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# NUMERICAL CALCULATION OF THE GENERALIZED SINE AND COSINE INTEGRAL

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ABSTRACT. A method is presented for the efficient calculation of the generalized sine and cosine integral. The evaluation is done by Čebyšev series with two nested recursions.

**1. Introduction.** The generalized sine and cosine integrals (see Table 1) are defined as

$$\text{Si}\,(x,\alpha) := \int_0^x \frac{\sin t}{t^\alpha}\,dt, \quad 0 \le x, 0 < \alpha < 2, \tag{1}$$

$$\text{Ci}\,(x,\alpha) := \int_0^x \frac{\cos t}{t^\alpha}\,dt, \quad 0 \le x, 0 < \alpha < 1. \tag{2}$$

Both functions were extensively studied in all details with respect to their analytical behaviour in Kreyszig [**5**]. In an earlier paper by Walther [**7**] the generalized sine integral was already used to study the Gibbs's phenomenon of Fourier series.

Special cases of both integrals are well known.

For $\alpha = 1$ we obtain the "ordinary" sine integral $\text{Si}\,(x) := \int_0^x (\sin t/t)\,dt$ and for $\alpha = (1/2)$ the Fresnel integrals

$$\int_0^x \frac{\sin t}{\sqrt{t}}\,dt = 2\int_0^{\sqrt{x}} \sin\tau^2\,d\tau, \qquad \int_0^x \frac{\cos t}{\sqrt{t}}\,dt = 2\int_0^{\sqrt{x}} \cos\tau^2\,d\tau.$$

A close relationship exists between both integrals to the hypergeometric function ${}_1\text{F}_2$, the incomplete Gamma function $\gamma$ and the confluent

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hypergeometric function ${}_1\mathrm{F}_1$. The following relations hold

$$\begin{aligned}
\mathrm{Si}\,(x,\alpha) &= \frac{x^{2-\alpha}}{2-\alpha}\,{}_1\mathrm{F}_2\left(\begin{matrix}\frac{1}{2}(2-\alpha)\\ \frac{1}{2}(4-\alpha)\,,\ \frac{3}{2}\end{matrix}\,\middle|\, -\frac{x^2}{4}\right), \quad \alpha<2,\\
\mathrm{Ci}\,(x,\alpha) &= \frac{x^{1-\alpha}}{1-\alpha}\,{}_1\mathrm{F}_2\left(\begin{matrix}\frac{1}{2}(1-\alpha)\\ \frac{1}{2}(3-\alpha)\,,\ \frac{1}{2}\end{matrix}\,\middle|\, -\frac{x^2}{4}\right), \quad \alpha<1,\\
\mathrm{Ci}\,(x,\alpha)+\mathrm{i}\,\mathrm{Si}(x,\alpha) &= e^{i(\pi/2)(1-\alpha)}\,\gamma\,(-ix,\,1-\alpha)\\
&= \frac{x^{1-\alpha}}{1-\alpha}\,{}_1\mathrm{F}_1\left(\begin{matrix}1-\alpha\\ 2-\alpha\end{matrix}\,\middle|\, i\,x\right), \quad \alpha<1.
\end{aligned}$$

We may evaluate both integrals to any prescribed accuracy via the hypergeometric functions by using a computer algebra program such as *Mathematica* [**4**]. However, this is not an efficient way to compute these integrals if a great number of function evaluations is needed in the course of simulations. The generalized sine and cosine integrals are used among others, for instance, in the simulation of the propagation of electromagnetic waves. The aircraft industry requires fast algorithms (written in advanced programming languages such as FORTRAN and C) for such extensive numerical simulations. In these programming languages each real variable occupies one computer word in single precision or two consecutive computer words if double precision is used; the numerical calculation is done with an a priori fixed number of words for each real variable.

The numerical calculation of both integrals is done in a fast and efficient way by truncated Čebyšev series including two nested recursions as we will show in the next sections.

