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A NEW A_n EXTENSION OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION WITH APPLICATIONS TO MULTILATERAL A_n SERIES

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ABSTRACT. In this article we derive some identities for multilateral basic hypergeometric series associated to the root system A_n. First, we apply Ismail's [**15**] argument to an A_n q-binomial theorem of Milne [**25**, Theorem 5.42] and derive a new A_n generalization of Ramanujan's ${}_1\psi_1$ summation theorem. From this new A_n ${}_1\psi_1$ summation and from an A_n ${}_1\psi_1$ summation of Gustafson [**9**], we deduce two lemmas for deriving simple A_n generalizations of bilateral basic hypergeometric series identities. These lemmas are closely related to the Macdonald identities for A_n. As samples for possible applications of these lemmas, we provide several A_n extensions of Bailey's ${}_2\psi_2$ transformations, and several A_n extensions of a particular ${}_2\psi_2$ summation.

1. Introduction. The theory of basic hypergeometric series (cf. [**8**]), consists of many known summation and transformation formulas. The most important of these is probably the q-binomial theorem, a summation first discovered by Cauchy [**6**]. Surprisingly, the q-binomial theorem admits a bilateral generalization, the ${}_1\psi_1$ summation theorem, first discovered by Ramanujan [**11**]. Other important identities for basic hypergeometric series include the q-Gauß summation and Heine's ${}_2\phi_1$ transformations. These and many other basic hypergeometric series identities conspicuously appear in combinatorics and in related areas, such as number theory, statistics, physics and representation theory of Lie algebras, see Andrews [**1**].

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Multiple basic hypergeometric series associated to the root system A_n, or equivalently associated to the unitary group $U(n+1)$, have been investigated by various authors. Many different types of such series exist in the literature. The multi-variable series we consider in this article have their origin in the work of the three mathematical physicists Biedenharn, Holman and Louck, see [**12**] and [**13**]. Their work was done in the context of the quantum theory of angular momentum, using methods relying on the representation theory of $U(n)$. In the sequel, substantial developments have taken place. Extensive investigations in the theory of multiple basic hypergeometric series associated to the root system A_n have been carried out by Gustafson, Milne and their co-workers. As a result many of the classical formulas for basic hypergeometric series (cf. [**8**]), have already been generalized to the setting of the A_n series. For some selected results on multiple basic hypergeometric series associated to A_n, see the references [**5, 7, 9, 10, 18**–**22**, **24-30**].

There are different methods for obtaining identities for A_n basic hypergeometric series. Partial fraction decompositions and q-difference equations are often involved in initially deriving such identities (e.g., [**5, 10**] and [**18**]). Further, where summations for multi-dimensional basic hypergeometric series are already known, multi-dimensional matrix inversions can often be utilized for obtaining new summation theorems for multi-dimensional basic hypergeometric series, see [**5, 25, 26, 28**–**30**]. But there is also another, simpler, way of obtaining identities for A_n basic hypergeometric series. By utilizing Lemma 7.3 of Milne [**25**] see Lemma 4.1 in this article, and by using identities of the classical one-dimensional theory, simple identities for A_n series can be derived.

In this article, we find two multilateral generalizations of [**25**, Lemma 7.3] (see Lemmas 4.3 and 4.9). These lemmas are closely related to the Macdonald [**17**] identities for the affine root system A_n. By using our lemmas combined with bilateral one-dimensional series identities, we are able to derive simple multi-lateral identities for A_n series. We give some particular applications of this method. The A_n ${}_2\psi_2$ transformations and summations given in this article are just samples of the possible applications. It must be said that the identities obtained by this method concern A_n series of "simpler type" and are apparently not as deep as many of the A_n identities in the above-mentioned references. Nevertheless, in spite of, or maybe even because of the "simplicity" of these A_n series, our formulas could be useful in future applications.

Our article is organized as follows: In Section 2 we introduce some notation and give some background information. In Section 3 we apply Ismail's [**15**] analytic continuation argument to an A_n q-binomial theorem of Milne [**25**, Theorem 5.42] to derive a new A_n extension of Ramanujan's [**11**] ${}_1\psi_1$ summation theorem. In [**19**] a similar argument was used to find the first $U(n)$ generalization of the ${}_1\psi_1$ summation. More recently, motivated by [**23**], Kaneko [**16**] utilized this type of argument to derive a ${}_1\psi_1$ summation theorem for multiple basic hypergeometric series of a Macdonald polynomial argument. In Section 4 we deduce from our new A_n ${}_1\psi_1$ summation and from Gustafson's [**9**, Theorem 1.17] ${}_1\psi_1$ summation two lemmas for deriving simple multilateral series identities in A_n. We discuss the connection of these lemmas with the Macdonald identities for A_n, partly following the similar analysis of [**19**]. Finally, in Section 5, we apply these lemmas to classical (one-dimensional) formulas for ${}_2\psi_2$ series. As a result, we deduce several (different) A_n extensions of Bailey's [**3**] ${}_2\psi_2$ transformations, and moreover, deduce several (different) A_n extensions of a particular summation for ${}_2\psi_2$ series.

