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ON LINEAR MEANS OF MULTIPLE FOURIER INTEGRALS DEFINED BY SPECIAL DOMAINS

E. LIFLYAND AND A. NAKHMAN

ABSTRACT. Weak and strong estimates in weighted L^p spaces are obtained for linear means of Fourier integrals defined by a single function with support in a specially organized set.

Introduction. For a function f integrable on the n-dimensional Euclidean space $\mathbf{R}^n$, written $f \in L^1(\mathbf{R}^n)$, its Fourier transform is well defined

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(u) e^{-ixu}\, du,$$

where $x = (x_1, x_2, \dots, x_n)$, $u = (u_1, u_2, \dots, u_n) \in \mathbf{R}^n$ and $xu = x_1u_1 + x_2u_2 + \cdots + x_nu_n$. Let

$$\int_D \hat{f}(x) e^{iux}\, dx$$

be the partial Fourier integral defined by a set D. The behavior of partial Fourier integrals with respect to a specifically organized family of such sets characterizes approximation properties of f. It is natural to define such a family as a sequence of dilations of a fixed set D. This has been extensively studied when D is the cube (cubic case)

$$D = \{x \in \mathbf{R}^n : |x_j| \le 1,\ j = 1, 2, \dots, n\},$$

or the ball (spherical case)

$$D = \{x \in \mathbf{R}^n : |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \le 1\}.$$

Their R-dilations are

$$RD = \{x \in \mathbf{R}^n : |x_j| \le R,\ j = 1, 2, \dots, n\}$$

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and

$$RD = \{x \in \mathbf{R}^n : |x| \le R\},$$

respectively. The other example of a family of sets is the family of rectangles

$$\{x \in \mathbf{R}^n : |x_j| \le R_j,\ R_j > 0,\ j = 1, 2, \dots, n\}$$

that cannot be expressed as a family of dilations of a fixed set. Numerous results on these (as well as references) may be found, e.g., in [**12**, Chapter 17] or [**11**], where similar problems are studied for multiple Fourier series as well.

In this paper we consider linear means of multiple Fourier integrals rather than partial sums. We define them by the family of dilations of a set D from some special class. The latter is closer to the spherical case rather than to the other ones. The estimates are obtained for the weighted L^p spaces.

The outline of the paper is as follows. In Section 1 we give necessary preliminaries and formulate main results. In Section 2 auxiliary results are given. In Section 3 we prove the main results. In the next section we give some concluding remarks and commentaries.

1. Notation and main results. Given functions $u(x) \ge 0$, $\omega(x) \ge 0$ measurable on each cube $\Pi(\theta, 0) = \{x : |x_j| \le \theta, \theta > 0, j = 1, 2, \dots, n\}$ and a set G of either finite or infinite Lebesgue measure, associate with them the measure

$$\mu(G) = \int_G u(x)\,dx$$

and the numbers

$$A_p^{u,\omega}(G) = \mu(G)\biggl(\int_G \omega^{-1/(p-1)}(x)\,dx\biggr)^{p-1}, \quad p > 1,$$

and

$$A_1^{u,\omega}(G) = \mu(G) \operatorname*{ess\,sup}_{x \in G} \frac{1}{\omega(x)}.$$

Define the weighted L^p_ω space, $p \geq 1$, as the space of functions g endowed with the norm

$$\|g\|_{p,\omega} = \left(\int_{\mathbf{R}^n} |g(x)|^p \omega(x)\, dx \right)^{1/p} < \infty.$$

As in [**5**], we take $0 \cdot \infty = 0$. If $\omega \equiv 1$, we obtain $L^p_\omega = L^p$, the usual L^p space.

Let D be a convex domain with nowhere vanishing principal curvatures of its boundary ∂D. Let D have compact closure with ∂D being a C^k-smooth hypersurface, $k \geq 1$.

Set

$$E = E(M, x) = \{u : x - u \in MD\}.$$

We write $(u, \omega) \in A_p$, $p \geq 1$, if a constant $C > 0$ exists such that for any $E = E(M, x)$

$$A_p^{u,\omega}(E) \leq C|E|^p, \tag{1}$$

and $\omega \in A_p$ for $u = \omega$; here and in what follows $|E|$ means the Lebesgue measure of E.

Throughout the paper we denote by C various constants, which are independent of functions f and parameters R, M, k (all of which will be introduced in the course of the work). We use subindices to emphasize the dependence of such constants on certain parameters, say C_p, C_η, etc.

A condition of type (1) was first introduced by Muckenhoupt [**5**] for weighted estimates of Hardy maximal functions.