The special cases $\alpha = 1/2$ and $\alpha = 1$ are already treated by Németh [**6**] and in [**2**].

**2. The numerical calculation of the generalized sine and cosine integral.** Both integrals are functions of two variables $x$ and $\alpha$. Without loss of generality, we may in the case of the sine integral restrict $\alpha$ to the range $0 < \alpha < 2$ and for the cosine integral to $0 < \alpha < 1$, respectively. For $x$ we have the range $0 < x < \infty$.

The range $0 < x < \infty$ is divided in an appropriate way into two subranges $0 < x \le \lambda$, $\lambda < x < \infty$. In each range we have the Čebyšev

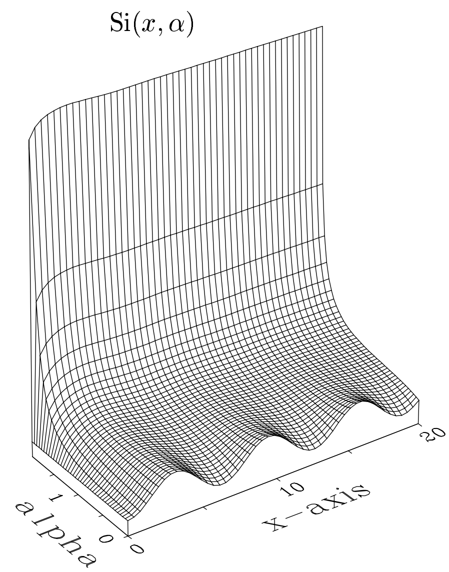


FIGURE 1a. Generalized sine and cosine integral.

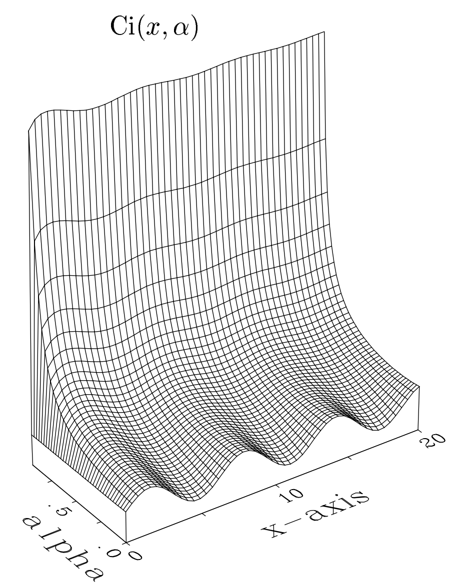


FIGURE 1b. Generalized sine and cosine integral.

expansions

$$\sin(\xi\tau) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(\varrho\,\tau)\,T_{2k+1}\left(\frac{\xi}{\varrho}\right), \tag{3}$$

$$\cos(\xi\tau) = 2\sum_{k=0}^{\infty}{}'(-1)^k J_{2k}(\varrho\,\tau)\,T_{2k}\left(\frac{\xi}{\varrho}\right), \quad \varrho \neq 0 \tag{4}$$

($\xi$, $\tau$ suitably chosen) where $J_l$ denotes the Bessel function and $T_l$ is the Čebyšev polynomial. The prime in $\sum{}'$ indicates that the first term of the sum is weighted by a half.

**2.1. The range $\mathbf{0 < x \leq \lambda}$, $\lambda < \infty$.** Using the Čebyšev expansion equations (3) and (4) we deduce

$$\begin{aligned} \mathrm{Si}\,(x,\alpha) &= x^{1-\alpha}\int_0^1 \frac{\sin(x\,\tau)}{\tau^\alpha}\,d\tau \\ &= 2\,x^{1-\alpha}\sum_{k=0}^{\infty}(-1)^k\,c_{2k+1}\,T_{2k+1}\left(\frac{x}{\lambda}\right), \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{Ci}\,(x,\alpha) &= x^{1-\alpha}\int_0^1 \frac{\cos(x\,\tau)}{\tau^\alpha}\,d\tau \\ &= 2\,x^{1-\alpha}\sum_{k=0}^{\infty}{}'(-1)^k\,c_{2k}\,T_{2k}\left(\frac{x}{\lambda}\right) \end{aligned} \tag{6}$$