2. Background and notation. Let us first recall some standard basic hypergeometric notation (cf. [**8**]). Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a)_\infty \equiv (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j)$$

and

$$(a)_k \equiv (a;q)_k := \frac{(a)_\infty}{(aq^k)_\infty}.$$

For brevity we employ the usual notation

$$(a_1,\dots,a_m)_k \equiv (a_1)_k \dots (a_m)_k,$$

where k is an integer or infinity. Further, we utilize the notations

$$(2.1) \qquad {}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z\right]$$

$$:= \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(q, b_1, \dots, b_s)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k,$$

and

$$(2.2) \qquad {}_r\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right]$$

$$:= \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(b_1, b_2, \dots, b_s)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{s-r} z^k,$$

for *basic hypergeometric* ${}_r\phi_s$ *series*, and *bilateral basic hypergeometric* ${}_r\psi_s$ *series*, respectively. See [**8**, pages 25 and 125] for the criteria of when these series terminate or, if not, when they converge. In this article, we make use of some of the elementary identities for q-shifted factorials listed in [**8**, Appendix I].

Next, we note the convention for naming the multiple series in this article as A_n basic hypergeometric series. We consider multiple series of the form

$$\sum_{k_1, \dots, k_n = -\infty}^{\infty} S(\mathbf{k}),$$

where $\mathbf{k} = (k_1, \dots, k_n)$, which reduce to classical basic hypergeometric series when $n = 1$. Such a multiple series is called an A_n basic hypergeometric series if the summand $S(\mathbf{k})$ contains the factor

$$(2.3) \qquad \prod_{1 \le i < j \le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right).$$

A typical example is the lefthand side of (3.3). A reason for naming these series as A_n series is that (2.3) is closely associated with the product side of the Weyl denominator formula for the root system A_n (see [**4**] and [**31**]).

For multi-dimensional series, we also employ the notation $|\mathbf{k}|$ for $(k_1 + \cdots + k_n)$ where $\mathbf{k} = (k_1, \dots, k_n)$. The convergence of multiple series can be checked by application of the multiple power series ratio test [**14**]. For explicit examples of how to use the multiple power series ratio test, see [**25**, Section 5].

3. An A_n extension of Ramanujan's ${}_1\psi_1$ summation. One of the most important summation theorems for basic hypergeometric series is the classical q-binomial theorem (cf. [**8**]),

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix} ; q, z\right] = \frac{(az)_\infty}{(z)_\infty}, \tag{3.1}$$

where $|z| < 1$.

A bilateral extension of (3.1) is Ramanujan's [**11**] ${}_1\psi_1$ summation theorem (cf. [**8**]),

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix} ; q, z\right] = \frac{(q, b/a, az, q/az)_\infty}{(b, q/a, z, b/az)_\infty}, \tag{3.2}$$

where $|b/a| < |z| < 1$. Clearly the $b = q$ case of (3.2) is (3.1).

Theorem 5.42 of [**25**] is one of the many multi-variable generalizations of (3.1). It can be stated as follows:

Theorem 3.1 (An A_n q-binomial theorem). *Let $a, x_1, \ldots, x_n$ and z be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (3.3) *vanishes. Then*

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} (a)_{|\mathbf{k}|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) = \frac{(az)_\infty}{(z)_\infty}, \tag{3.3}$$

provided $|z| < |q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^{n} x_i|$ for $j = 1, \ldots, n$.

We now apply Ismail's [**15**] argument and extend Theorem 3.1 to

Theorem 3.2 (An A_n ${}_1\psi_1$ summation). *Let $a, b_1, \ldots, b_n, x_1, \ldots, x_n$ and z be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (3.4) *vanishes. Then*

$$\sum_{k_1,\ldots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} b_j \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} (a)_{|k|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^{n} \binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) = \frac{(az, q/az, b_1 \ldots b_n q^{1-n}/a)_\infty}{(z, b_1 \ldots b_n q^{1-n}/az, q/a)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q)_\infty}{((x_i/x_j)b_j)_\infty}, \tag{3.4}$$

provided $|b_1 \ldots b_n q^{1-n}/a| < |z| < |q^{(n-1)/2} x_j^{-n} \prod_{i=1}^{n} x_i|$ *for* $j = 1, \ldots, n$.

Proof. We apply Ismail's argument successively to the parameters $b_1, \ldots, b_n$ using (3.3). The multiple series identity in (3.4) is analytic in each of the parameters $b_1, \ldots, b_n$ in a domain around the origin. Now the identity is true for $b_1 = q^{1+m_1}$, $b_2 = q^{1+m_2}, \ldots$ and $b_n = q^{1+m_n}$ by the A_n q-binomial theorem in Theorem 3.1 (see below for the details). This holds for all $m_1, \ldots, m_n \geq 0$. Since $\lim_{m_1\to\infty} q^{1+m_1} = 0$ is an interior point in the domain of analyticity of b_1, by analytic continuation we obtain an identity for b_1. By iterating this argument for $b_2, \ldots, b_n$, we establish (3.4) for general $b_1, \ldots, b_n$.

