Approximation of fixed points of strongly pseudo-contractive mappings*, Proc. Amer. Math. Soc. **120** (1994), 545–551.

7. ———, *Iterative solution of nonlinear equations with strongly accretive operators*, J. Math. Anal. Appl. **192** (1995), 502–518.

8. ———, *Iterative solution of nonlinear equations in smooth Banach spaces*, Nonlinear Anal. **26** (1996), 1823–1834.

9. C.E. Chidume and M.O. Osilike, *Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings*, J. Math. Anal. Appl. **192** (1995), 727–741.

10. ———, *Nonlinear accretive and pseudo-contractive operator equations in Banach spaces*, Nonlinear Anal. **31** (1998), 779–789.

11. ———, *Iterative solution of nonlinear accretive operator equations in arbitrary Banach spaces*, Nonlinear Anal. **36** (1999), 863–872.

12. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

13. L. Deng, *On Chidume's open questions*, J. Math. Anal. Appl. **174** (1993), 441–449.

14. ———, *An iterative process for nonlinear Lipschitz and strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces*, Acta Appl. Math. **32** (1993), 183–196.

15. ———, *Iteration processes for nonlinear Lipschitzian strongly accretive mappings in L_p spaces*, J. Math. Anal. Appl. **188** (1994), 128–140.

16. L. Deng and X.P. Ding, *Iterative approximation of Lipschitz strictly pseudo-contractive mappings in uniformly smooth Banach spaces*, Nonlinear Anal. **24** (1995), 981–987.

17. S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

18. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.

19. L.S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

20. L.W. Liu, *Approximation of fixed points of a strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **125** (1997), 1363–1366.

21. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

22. R.H. Martin, Jr., *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc. **26** (1970), 307–314.

23. M.O. Osilike, *Iterative solution of nonlinear equations of the ϕ-strongly accretive type*, J. Math. Anal. Appl. **200** (1996), 259–271.

24. ———, *Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type*, J. Math. Anal. Appl. **213** (1997), 91–105.

25. ———, *Iterative solution of nonlinear ϕ-strongly accretive operator equations in arbitrary Banach spaces*, Nonlinear Anal. **36** (1999), 1–9.

26. K.K. Tan and H.K. Xu, *Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces*, J. Math. Anal. Appl. **178** (1993), 9–21.

27. ———, *Aproximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

28. L.C. Zeng, *Iterative approximation of solutions to nonlinear equations of strongly accretive operators in Banach spaces*, Nonlinear Anal. **31** (1998), 589–598.

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