where

$$c_l = \int_0^1 \frac{J_l(\lambda\,\tau)}{\tau^\alpha}\,d\tau = \lambda^{\alpha-1}\int_0^\lambda \frac{J_l(t)}{t^\alpha}\,dt \quad \text{and} \quad t = \lambda\,\tau. \tag{7}$$

We now have

**Lemma 1.** *The coefficients* (7) *satisfy the recursion*

$$(l+1-\alpha)\,c_l - (l+1+\alpha)\,c_{l+2} = J_l(\lambda) + J_{l+2}(\lambda). \tag{8}$$

*Proof.* We use the well known relations

$$J_{l-1}(z) + J_{l+1}(z) = \frac{2l}{z}\,J_l(z), \qquad J_{l-1}(z) - J_{l+1}(z) = 2\,J_l'(z).$$

Equation (7) yields

$$c_l - c_{l+2} = \lambda^{\alpha-1} \int_0^\lambda \frac{J_l(t) - J_{l+2}(t)}{t^\alpha}\, dt = 2\,\lambda^{\alpha-1} \int_0^\lambda \frac{J'_{l+1}(t)}{t^\alpha}\, dt. \tag{9}$$

Integration by parts of the righthand side of Equation (9) now gives

$$\begin{aligned} c_l - c_{l+2} &= 2\,\lambda^{\alpha-1} \left[\frac{J_{l+1}(t)}{t^\alpha}\right]\Bigg|_0^\lambda + 2\,\lambda^{\alpha-1}\,\alpha \int_0^\lambda \frac{J_{l+1}(t)}{t^{\alpha+1}}\, dt \\ &= 2\,\frac{J_{l+1}(\lambda)}{\lambda} + \frac{2\,\lambda^{\alpha-1}\,\alpha}{2\,(l+1)} \int_0^\lambda \frac{J_l(t) + J_{l+2}(t)}{t^\alpha}\, dt \\ &= 2\,\frac{J_{l+1}(\lambda)}{\lambda} + \frac{\alpha}{l+1}\,(c_l + c_{l+2})\,. \end{aligned}$$

This results in the recursion formula (8). □

For the Bessel functions $J_\nu$, $\nu \ge -1/2$, we have upper bounds (see [**1**])

$$|\,J_\nu(z)| \le \frac{|z/2|^\nu\; e^{|\,\mathrm{Im}\, z\,|}}{\Gamma(\nu+1)}, \quad \nu \ge -\frac{1}{2}, \quad z \in \mathbf{C}.$$

These upper bounds yield an upper bound for the Čebyšev coefficients $c_l$ of the sine and cosine integral

$$|\,c_l\,| < \frac{|\lambda/2|^l}{(l-1)\,\Gamma(l+1)} =: UB_1(l,\lambda), \quad 2 \le l. \tag{10}$$

**2.1.1 Numerical solution in the range $0 < x \le \lambda$.** It is a well-known fact that truncated Čebyšev series provide quite efficient approximations. We have for the integrals above

$$\begin{aligned} \mathrm{Si}\,(x,\alpha) &\doteq 2\,x^{1-\alpha} \sum_{k=0}^{N_1} (-1)^k\, c_{2k+1}\, T_{2k+1}\left(\frac{x}{\lambda}\right), \\ \mathrm{Ci}\,(x,\alpha) &\doteq 2\,x^{1-\alpha} \sum_{k=0}^{N_1}{}' (-1)^k\, c_{2k}\, T_{2k}\left(\frac{x}{\lambda}\right) \end{aligned} \tag{11}$$