The details are displayed as follows: Setting $b_i = q^{1+m_i}$, for $i = 1, \ldots, n$, the left side of (3.4) becomes

$$\sum_{\substack{-m_i \le k_i \le \infty \\ i=1,\ldots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q^{1+m_j} \right)_{k_i}^{-1} \times \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} (a)_{|\mathbf{k}|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^{n} \binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg). \tag{3.5}$$

We shift the summation indices in (3.5) by $k_i \mapsto k_i - m_i$ for $i = 1, \ldots, n$

and obtain
(3.6)

$$
\begin{aligned}
&q^{-\binom{|\mathbf{m}|+1}{2}+n\sum_{i=1}^n\binom{m_i+1}{2}}(-1)^{(n-1)|\mathbf{m}|}(a)_{-|\mathbf{m}|}z^{-|\mathbf{m}|}\prod_{i=1}^n x_i^{|\mathbf{m}|-nm_i}\\
&\quad\times\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j}\right)^{-1}_{-m_i}\sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{1\le i<j\le n}\left(\frac{x_iq^{-m_i+k_i}-x_jq^{-m_j+k_j}}{x_i-x_j}\right)\\
&\quad\times\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j-m_i}\right)^{-1}_{k_i}(aq^{-|\mathbf{m}|})_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\\
&\qquad\qquad\qquad\qquad\times\prod_{i=1}^n(x_iq^{-m_i})^{nk_i-|\mathbf{k}|}\Bigg)\\
&=q^{n\sum_{i=1}^n\binom{m_i+1}{2}}(-1)^{n|\mathbf{m}|}(az)^{-|\mathbf{m}|}(q/a)^{-1}_{|\mathbf{m}|}\\
&\quad\times\prod_{i=1}^n x_i^{|\mathbf{m}|-nm_i}\prod_{i,j=1}^n\frac{((x_i/x_j)q)_{m_j}}{((x_i/x_j)q)_{m_j-m_i}}\prod_{1\le i<j\le n}\left(\frac{x_iq^{-m_i}-x_jq^{-m_j}}{x_i-x_j}\right)\\
&\quad\times\sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{1\le i<j\le n}\left(\frac{x_iq^{-m_i+k_i}-x_jq^{-m_j+k_j}}{x_iq^{-m_i}-x_jq^{-m_j}}\right)\\
&\qquad\qquad\times\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j-m_i}\right)^{-1}_{k_i}\\
&\quad\times(aq^{-|\mathbf{m}|})_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\\
&\qquad\qquad\times\prod_{i=1}^n(x_iq^{-m_i})^{nk_i-|\mathbf{k}|}\Bigg).
\end{aligned}
$$

Next, we apply the $y_i\mapsto -m_i$, $i=1,\dots,n$, case of [**25**], specifically
(3.7)

$$
\begin{aligned}
\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q\right)_{m_j-m_i}&=(-1)^{(n-1)|\mathbf{m}|}q^{-\binom{|\mathbf{m}|+1}{2}+n\sum_{i=1}^n\binom{m_i+1}{2}}\\
&\qquad\times\prod_{i=1}^n x_i^{|\mathbf{m}|-nm_i}\prod_{1\le i<j\le n}\left(\frac{x_iq^{-m_i}-x_jq^{-m_j}}{x_i-x_j}\right),
\end{aligned}
$$

and the $a\mapsto aq^{-|\mathbf{m}|}$, $x_i\mapsto x_iq^{-m_i}$, $i=1,\dots,n$, case of the multidimensional summation theorem in (3.3) to simplify the expression obtained in (3.6) to

$$q^{\binom{|\mathbf{m}|+1}{2}}(-az)^{-|\mathbf{m}|}\frac{(azq^{-|\mathbf{m}|})_\infty}{(q/a)_{|\mathbf{m}|}(z)_\infty}\prod_{i,j=1}^{n}\left(\frac{x_i}{x_j}q\right)_{m_j}.$$

Now this can easily be further transformed into

$$\frac{(q^{1+|\mathbf{m}|}/a, az, q/az)_\infty}{(q/a, z, q^{1+|\mathbf{m}|}/az)_\infty}\prod_{i,j=1}^{n}\frac{((x_i/x_j)q)_\infty}{((x_i/x_j)q^{1+m_j})_\infty},$$

which is exactly the $b_i = q^{1+m_i}$, $i = 1, \dots, n$, case of the right side of (3.4). □

If we set $z \mapsto -z/a$ and $b_i = 0$, $i = 1, \dots, n$ in (3.4) and then let $a \to \infty$, we obtain an A_n generalization of Jacobi's triple product identity, equivalent to Theorem 3.7 of [**19**].

4. Two lemmas for deriving multilateral A_n series identities. As an immediate consequence of a fundamental theorem for A_n series [**18**], the first author [**25**] of this article derived the following lemma, which is

Lemma 4.1 (Milne). *Let $a_1, \dots, a_n$ and $x_1, \dots, x_n$ be indeterminate, let N be a nonnegative integer, let $n \geq 1$, and suppose that none of the denominators in* (4.1) *vanishes. Then if $f(m)$ is an arbitrary function of nonnegative integers m, we have*

$$\sum_{m=0}^{N}\frac{(a_1a_2\dots a_n)_m}{(q)_m}f(m) = \sum_{\substack{k_1,\dots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq N}}\prod_{1\leq i<j\leq n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \times \prod_{i,j=1}^{n}\frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)q)_{k_i}}\cdot f(|\mathbf{k}|). \tag{4.1}$$

With Lemma 4.1 and one-dimensional basic hypergeometric series identities, (simple) identities for A_n series can be derived. Some examples are given in [**25**].

In this section we provide two new lemmas see Lemmas 4.3 and 4.9, which similarly can be used for deriving simple A_n generalizations of *bilateral* basic hypergeometric series identities. We make use of our A_n extension of Ramanujan's ${}_1\psi_1$ summation in Theorem 3.2 and of an A_n ${}_1\psi_1$ summation by Gustafson [**9**] see Theorem 4.5.