Let λ be a function whose support is the closure of D and that is C^k-smooth inside D, of the form

$$\lambda(x) = \rho(x)^\alpha \varphi(x), \tag{2}$$

where $\varphi \in C^k(\mathbf{R}^n)$ and does not vanish on ∂D and ρ is a regularized distance to the boundary (see [**10**, Chapter 6, Theorem 2]), that is, $\rho \in C^\infty$ outside ∂D and

$$C_1 \operatorname{dist}(x, \partial D) \leq \rho(x) \leq C_2 \operatorname{dist}(x, \partial D)$$

for some positive constants C_1 and C_2. In addition, assume that $\rho(x) = 0$ when $x \notin D$.

Define the linear means of the Fourier integral

$$\sigma_R(f;x) = \sigma_R(f;x;\lambda) = (2\pi)^{-n} \int_{\mathbf{R}^n} f(x-s) R^n \hat{\lambda}(-Rs)\, ds. \tag{3}$$

Indeed, for f smooth enough we have, by Fubini's theorem,

$$\begin{aligned}
\sigma_R(f;x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} f(x-s) R^n \hat{\lambda}(-Rs)\, ds \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} f(x-s) R^n \int_{\mathbf{R}^n} \lambda(v) e^{iRsv}\, dv\, ds \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} f(x-s) \int_{\mathbf{R}^n} \lambda(v/R) e^{ivs}\, dv\, ds \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} \lambda(v/R) e^{ixv}\, dv \int_{\mathbf{R}^n} f(s) e^{-ivs}\, ds \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} \lambda(v/R) \hat{f}(v) e^{ixv}\, dv,
\end{aligned}$$

that is, the linear means are defined in a usual (multiplier) way. We see that they are defined by means of the function λ that, in turn, strongly depends on geometric properties of D. We will write $\sigma_R(f)$ if the argument is of no importance for us. The representation (3) is a usual way to avoid problems of definition of the Fourier transform of f.

In this paper we are going to restrict ourselves to the case $\alpha > (n-1)/2$. By this (3) is a generalization of the Bochner-Riesz means of order greater than critical one, $(n-1)/2$. They were first introduced in the celebrated Bochner's paper [**1**]. In our notation this is the case when $D = \{x : |x| \le 1\}$ and $\lambda(x) = (1-|x|^2)_+^\alpha$.

Also we fix arbitrary k which satisfies $k > \alpha + (n-1)/2$. We need the following notation convenient for presentation of weak type estimates:

$$S_g = S_g(\xi) = \{x : |g(x)| > \xi > 0\}.$$

We will prove the following results. Set

$$\sigma_*(f;x) = \sup_{R>0} |\sigma_R(f;x)|.$$

Theorem 1. *Let* $\omega \in A_p$. *Then*

i) $\mu(S_{s^*} \leq C\xi^{-p}\|f\|_{p,\omega}^p$, $p \geq 1$;

ii) $\|\sigma_*(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}$, $p > 1$.

iii) *If we have in addition*

$$\lambda(0) = 1 \tag{4}$$

then

$$\lim_{R\to\infty} \sigma_R(f;x) = f(x)$$

μ*-almost everywhere for each* $f \in L_\omega^p$, $p \geq 1$.

Theorem 2. *Let* $(u,\omega) \in A_p$, $p \geq 1$; *then*

$$\|\sigma_R(f)\|_{p,u} \leq C\|f\|_{p,\omega}, \quad p > 1. \tag{5}$$

If, in addition, (4) *and* $A_p^{u,\omega}(\mathbf{R}^n) < \infty$ *are satisfied, then for every* $f \in L_u^p \cap L_\omega^p$ *the estimate* (5) *is equivalent to*

$$\lim_{R\to\infty} \|\sigma_R(f) - f\|_{p,u} = 0. \tag{6}$$

2. Auxiliary results. First let us give the following asymptotic estimate for λ (see [**4**, Theorem 3] or [**9**]).

Theorem A. *Let* $\alpha > 0$, $k > \max(1, (n-1)/2 + \alpha)$, $\eta \in \mathbf{R}^n$ *be a unit vector and* $x^+(\eta)$ *and* $x^-(\eta)$ *be the* (*uniquely defined*) *points of* ∂D *at which the function* $\eta x = \eta_1 x_1 + \cdots + \eta_n x_n$ *attains maximum and minimum on* ∂D*, respectively. Then, for* $t \to \infty$,

$$\hat{\lambda}(t\eta) = t^{-\alpha-(n+1)/2}[e^{itx^+(\eta)\eta\Xi^+} + e^{itx^-(\eta)\eta\xi^-} + o(1)],$$

where

$$\Xi^\pm = e^{\pm\pi i(2\alpha+n+1)/4}\varphi(x^\pm)(\varkappa^\pm)^{-1/2}(2\pi)^{(n-1)/2}\Gamma(\alpha+1),$$

the remainder term is small uniformly in η, *and* $\varkappa^\pm$ *are the Gaussian curvatures of* ∂D *at the points* $x^\pm$, *respectively.*

Observe that for $\alpha > (n-1)/2$ the Fourier transform $\hat{\lambda}$ is integrable on $\mathbf{R}^n$; we also have $k > \alpha + (n-1)/2$ which justifies the above specification.