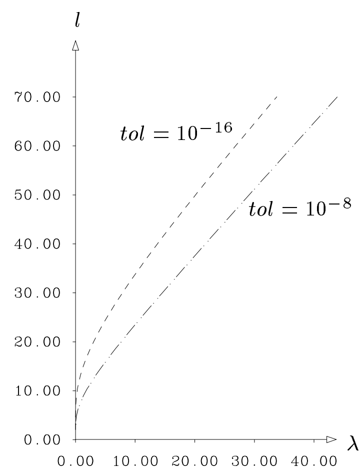


FIGURE 2. $UB_1(l,\lambda) = \text{tol}$, $\text{tol} = 10^{-8}, 10^{-16}$.

where the upper index $N_1$ will depend on $\lambda$ and the prescribed accuracy.

First we have to compute approximate values for the Čebyšev coefficients $c_l = c_l(\alpha,\lambda)$. We do this by solving Equation (8) by backward recursion.

Let us denote the approximate value of $c_l$ by $\tilde{c}_l$. Then we have

$$\tilde{c}_{2N_1+3} = \tilde{c}_{2N_1+2} = 0$$

and

$$\tilde{c}_l = \left[(l+1+\alpha)\,\tilde{c}_{l+2} + \tilde{J}_l + \tilde{J}_{l+2}\right] \Big/ (l+1-\alpha)$$
$$\text{for } l = 2\,N_1+1, \dots, 1 \quad \text{or } 0.$$

$\tilde{J}_l$ are numerical approximations of $J_l(\lambda)$, $l = 2N_1+3, \dots, 0$, see for instance Gautschi [**3**].

The summation of the truncated Čebyšev series Equation (11) is done by Clenshaw's recurrence algorithm

$$A_{N_1+1} = A_{N_1+2} = B_{N_1+1} = B_{N_1+2} = 0, \quad y = 2\left(2\left(\frac{x}{\lambda}\right)^2 - 1\right)$$

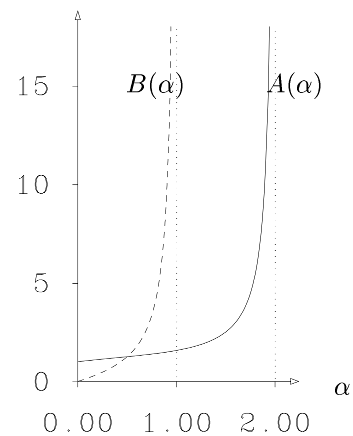


FIGURE 3. $B(\alpha)$ and $A(\alpha)$.

and

$$A_l = \tilde{c}_{2l+1} - y * A_{l+1} - A_{l+2},$$
$$B_l = \tilde{c}_{2l} - y * B_{l+1} - B_{l+2} \qquad (\text{if } \alpha < 1) \text{ for } l = N_1, \dots, 0.$$

So we finally have

$$\mathrm{Si}\,(x, \alpha) \doteq 2\, x^{1-\alpha}\, \frac{x}{\lambda}\, (A_0 + A_1)\,,$$
$$\mathrm{Ci}\,(x, \alpha) \doteq x^{1-\alpha}\, (B_0 - B_2) \quad \text{if } \alpha < 1.$$

**2.2 The range $\lambda \le \mathbf{x} < \infty$.** Let $x$ tend to $\infty$. In that case

$$\lim_{x\to\infty} \mathrm{Si}\,(x, \alpha) = A(\alpha) = \Gamma(2-\alpha)\, \frac{\sin\left((1-\alpha)\,(\pi/2)\right)}{1-\alpha}, \tag{12}$$

$$\lim_{x\to\infty} \mathrm{Ci}\,(x, \alpha) = B(\alpha) = \Gamma(1-\alpha)\, \sin\left(\alpha\, \frac{\pi}{2}\right). \tag{13}$$