Since for $|b_1\cdots b_n q^{1-n}/a| < |z| < 1$,

$$ {}_1\psi_1\left[\begin{matrix} a \\ b_1\dots b_n q^{1-n} \end{matrix};q,z\right] = \frac{(q, b_1\dots b_n q^{1-n}/a, az, q/az)_\infty}{(b_1\dots b_n q^{1-n}, q/a, z, b_1\dots b_n q^{1-n}/az)_\infty}, $$

by Ramanujan's ${}_1\psi_1$ summation (3.2) we immediately see from (3.4) that

$$ \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} b_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} $$

(4.2)

$$ \times (a)_{|\mathbf{k}|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) $$

$$ = \frac{(b_1\dots b_n q^{1-n})_\infty}{(q)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q)_\infty}{((x_i/x_j)b_j)_\infty} \sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b_1\dots b_n q^{1-n})_k} z^k $$

(provided $|z| < 1$ and $|b_1\dots b_n q^{1-n}/a| < z| < |q^{n-1/2} x_j^{-n} \prod_{i=1}^n x_i|$ for $j = 1,\dots,n$).

In (4.2) we equate coefficients of $(a)_m z^m$ and extract

Proposition 4.2. *Let $b_1,\dots,b_n$ and $x_1,\dots,x_n$ be indeterminate, let m be an integer, let $n \ge 1$, and suppose that none of the denominators in* (4.3) *vanishes. Then*

(4.3)

$$ \sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty \\ |\mathbf{k}|=m}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} b_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} $$

$$ \times (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} \Bigg) $$

$$ = \frac{(b_1\dots b_n q^{1-n})_\infty}{(q)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q)_\infty}{((x_i/x_j)b_j)_\infty} \frac{1}{(b_1\dots b_n q^{1-n})_m}. $$

We state Proposition 4.2 although it is just a special case of Proposition 4.6. We utilize the $m=0$ case of Proposition 4.2 in the proof of Theorem 5.7.

Now, if we multiply both sides of (4.3) by

$$\frac{(q)_\infty}{(b_1\dots b_n q^{1-n})_\infty}\prod_{i,j=1}^n \frac{((x_i/x_j)b_j)_\infty}{((x_i/x_j)q)_\infty}\cdot f(m),$$

for suitable $f(m)$ and sum over all integers m, we obtain

Lemma 4.3. *Let $b_1,\dots,b_n,x_1,\dots,x_n$ be indeterminate, let $n\geq 1$, and suppose that none of the denominators in* (4.4) *vanishes. Then if $f(m)$ is an arbitrary function of integers m, we have*

(4.4)

$$\sum_{m=-\infty}^{\infty}\frac{f(m)}{(b_1\dots b_n q^{1-n})_m} = \frac{(q)_\infty}{(b_1\dots b_n q^{1-n})}\prod_{i,j=1}^n \frac{(\frac{x_i}{x_j}b_j)_\infty}{(\frac{x_i}{x_j}q)_\infty}$$
$$\times\sum_{k_1,\dots,k_n=-\infty}^{\infty}\Bigg(\prod_{1\leq i<j\leq n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}b_j\right)_{k_i}^{-1}$$
$$\times\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}\cdot f(|\mathbf{k}|)\Bigg),$$

provided the series converge.

Thus, with Lemma 4.3, we can use one-dimensional bilateral series identities to obtain identities for multilateral A_n series.

The special case of Lemma 4.3, where $b_i=q$, for $i=1,\dots,n$ is worth noting:

Corollary 4.4. *Let $x_1,\dots,x_n$ be indeterminate, let $n\geq 1$ and suppose that none of the denominators in* (4.5) *vanishes. Then if $f(m)$*

is an arbitrary function of nonnegative integers m, we have

$$\sum_{m=0}^{\infty}\frac{f(m)}{(q)_m}=\sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\times\prod_{i,j=1}^{n}\left(\frac{x_i}{x_j}q\right)_{k_i}^{-1}\prod_{i=1}^{n}x_i^{nk_i-|\mathbf{k}|}\times(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^{n}\binom{k_i}{2}}\cdot f(|\mathbf{k}|)\Bigg),\tag{4.5}$$

provided the series converge.

Corollary 4.4 can also be obtained by specializing Lemma 4.1. Namely, by setting

$$f(m)\longmapsto(-1)^mq^{-\binom{m}{2}}(a_1a_2\dots a_n)^{-m}f(m)$$

in Lemma 4.1, and then letting $N\to\infty$ and $a_i\to\infty$, for $i=1,\dots,n$, we also obtain Corollary 4.4.

Next, we put our attention towards the derivation of another lemma for deriving multilateral series identities. For this we utilize Gustafson's [**9**] multivariable generalization of Ramanujan's ${}_1\psi_1$ summation (3.2).

Theorem 4.5 ((Gustafson) An A_n ${}_1\psi_1$ summation). *Let $a_1,\dots,a_n$, $b_1,\dots,b_n,z$ and $x_1,\dots,x_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in* (4.6) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\times\prod_{i,j=1}^{n}\frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j)_{k_i}}z^{|\mathbf{k}|}=\frac{(a_1\cdots a_nz,q/a_1\dots a_nz)_\infty}{(z,b_1\dots b_nq^{1-n}/a_1\dots a_nz)_\infty}\prod_{i,j=1}^{n}\frac{((x_i/x_j)q,x_ib_j/x_ja_i)_\infty}{((x_i/x_j)b_j,x_iq/x_ja_i)_\infty},\tag{4.6}$$

provided $|b_1\dots b_nq^{1-n}/a_1\dots a_n|<z|<1$.