Set

$$f_M(x) = \frac{1}{|E(M,x)|} \int_{E(M,x)} |f(u)|\, du,$$

and

$$f^*(x) = \sup_{M>0} f_M(x).$$

Lemma 1. *The following inequality*

$$\sigma_*(f;x) \le C_{n,\alpha} f^*(x)$$

holds.

Proof. For $R|s|$ small enough, it suffices to make use of the following inequality

$$\left| \int_{\mathbf{R}^n} \lambda(u) e^{iRsu}\, du \right| = \left| \int_D \lambda(u) e^{iRsu}\, du \right| \le C, \tag{7}$$

while for $R|s|$ large enough, Theorem A yields

$$\left| \int_{\mathbf{R}^n} \lambda(u) e^{iRsu}\, du \right| \le C(R|s|)^{-\alpha-(n+1)/2}. \tag{8}$$

We have

$$\begin{aligned} \sigma_R(f;x) &= (2\pi)^{-n} \int_{E(2^M/R,0)} f(x-s) R^n \hat{\lambda}(-Rs)\, ds \\ &\quad + (2\pi)^{-n} \sum_{k=M}^{\infty} \int_{\mathbf{D}_k(R)} f(x-s) R^n \hat{\lambda}(-Rs)\, ds \end{aligned} \tag{9}$$

where $\mathbf{D}_k(R) = E(2^{k+1}/R,0) \setminus E(2^k/R,0)$ and M is such that $x - s \in E(2^M/R,0)$. For the first integral on the righthand side of (9) we have by (7)

$$\begin{aligned} &\left| \int_{E(2^M/R,0)} f(x-s)\, ds R^n \hat{\lambda}(-Rs) \right| \\ &\quad \le C \frac{1}{|E(2^M/R,0)|} \int_{E(2^M/R,0)} |f(x-s)|\, ds \le C f_{2^M/R}(x). \end{aligned} \tag{10}$$

Furthermore, by (8),

$$
\begin{aligned}
(11)\quad & \left| \int_{\mathbf{D}^k(R)} f(x-s)\,dsR^n\hat{\lambda}(-Rs) \right| \\
&\le R^{(n-1)/2-\alpha} \int_{\mathbf{D}_k(R)} |f(x-s)||s|^{-a-(n+1)/2}\,ds \\
&\le CR^{(n-1)/2-\alpha}(2^k/R)^{-\alpha-(n+1)/2} \int_{E(2^{k+1}/R,0)} |f(x-s)|\,ds \\
&\le CR^n 2^{-k(\alpha+(n+1)/2)} \int_{E(2^{k+1}/R,0)} |f(x-s)|\,ds \\
&\le CR^n 2^{-k(\alpha+(n+1)/2)}(2^{k+1}/R)^n \frac{1}{|E(2^{k+1}/R,0)|} \\
&\quad \cdot \int_{E(2^{k+1}/R,0)} |f(x-s)|\,ds \\
&\le C2^{k((n-1)/2-\alpha)} f_{2^{k+1}/R}(x).
\end{aligned}
$$

It follows from the representation (9) and estimates (10) and (11)

$$
(12)\qquad \sigma_*(f,x) \le C\left\{1 + \sum_{k=M}^{\infty} 2^{k((n-1)/2-\alpha)}\right\} f^*(x) \le C_{n,\alpha} f^*(x),
$$

since the series in (12) converges just for $\alpha > (n-1)/2$. The lemma is proved. □

Lemma 2. *Let* $\omega \in A_p$.

i) *If* $p \ge 1$, *then*

$$
(13)\qquad \mu(S_{f^*}) \le C\xi^{-p}\|f\|_{p,\omega}^p
$$

ii) *For* $p > 1$,

$$
(14)\qquad \|f^*\|_{p,\omega} \le C_p\|f\|_{p,\omega}.
$$

Proof. This assertion is proved in [**3**] with f^* and A_p defined by means of the cubes

$$
(15)\qquad \Pi(M,x) = \{u \in \mathbf{R}^n : |u_j - x_j| \le M,\ j = 1,2,\dots,n\}.
$$