The representations hold

$$\mathrm{Si}\,(x, \alpha) = A(\alpha) - \frac{1}{x^\alpha}\left[\, P(x)\, \sin x + Q(x)\, \cos x\,\right], \tag{14}$$

$$\mathrm{Ci}\,(x, \alpha) = B(\alpha) + \frac{1}{x^\alpha}\left[\, Q(x)\, \sin x - P(x)\, \cos x\,\right] \tag{15}$$

where

$$P(x) = \frac{2}{\Gamma(\alpha)} \int_0^\infty (\sqrt{t})^{\alpha-1} K_{\alpha-1}(2\sqrt{t}) \sin\left(\frac{t}{x}\right) dt, \tag{16}$$

$$Q(x) = \frac{2}{\Gamma(\alpha)} \int_0^\infty (\sqrt{t})^{\alpha-1} K_{\alpha-1}(2\sqrt{t}) \cos\left(\frac{t}{x}\right) dt \tag{17}$$

($K_\mu(\ldots)$ denote the modified Bessel functions).

In Equations (16) and (17) we again make use of the decomposition rules

$$\sin\left(\frac{t}{x}\right) = \sin\left(\frac{t}{\lambda}\,\frac{\lambda}{x}\right) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}\left(\frac{t}{\lambda}\right) T_{2k+1}\left(\frac{\lambda}{x}\right),$$

$$\cos\left(\frac{t}{x}\right) = \cos\left(\frac{t}{\lambda}\,\frac{\lambda}{x}\right) = 2\sum_{k=0}^{\infty}{}'(-1)^k J_{2k}\left(\frac{t}{\lambda}\right) T_{2k}\left(\frac{\lambda}{x}\right)$$

and from this we obtain the Čebyšev expansions

$$P(x) = \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)} \sum_{k=0}^{\infty}(-1)^k d_{2k+1}\, T_{2k+1}\left(\frac{\lambda}{x}\right), \tag{18}$$

$$Q(x) = \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)} \sum_{k=0}^{\infty}{}'(-1)^k d_{2k}\, T_{2k}\left(\frac{\lambda}{x}\right) \tag{19}$$

where the Čebyšev coefficients are given by

$$d_l = \int_0^\infty f_{\alpha-1}(2\sqrt{\lambda\, t}\,) J_l(t)\, dt, \quad f_\mu(\tau) = \tau^\mu K_\mu(\tau). \tag{20}$$

We may now state

**Lemma 2.** *The Čebyšev coefficients $d_l$ satisfy the recursion*

$$(l+\alpha)(d_l - d_{l+2}) - (l+4-\alpha)(d_{l+2} - d_{l+4}) = 2\,\lambda\,(d_{l+1} + d_{l+3}). \tag{21}$$

*Proof.* The relations for the modified Bessel functions yield

$$\frac{d}{d\tau} f_\mu(\tau) = -\tau\, f_{\mu-1}(\tau),$$
$$f_{\mu+1}(\tau) - \tau^2 f_{\mu-1}(\tau) = 2\,\mu\, f_\mu(\tau)$$

from which we obtain in turn

$$d_l - d_{l+2} = 2 \int_0^\infty f_{\alpha-1}(2\sqrt{\lambda\, t}\,) J'_{l+1}(t)\, dt,$$

$$d_l - d_{l+2} = \frac{2\,\lambda}{\alpha - 2}\,(d_{l+1} - h_{l+1})$$

where

$$h_{l+1} = \int_0^\infty (4\,\lambda\, t)\, f_{\alpha-3}(2\sqrt{\lambda\, t}\,) J_{l+1}(t)\, dt.$$

This gives

$$(d_l - d_{l+2}) - (d_{l+2} - d_{l+4}) = \frac{2\,\lambda}{l+2}\,(h_{l+1} + h_{l+3}).$$

The last equation results from integration by parts; this is the recursion formula (21). □