Since, for $|b_1\dots b_n q^{1-n}/a_1\dots a_n| < |z| < 1$,

$$
{}_1\psi_1\left[\begin{matrix} a_1\dots a_n \\ b_1\dots b_n q^{1-n} \end{matrix} ; q, z\right]
= \frac{(q, b_1\dots b_n q^{1-n}/a_1\dots a_n, a_1\dots a_n z, q/a_1\dots a_n z)_\infty}{(b_1\dots b_n q^{1-n}, q/a_1\dots a_n, z, b_1\dots b_n q^{1-n}/a_1\dots a_n z)_\infty},
$$

by Ramanujan's ${}_1\psi_1$ summation (3.2) we immediately see from (4.6) that

(4.7)
$$
\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n}\left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j)_{k_i}} z^{|\mathbf{k}|}
= \frac{(b_1\dots b_n q^{1-n}, q/a_1\dots a_n)_\infty}{(q, b_1\dots b_n q^{1-n}/a_1\dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, x_i b_j/x_j a_i)_\infty}{((x_i/x_j)b_j, x_i q/x_j a_i)_\infty}
\times \sum_{k=-\infty}^{\infty} \frac{(a_1\dots a_n)_k}{(b_1\dots b_n q^{1-n})_k} z^k
$$

(provided $|b_1\dots b_n q^{1-n}/a_1\dots a_n| < |z| < 1$).

In (4.7) we equate coefficients of z^m and extract

Proposition 4.6. *Let $a_1,\dots,a_n$, $b_1,\dots,b_n$, and $x_1,\dots,x_n$ be indeterminate, let m be an integer, let $n\ge 1$, and suppose that none of the denominators in* (4.8) *vanishes. Then*

$$
\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty \\ |\mathbf{k}|=m}} \prod_{1\le i<j\le n}\left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j)_{k_i}}
= \frac{(b_1\dots b_n q^{1-n}, q/a_1\dots a_n)_\infty}{(q, b_1\dots b_n q^{1-n}/a_1\dots a_n)_\infty}
\times \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, x_i b_j/x_j a_i)_\infty}{((x_i/x_j)b_j, x_i q/x_j a_i)_\infty} \cdot \frac{(a_1\dots a_n)_m}{(b_1\dots b_n q^{1-n})_m}, \tag{4.8}
$$

provided $|b_1\dots b_n q^{1-n}/a_1\dots a_n| < 1$.

The $b_i = b$, $i = 1, \dots, n$, case of Proposition 4.6 was established in [**19**, Theorem 1.21].

A specialization of Proposition 4.6 gives Proposition 4.2. Namely, if we divide both sides of (4.8) by $(a_1 \dots a_n)_m$ and then let $a_i \to \infty$, $i = 1, \dots, n$, we obtain (4.3).

The $m = 0$ case of Proposition 4.6 was established by Gustafson in [**9**, Theorem 1.15]:

Theorem 4.7 ((Gustafson) An A_{n-1} ${}_6\psi_6$ summation). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (4.9) *vanishes. Then*

$$\sum_{\substack{-\infty \leq k_1, \dots, k_n \leq \infty \\ |\mathbf{k}|=0}} \prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j)_{k_i}} = \frac{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, x_i b_j / x_j a_i)_\infty}{((x_i/x_j)b_j, x_i q / x_j a_i)_\infty}, \tag{4.9}$$

provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < 1$.

The $n = 2$ case of Theorem 4.7 is equivalent to Bailey's [**2**] very well-poised ${}_6\psi_6$ summation (cf. [**8**]).

We utilize Theorem 4.7 in the proof of Theorem 5.9.

From Theorem 4.7, we immediately deduce a ${}_1\psi_1/{}_6\psi_6$ generalization of the Macdonald identities for A_n, generalizing Theorem 1.24 of [**19**]. The analysis is similar to that in [**19**] where the $b_i = b$, $i = 1, \dots, n$, case of Theorem 4.7 was utilized to obtain [**19**, Theorem 1.24]. The following result appears implicitly in [**10**, Section 7].

Theorem 4.8 ((Gustafson) A ${}_1\psi_1$ generalization of the Macdonald identities for A_n). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (4.10) *vanishes. Then*

(4.10)

$$\sum_{\sigma\in\mathcal{S}_n}\varepsilon(\sigma)\prod_{i=1}^n x_{\sigma(i)}^{i-\sigma(i)} \sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |\mathbf{k}|=0}} q^{\sum_{i=1}^r (i-1)k_{\sigma(i)}} \prod_{i,j=1}^n \frac{((x_i/x_j)a_j)_k}{(x_i/x_j)b_j)_{k_i}}$$
$$= \frac{(b_1\dots b_n q^{1-n}, q/a_1\dots a_n)_\infty}{(q, b_1\dots b_n q^{1-n}/a_1\dots a_n)_\infty} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)$$
$$\times \prod_{i,j=1}^n \frac{((x_i/x_j)q, (x_i b_j/x_j a_i))_\infty}{((x_i/x_j)b_j, (x_i q/x_j a_i))_\infty},$$

provided $|b_1\dots b_n q^{1-n}/a_1\dots a_n| < 1$, *where* $\mathcal{S}_n$ *is the symmetric group of order* n *and* $\varepsilon(\sigma)$ *is the sign of the permutation* σ.