It remains to observe that two positive constants C_1 and C_2 exist, $C_1 < C_2$, such that

$$C_1 \le |\Pi(M,x)|/|E(M,x)| \le C_2. \qquad \square \tag{16}$$

Lemma 3. *Let* $(u,\omega) \in A_p$, $p \ge 1$. *Then*

$$\|f_M\|_{p,u} \le C\|f\|_{p,\omega}.$$

Proof. If $f_M(x)$ and A_p are defined by means of the cubes (15) provided that f is a periodic function, this assertion is proved in [**7**] (as an extension of the corresponding one-dimensional assertion proved in [**6**]). This proof remains valid for the nonperiodic case as well. The results follows now from relation (16). $\square$

Observe that in Lemmas 2 and 3 as well as in what follows, if $(u,\omega) \in A_p$, we have $f \in L^1$ on each cube $\Pi(\theta,0)$. Indeed, $|f| = |f|\omega^{1/p}\omega^{-1/p}$, and it follows by Hölder's inequality that

$$\int_{\Pi(\theta,0)} |f(x)|\,dx \le \|f\|_{p,\omega}\left(\int_{\Pi(\theta,0)} \omega^{-1/(p-1)}(x)\,dx\right)^{(p-1)/p}. \tag{17}$$

The integral on the right-hand side of (17) is finite; otherwise, by the A_p condition, we have $u \sim 0$ and the assertions of Lemma 3 and below are trivial.

3. Proofs of the main results.

Proof of Theorem 1. The assertions i) and ii) immediately follow from Lemma 1 and (13), (14), respectively. To prove iii), a standard argument is used. First, observe that the integral

$$(2\pi)^{-n}\int_{\mathbf{R}^n} R^n\hat{\lambda}(-Rs)\,ds = (2\pi)^{-n}\int_{\mathbf{R}^n} \hat{\lambda}(-s)\,ds = 1,$$

because of the condition (4). Since the Fourier transform of λ is integrable, the Lebesgue constants are uniformly bounded, and by Theorem 1.18 from [**11**]

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}^n} |g(x) - \sigma_R(g;x)| = o(1), \tag{18}$$

as $R \to \infty$, for any continuous function g with compact support. Approximating f by such functions g in L^p_ω-norm and denoting $\phi = f - g$, we arrive at the estimate

$$|\sigma_R(f,x) - f(x)| \le o(1) + \sigma_*(\phi,x) + |\phi(x)|$$

which holds almost everywhere. By i) and Chebyshev's inequality, the sum $\sigma_*(\phi,x) + |\phi(x)|$ is small up to a set of small μ-measure. This completes the proof of Theorem 1. □

Proof of Theorem 2. The assertion (5) follows from Lemma 3, representation (9), estimates (10), (11) and Minkowski's inequality for integrals.

Now let $f \in L^p_u \cap L^p_\omega$. If we approximate f in the L^p_u norm by a continuous function g with compact support, then by (18)

$$\|\sigma_R(f,x) - f(x)\|_{p,u} \le o(1)\biggl(\int_{\mathbf{R}^n} u(x)\,dx\biggr)^{1/p} + \|\sigma_R(\phi)\|_{p,u} + \|\phi\|_{p,u}. \tag{19}$$

The integral $\int_{\mathbf{R}^n} u(x)\,dx$ is finite provided $A_p^{u,\omega}(\mathbf{R}^n) < \infty$. Indeed, otherwise $\omega = \infty$ almost everywhere, and thus $f \in L^p_\omega$ leads to the trivial case $f = 0$ almost everywhere; the second assertion of Theorem 2 is trivial in this case. The additional assumption of Theorem 2 yields

$$\|f\|_{p,u} \le C\|f\|_{p,\omega}. \tag{20}$$

This inequality was proved in [**7**] for $f \in L^p_u \cap L^p_\omega$ with support in $\Pi(\theta,0)$; here θ is arbitrary and C does not depend on f. Then we obtain (20) as $\theta \to \infty$. To obtain (6), it now remains now to combine (5), (19) and (20).