We have an upper bound for the Čebyšev coefficients $d_l$ in

**Lemma 3.** *If* $l \geq 2$ *and* $l\,\lambda \geq 2$ *then*

$$|\, d_l \,| \leq \frac{2.32}{(2\,\lambda)^{l+1}\,\Gamma(l+2)} + 1.77\,(2\,\sqrt{\lambda\, l})^{-1/2} e^{-0.45\, l} + \frac{1.1}{\lambda}\,(\sqrt{2\,\lambda\, l}\,)^{3/2} e^{-\sqrt{2\,\lambda\, l}} =: UB_2(l, \lambda). \tag{22}$$

*Proof.* We split the integral in Equation (20) into three parts

$$d_l = \left( \int_0^{1/\lambda} + \int_{1/\lambda}^{l/2} + \int_{l/2}^\infty \right) f_{\alpha-1}(2\sqrt{\lambda\, t}\,) J_l(t)\, dt, \quad f_\mu(\tau) = \tau^\mu\, K_\mu(\tau)$$

and calculate upper bounds for the absolute value of the three integrals.

We use the following relations (see [**1**])

$$0 < K_\mu(\tau) \leq K_1(\tau), \quad |\mu| \leq 1, \tau > 0, \tag{23}$$

$$0 < \tau\, K_1(\tau) \leq 1.16, \quad 0 < \tau \leq 2, \tag{24}$$

$$0 < \sqrt{\tau}\, e^\tau\, K_1(\tau) \leq 1.51, \quad 2 \leq \tau, \tag{25}$$

$$|J_l(t)| \leq \frac{(t/2)^l}{\Gamma(l+1)}, \quad 0 \leq t \leq l, \tag{26}$$

$$|\, J_l(t)\,| \leq \frac{1}{\sqrt{2}}, \quad l \geq 1,\ 0 \leq t, \tag{27}$$

$$
\begin{aligned}
|\,J_l(l\,\zeta)\,| &\le \left[\frac{\zeta\,\exp\left(\sqrt{1-\zeta^2}\right)}{1+\sqrt{1-\zeta^2}}\right]^l \\
&\le \left[\frac{e}{2}\,\zeta\,\left(1-\frac{\zeta^2}{4}\right)\right]^l, \quad 0 \le \zeta \le 1.
\end{aligned} \tag{28}
$$

a) With the relations (23), (24) and (25) we obtain for the first integral

$$
\left|\int_0^{1/\lambda} f_{\alpha-1}\left(2\sqrt{\lambda\,t}\right) J_l(t)\,dt\right| \le \frac{2.32}{(2\,\lambda)^{l+1}\,\Gamma(l+2)}.
$$

b) With the relations (23), (25), (28) and $w = 2\sqrt{\lambda\,l}$ we deduce for the second integral

$$
\begin{aligned}
&\left|\int_{1/\lambda}^{l/2} f_{\alpha-1}\left(2\sqrt{\lambda\,t}\right) J_l(t)\,dt\right| \\
&\quad\le \frac{1.51}{4\,\lambda}\,w^{\alpha+\frac{1}{2}} \int_0^{1/2} (\sqrt{\zeta}\,)^{\alpha-(3/2)} e^{-w\sqrt{\zeta}}\;|\,J_l(l\,\zeta)\,|\;d\zeta \\
&\quad\le \frac{1.51}{2\,\lambda}\,w^{\alpha+(1/2)}\left(\frac{e}{2}\right)^l \int_0^{1/\sqrt{2}} e^{-w\,\rho}\,\rho^{2l+\alpha-(1/2)}\left(1-\frac{\rho^4}{4}\right)^l d\rho \\
&\quad\le 1.77\,w^{\alpha-(5/2)}\,e^{-0.45\,l}.
\end{aligned}
$$

c) The relations (23), (25) and (27) give for the third integral

$$
\left|\int_{l/2}^{\infty} f_{\alpha-1}(2\sqrt{\lambda\,t}\,) J_l(t)\,dt\right| \le \frac{1.1}{\lambda}\,(\sqrt{2\,\lambda\,l}\,)^{\alpha-(1/2)}\,e^{-\sqrt{2\,\lambda\,l}}.
$$

In the worst case $\alpha = 2$ we get the relation (22). □

The relation (22) leads to $\lim_{l\to\infty} d_l = 0$.