Replacing a_i and b_i by $-1/c$ and 0, respectively, for $i = 1,\dots,n$, in Theorem 4.8, simplifying and then letting $c\to 0$ yields Equation (4.3) of [**18**] which is equivalent to the Macdonald identities for A_n (see [**18**, Section 4]). Thus, Theorem 4.8 may be viewed as a generalization of the Macdonald identities for A_n with the extra parameters $a_1,\dots,a_n$ and $b_1,\dots,b_n$.

For future reference, we write down the $b_i = a_i q$, $i = 1,\dots,n$, case of Theorems 4.8 and 4.6. Note that this case is valid since the convergence condition $|b_1\dots b_n q^{1-n}/a_1\dots a_n| < 1$ becomes $|q| < 1$. After a routine simplification, (4.10) becomes

(4.11)

$$\sum_{\sigma\in\mathcal{S}_n}\varepsilon(\sigma)\prod_{i=1}^n x_{\sigma(i)}^{i-\sigma(i)}$$
$$\times \sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |\mathbf{k}|=0}} q^{\sum_{i=1}^r (i-1)k_{\sigma(i)}} \prod_{i,j=1}^n \frac{(1-(x_i/x_j)a_j)}{(1-(x_i/x_j)a_j q^{k_i})}$$
$$= \frac{(a_1\dots a_n q, q/a_1\dots a_n)_\infty}{(q,q)_\infty} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)$$
$$\times \prod_{i,j=1}^n \frac{((x_i/x_j)q, (x_i a_j/x_j a_i)q)_\infty}{((x_i/x_j)a_j q, (x_i q/x_j a_i))_\infty}.$$

Similarly, (4.8) becomes

(4.12)
$$\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |\mathbf{k}|=m}} \prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{n}\frac{(1-(x_i/x_j)a_j)}{(1-(x_i/x_j)a_jq^{k_i})}$$
$$=\frac{(a_1\dots a_n,q/a_1\dots a_n)_\infty}{(1-a_1\dots a_nq^m)(q,q)_\infty}\prod_{i,j=1}^{n}\frac{((x_i/x_j)q,(x_ia_j/x_ja_i)q)_\infty}{((x_i/x_j)a_jq,(x_iq/x_ja_i))_\infty}.$$

Equations (4.11) and (4.12) extend (3.16) and (3.17) of [**19**], respectively, to which they reduce when $a_i=a$ for $i=1,\dots,n$.

Now let us return to our objective of finding a multilateral generalization of Lemma 4.1. If we multiply both sides of (4.8) by

$$\frac{(q,b_1\dots b_nq^{1-n}/a_1\dots a_n)_\infty}{(b_1\dots b_nq^{1-n},q/a_1\dots a_n)_\infty}\prod_{i,j=1}^{n}\frac{((x_i/x_j)b_j,x_iq/x_ja_i)_\infty}{((x_i/x_j)q,x_ib_j/x_ja_i)_\infty}\cdot g(m),$$

for suitable $g(m)$ and sum over all integers m, we obtain

Lemma 4.9. *Let $a_1,\dots,a_n,b_1,\dots,b_n,x_1,\dots,x_n$ be indeterminate, let $n\ge1$ and suppose that none of the denominators in* (4.13) *vanishes. Then, if $g(m)$ is an arbitrary function of integers m, we have*

(4.13)
$$\sum_{m=-\infty}^{\infty}\frac{(a_1\dots a_n)_m}{(b_1\dots b_nq^{1-n})_m}g(m)=\frac{(q,b_1\dots b_nq^{1-n}/a_1\dots a_n)_\infty}{(b_1\dots b_nq^{1-n},q/a_1\dots a_n)_\infty}$$
$$\times\prod_{i,j=1}^{n}\frac{((x_i/x_j)b_j,x_iq/x_ja_i)_\infty}{((x_i/x_j)q,x_ib_j/x_ja_i)_\infty}$$
$$\times\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)$$
$$\times\prod_{i,j=1}^{n}\frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j)_{k_i}}\cdot g(|\mathbf{k}|),$$

provided the series converge.

Hence, besides Lemma 4.3, we can also use Lemma 4.9 with one-dimensional bilateral series identities to obtain identities for multilateral A_n series. Lemma 4.9 generalizes the $N \to \infty$ case of Lemma 4.1 by additional parameters $b_1, \dots, b_n$, since the special case $b_i = q$ for $i = 1, \dots, n$ of Lemma 4.9 boils down to the $N \to \infty$ case of Lemma 4.1.

5. Applications: Some ${}_2\psi_2$ formulas in A_n. In this section we illustrate the usefulness of the lemmas of the preceding section and provide some multi-dimensional extensions of Bailey's [**3**] ${}_2\psi_2$ transformations, associated to the root system A_n. Further, as interesting special cases of these ${}_2\psi_2$ transformations in A_n, we provide some ${}_2\psi_2$ summation sin A_n.