The converse implication (6) $\Rightarrow$ (5) is provided by the Banach-Steinhaus theorem since, for each R, the operator σ_R taking L^p_ω into L^p_u is bounded. Indeed, as in the proof of Lemma 1,

$$\begin{aligned}\|\sigma_R(f)\|^p_{p,u} &\le C\int_{\mathbf{R}^n}\left[\int_{\mathbf{R}^n}|f(x-s)|R^n|\hat{\lambda}(-Rs)|\,ds\right]^p u(x)\,dx\\ &\le CR^{np}\mu(\mathbf{R}^n)\bigg(\int_{\mathbf{R}^n}|f(s)|\,ds\bigg)^p,\end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned}\|\sigma_R(f)\|^p_{p,u} &\le CR^{np}\mu(\mathbf{R}^n)\bigg[\int_{\mathbf{R}^n}|f(s)|^p\omega(s)\,ds\bigg]\\ &\quad\cdot\bigg[\int_{\mathbf{R}^n}\omega^{-1/(p-1)}(s)\,ds\bigg]^{p-1}\\ &\le CR^{np}A^{u,\omega}_p(\mathbf{R}^n)\|f\|^p_{p,\omega},\end{aligned}$$

which completes the proof. □

4. Concluding remarks.

Remark 1. Of course, the case $u=\omega\equiv 1$, that is, $L^p_\omega = L^p$, is of special interest. Note that Theorem 1 is true for $\omega\equiv 1$ as well. As for (6), the condition $A^{u,\omega}_p(\mathbf{R}^n)<\infty$ is no longer valid for $u=\omega\equiv 1$; nevertheless, (6) follows from (5) by the same Theorem 1.18 from [**11**].

Remark 2. Similar results are true for the linear means of multiple Fourier series defined by λ:

$$\sum_{k\in\mathbf{Z}^n}\lambda(k/R)\hat{f}(k)e^{ikx},$$

where f is a 2π-periodic function in each variable, and

$$\hat{f}(k) = (2\pi)^{-n}\int_{\mathbf{T}^n} f(x)e^{-ikx}\,dx,$$

$\mathbf{T} = (-\pi,\pi]$, is its kth Fourier coefficient. The same tools can be involved with the help of the Poisson summation formula (see, e.g., [**11**, Chapter 7]).

We mention that in the case of L^1-space, precise estimates from above were obtained in [**2**] and from below in [**4**].

Remark 3. For dimension 2 in a special case when D is an ellipse and λ is elliptically symmetric, more subtle estimates were obtained in [**8**]. The point is that Theorem A gives estimates of the Fourier transform only in power scale, while in the special case considered in [**8**] less restrictive assumptions were posed on λ. Any improvement of Theorem A will lead to an immediate extension of Theorem 1 to a wider class of linear means defined by λ.

Remark 4. The "strong" two-weighted norm estimate

$$\|\sigma_*(f)\|_{p,u} \le C\|f\|_{p,\omega}$$

does not follow from $(u,\omega) \in A_p$, $u \ne \omega$. The same fact was mentioned by Muckenhoupt [**5**] for maximal function $f^*(x)$, where $f \ge 0$ and $f^*(x)$ and the A_p-condition are defined by means of the cubes $\Pi(M,x)$. Hence, to prove our assertion, it remains to establish the equivalence (two-sided estimate)

$$\sigma_*(f,x) \asymp f^*(x)$$

for some λ and D. Consider $D = \{x \in \mathbf{R}^n : |x| \le 1\}$ and the Bochner-Riesz means

$$\lambda(x) = (1-|x|^2)_+^{\alpha}, \quad \alpha > \frac{n-1}{2}.$$

In view of Lemma 1, it suffices to prove only the estimate

$$\sigma_*(f,x) \ge Cf^*(x), \quad C > 0. \tag{21}$$

Apply the relations (see, e.g., [**11**, Chapter 4, Theorem 4.15]):

$$\hat{\lambda}(s) \ge C_\alpha |s|^{-(n/2+\alpha)} J_{n/2+\alpha}(2\pi|s|), \quad C_\alpha > 0, \tag{22}$$

where $J_k(s)$ are the Bessel functions and $J_k(s) \ge Cs^k$ for s small enough, $k \ge 0$. The estimate (22) holds for $\hat{\lambda}(-Rs)$, $s \in E(M,x)$, if $R = C/M$. Hence, for every $f(s) > 0$ with support in $E(M,0)$, we obtain

$$\begin{aligned}\sigma_*(f,x) \ge |\sigma_R(f,x)| > (2\pi)^{-n} \int_{E(M,0)} f(s) R^n \hat{\lambda}(-R(x-s))\, dx \\ > Cf_M(x).\end{aligned}$$

This inequality yields (21).

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Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: liflyand@macs.biu.ac.il

Department of Applied Mathematics and Mechanics, Tambov State Technical University, Tambov 392620, Russia
E-mail address: nakhman@apmath.tstu.ru