The Čebyšev coefficients $d_l$ are the minimal solutions of the recurrence relation (21) and satisfy the relations

$$
\frac{2^{3-\alpha}\lambda^2}{\Gamma(\alpha)} \sum_{k=0}^{\infty}(2k+1)\,d_{2k+1} = \lim_{x\to\infty}(x\,P(x)) = \alpha, \tag{29}
$$

$$
\frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)} \sum_{k=0}^{\infty}{}' \,d_{2k} = \lim_{x\to\infty} Q(x) = 1. \tag{30}
$$

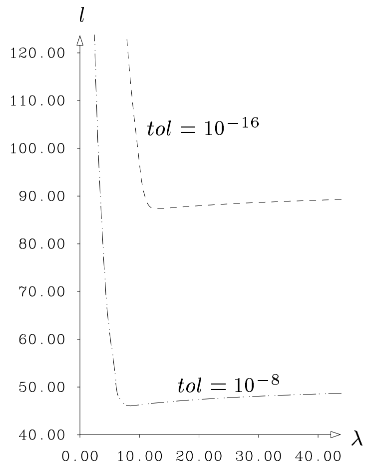


FIGURE 4. $4\,\lambda\, UB_2(l,\lambda) = \text{tol}$, $\text{tol} = 10^{-8}$, $10^{-16}$.

**2.2.1 Numerical solution in the range** $0 < \lambda < x$**.** As shown above, the functions $P(x)$, $Q(x)$, cf. Equations (18) and (19), can be approximated by the truncated Čebyšev series

$$(31)\qquad \begin{aligned} P(x) &\doteq \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)} \sum_{k=0}^{N_2} (-1)^k\, d_{2k+1}\, T_{2k+1}\left(\frac{\lambda}{x}\right), \\ Q(x) &\doteq \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)} \sum_{k=0}^{N_2}{}' (-1)^k\, d_{2k}\, T_{2k}\left(\frac{\lambda}{x}\right) \end{aligned}$$

where the upper index $N_2$ will depend on $\lambda$ and the prescribed accuracy.

We obtain numerical approximations for the Čebyšev coefficients $d_l = d_l(\alpha,\lambda)$ by solving Equation (21) by a backward recursion; let $\hat{d}_l$ denote the numerical approximations for the $d_l$ , then

$$\tilde{d}_{2N_2+5} = \tilde{d}_{2N_2+4} = \tilde{d}_{2N_2+3} = 0, \qquad \tilde{d}_{2N_2+2} = \varepsilon > 0$$

and

$$\tilde{d}_l = \tilde{d}_{l+2} + \big((l+4-\alpha)(\tilde{d}_{l+2} - \tilde{d}_{l+4}) + 2\,\lambda\,(\tilde{d}_{l+1} + \tilde{d}_{l+3})\big)\big/(l+\alpha)$$
$$\text{for } l = 2\,N_2+1, \ldots, 0.$$

Normalization according to Equation (30) yields

$$\sigma = \sum_{k=0}^{N_2+1}{}' \tilde{d}_{2k}, \qquad l = 0, \dots, 2N_2+1 \,:\, \hat{d}_l = \frac{\Gamma(\alpha)}{2^{3-\alpha}\lambda\,\sigma}\,\tilde{d}_l.$$