Using Ramanujan's ${}_1\psi_1$ summation (3.2) and elementary manipulations of series, Bailey [**3**] derived the transformation

$$(5.1)\qquad {}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] = \frac{(az, \frac{d}{a}, \frac{c}{b}, \frac{dq}{abz})_\infty}{(z, d, \frac{q}{b}, \frac{cd}{abz})_\infty} {}_2\psi_2\left[\begin{matrix} a, \frac{abz}{d} \\ az, c \end{matrix}; q, \frac{d}{a}\right],$$

were $\max(|z|, |cd/abz|, |d/a|, |c/b|) < 1$.

Bailey's ${}_2\psi_2$ transformation can be iterated. The result is [**3**, (2.4)]

$$(5.2)\qquad {}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] = \frac{(az, bz, \frac{cq}{abz}, \frac{dq}{abz})_\infty}{(\frac{q}{a}, \frac{q}{b}, c, d)_\infty} {}_2\psi_2\left[\begin{matrix} \frac{abz}{c}, \frac{abz}{d} \\ az, bz \end{matrix}; q, \frac{cd}{abz}\right],$$

where $\max(|z|, |cd/abz|) < 1$.

We can specialize (5.1), or (5.2), to obtain a summation theorem for a particular ${}_2\psi_2$ series. If $d = bq$ and $z = q/a$ in (5.1), then the series on the right side reduces just to one term, 1, and we have the summation

$$(5.3)\qquad {}_2\psi_2\left[\begin{matrix} a, b \\ c, bq \end{matrix}; q, \frac{q}{a}\right] = \frac{(q, q, bq/a, c/b)_\infty}{(q/a, bq, q/b, c)_\infty},$$

where $\max(|q/a|, |c|) < 1$.

In the following subsections, we combine our Lemmas 4.3 and 4.9 from Section 4 together with the above one-dimensional ${}_2\psi_2$ formulas.

In subsection 5.1 we derive several multivariable extensions of Bailey's ${}_2\psi_2$ transformation formulas (5.1) and (5.2). In subsection 5.2 we derive multivariable extensions of the ${}_2\psi_2$ summation in (5.3).

5.1 *Some A_n extensions of Bailey's ${}_2\psi_2$ transformations.* We give several, but not all, of the possible A_n ${}_2\psi_2$ transformations which arise from Lemmas 4.3 and 4.9.

We start with two multivariable extensions of (5.1) which arise from Lemma 4.3.

Theorem 5.1 (An A_n ${}_2\psi_2$ transformation). *Let $a, b, c_1, \dots, c_n$, d, $x_1, \dots, x_n, y_1, \dots, y_n$ and z be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.4) *vanishes. Then*

(5.4)
$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j \right)_{k_i}^{-1} \times \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} \frac{(a,b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n} \binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg)$$
$$= \frac{(az, d/a, c_1 \dots c_n q^{1-n}/b, dq/abz)_\infty}{(z, d, q/b, c_1 \dots c_n d q^{1-n}/abz)_\infty} \times \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, (y_i, y_j)c_j)_\infty}{((y_i/y_j)q, (x_i/x_j)c_j)_\infty}$$
$$\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^{n} y_i^{nk_i - |\mathbf{k}|} \times \frac{(a, abz/d)_{|\mathbf{k}|}}{(az)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n} \binom{k_i}{2}} \left(\frac{d}{a} \right)^{|\mathbf{k}|} \Bigg),$$

provided $|c_1 \dots c_n d q^{1-n}/ab| < |z| < |q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^{n} x_i|$ *and* $|c_i \dots c_n d z^{1-n}/ab| < |d/a| < |q^{\frac{n-1}{2}} y_j^{-n} \prod_{i=1}^{n} y_i|$ *for* $j = 1, \dots, n$.

Proof. We have for $\max(|z|, |c_1 \dots c_n\, dq^{1-n}/abz|, |d/a|, |c_1 \dots c_n q^{1-n}/b|) < 1$,

(5.5)
$$ {}_2\psi_2 \left[\begin{matrix} a, b \\ c_1 \dots c_n q^{1-n}, d \end{matrix} ; q, z \right] = \frac{(az, d/a, c_1 \dots c_n q^{1-n}/b, dq/abz)_\infty}{(z, d, q/b, c_1 \dots c_n dq^{1-n}/abz)_\infty} \times {}_2\psi_2 \left[\begin{matrix} a, abz/d \\ az, c_1 \dots c_n q^{1-n} \end{matrix} ; q, \tfrac{d}{a} \right], $$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.3 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.5) by the $b_i \mapsto c_i$, $i = 1, \dots, n$ and

$$ f(m) = \frac{(a, b)_m}{(d)_m} z^m $$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.5) is rewritten by the $b_i \mapsto c_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$, and

$$ f(m) = \frac{(a, abz/d)_m}{(az)_m} \left(\frac{d}{a} \right)^m $$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by

$$ \frac{(q)_\infty}{(c_1 \dots c_n q^{1-n})_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)c_j)_\infty}{((x_i/x_j)q)_\infty} \tag{5.6} $$

and simplify to obtain (5.4). □

Theorem 5.2 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, a_2, \dots, a_n, b$, $c_1, \dots, c_n, d, x_1, \dots, x_n, y_1, \dots, y_n$ and $z_1, \dots, z_n$ be indeterminate, let $n \geq 1$ and suppose that none of the denominators in (5.7) vanishes.*

Write $A \equiv a_1 \dots a_n$, $C \equiv c_1 \dots c_n$ and $Z \equiv z_1 \dots z_n$ for short. Then
(5.7)