Again the truncated Čebyšev series Equation (31) are evaluated by Clenshaw's recurrence algorithm

$$A_{N_2+1} = A_{N_2+2} = B_{N_2+1} = B_{N_2+2} = 0, \qquad y = 2\left(2\left(\frac{\lambda}{x}\right)^2 - 1\right)$$

and

$$A_l = \hat{d}_{2l+1} - y * A_{l+1} - A_{l+2}, \quad B_l = \hat{d}_{2l} - y * B_{l+1} - B_{l+2} \quad \text{for } l = N_2, \dots, 0.$$

The approximations for the auxiliary functions $P(x)$, $Q(x)$

$$P(x) \doteq \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)}\,\frac{\lambda}{x}\left(A_0 + A_1\right),$$
$$Q(x) \doteq \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)}\,\frac{1}{2}\left(B_0 - B_2\right)$$

and $\mathrm{Si}\,(x,\alpha)$, $\mathrm{Ci}\,(x,\alpha)$ are computed from Equations (14) and (15).

Incidentally, by a change of variable, i.e.,

$$d_l \longrightarrow \delta_l := \frac{2^{3-\alpha}\lambda}{\Gamma(\alpha)}\,d_l,$$

we may save on the number of required multiplications, i.e., computing time.

**2.3 Appropriate choice of $\lambda$ and the truncation indices $N_1$, $N_2$.** There are two objectives partly contradicting each other. The absolute truncation error in the interval $0 < x \le \lambda$ is given by the $(2N_1+3)$th Čebyšev coefficient $c_{2N_1+3}$ in the truncated Čebyšev expansions of $x^{\alpha-1}\,\mathrm{Si}\,(x,\alpha)$ and $x^{\alpha-1}\,\mathrm{Ci}\,(x,\alpha)$; and in the interval $\lambda < x < \infty$ by the $(2N_2+3)$th Čebyšev coefficient $\delta_{2N_2+3}$ in the truncated Čebyšev expansions of $P(x)$ and $Q(x)$. The errors should be less than a given

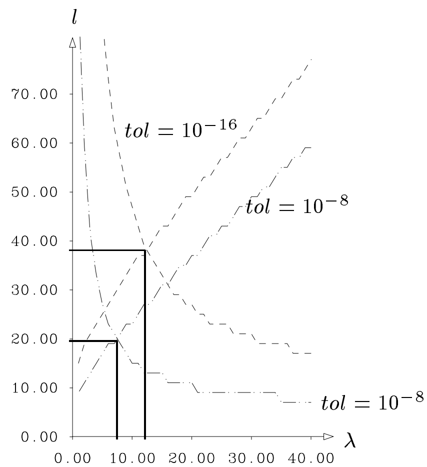


FIGURE 5. Curves $[l](\lambda)$.

tolerance tol. Computing time should be the same in the intervals $0 < x \leq \lambda$ and $\lambda < x < \infty$ and this requires $N_1 \approx N_2$.

In the worst case $\alpha = 2$, for fixed tolerance tol the number $N_1$ increases with increasing $\lambda$ and $N_2$ decreases with increasing $\lambda$.

The upper bound $UB_2$ is too conservative for estimating optimal values, cf., Table 4 and Table 5.

Therefore, for a fixed tolerance $tol$ and the worst case $\alpha = 2$, we have numerically computed the curve $[l]_1(\lambda)$ with

$$\left|c_{[l]_1(\lambda)}\right| \leq \text{tol} < \left|c_{[l]_1(\lambda)+2}\right|$$

and the curve $[l]_2(\lambda)$ with

$$\left|\delta_{[l]_2(\lambda)}\right| \leq \text{tol} < \left|\delta_{[l]_2(\lambda)+2}\right|.$$

The optimal values for $\lambda, N_1, N_2$ are given by the intersection of the two curves, see Table 5.

We obtain

| | |
|---|---|
| $\text{tol} = 10^{-8}$ | $\lambda = 7.25$ and $N_1 = N_2 = 9$ |
| $\text{tol} = 10^{-16}$ | $\lambda = 12.5$ and $N_1 = N_2 = 19$ |

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