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|}$$
$$\times \frac{(Aq^{1-n}, b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} Z^{|\mathbf{k}|} \Bigg)$$
$$= \frac{(Cq^{1-n}, dq^{n-1}/A, Cq^{1-n}/b, dq^n/AbZ)_\infty}{(Z, d, q/b, Cd/AbZ)_\infty} \prod_{i,j=1}^{n} \frac{(\frac{x_i}{x_j} q, \frac{y_i}{y_j} a_j z_j)_\infty}{(\frac{y_i}{y_j} q, \frac{x_i}{x_j} c_j)_\infty}$$
$$\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} a_j z_j \right)_{k_i}^{-1}$$
$$\times \prod_{i=1}^{n} y_i^{nk_i - |\mathbf{k}|}$$
$$\times \frac{(Aq^{1-n}, \frac{AbZq^{1-n}}{d})_{|\mathbf{k}|}}{(Cq^{1-n})_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} \left(\frac{dq^{n-1}}{A} \right)^{|\mathbf{k}|} \Bigg),$$

provided that $|Cd/Ab| < |Z| < |q^{(n-1)/2} x_j^{-n} \prod_{i=1}^{n} x_i|$ and $|Cd/Ab| < |dq^{n-1}/A| < |q^{(n-1)/2} y_j^{-n} \prod_{i=1}^{n} y_i|$ for $j = 1, \dots, n$.

Proof. We have for $\max(|Z|, |Cd/AbZ|, |dq^{n-1}/A|, |Cq^{1-n}/b|) < 1$,
(5.8)

$${}_2\psi_2 \left[\begin{matrix} Aq^{1-n}, b \\ Cq^{1-n}, d \end{matrix} ;q,Z \right] = \frac{(AZq^{1-n}, dq^{n-1}/A, Cq^{1-n}/b, dq^n/AbZ)_\infty}{(Z, d, q/b, Cd/AbZ)_\infty}$$
$$\times {}_2\psi_2 \left[\begin{matrix} Aq^{1-n}, AbZq^{1-n}/d \\ AZq^{1-n}, Cq^{1-n} \end{matrix} ; q, \frac{dq^{n-1}}{A} \right],$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.3 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.8) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$f(m) = \frac{(Aq^{1-n}, b)_m}{(d)_m} Z^m$$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.8) is rewritten by the $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$ and

$$f(m) = \frac{(Aq^{1-n}, AbZq^{1-n}/d)_m}{(Cq^{1-n})_m} \left(\frac{dq^{n-1}}{A}\right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.7). □

Next, we give two multivariable extensions of (5.1), which arise from Lemma 4.9.

Theorem 5.3 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, a_2, \dots, a_n, b, c_1, \dots, c_n, d, x_1, \dots, x_n, y_1, \dots, y_n$ and z be indeterminate, let $n \geq 1$ and suppose that none of the denominators in* (5.9) *vanishes. Write $A \equiv a_1 \dots a_n$ and $C \equiv c_1 \dots c_n$ for short. Then*

(5.9)
$$\begin{aligned}
\sum_{k_1,\dots,k_n=-\infty}^{\infty} & \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)a_j)_{k_i}}{(x_i/x_j c_j)_{k_i}} \frac{(b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} z^{|\mathbf{k}|} \\
&= \frac{(Az, d/A, Cq^{1-n}/b, dq/Abz)_\infty}{(z, d, q/b, Cdq^{1-n}/Abz)_\infty} \\
&\times \prod_{i,j=1}^{n} \frac{((y_i/y_j)c_j, (y_i q/y_j a_i), (x_i/x_j)q, (x_i c_j/x_j a_i))_\infty}{((x_i/x_j)c_j, (x_i q/x_j a_i), (y_i/y_j)q, (y_i c_j/y_j a_i))_\infty} \\
&\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \\
&\times \prod_{i,j=1}^{n} \frac{((y_i/y_j)a_j)_{k_i}}{((y_i/y_j)c_j)_{k_i}} \frac{(Abz/d)_{|\mathbf{k}|}}{(Az)_{|\mathbf{k}|}} \left(\frac{d}{A}\right)^{|\mathbf{k}|},
\end{aligned}$$

provided $|Cdq^{1-n}/Ab| < |z| < 1$ and $|Cdq^{1-n}/Ab| < |d/A| < 1$.

Proof. We have for $\max(|z|, |Cdq^{1-n}/Abz, |d/A|, |Cq^{1-n}/b|) < 1$,

$$
{}_2\psi_2\left[\begin{matrix} A, b \\ Cq^{1-n}, d \end{matrix}; q, z\right] = \frac{(Az, d/A, Cq^{1-n}/b, dq/Abz)_\infty}{(z, d, q/b, Cdq^{1-n}/Abz)_\infty} \times {}_2\psi_2\left[\begin{matrix} A, Abz/d \\ Az, Cq^{1-n} \end{matrix}; q, \tfrac{d}{A}\right], \tag{5.10}
$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.9 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.10) by the $b_i \mapsto c_i$, $i = 1, \ldots, n$ and

$$
g(m) = \frac{(b)_m}{(d)_m} z^m
$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.10) is rewritten by the $b_i \mapsto c_i$, $x_i \mapsto y_i$, $i = 1, \ldots, n$, and

$$
g(m) = \frac{(Abz/d)_m}{(Az)_m}\left(\frac{d}{A}\right)^m
$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$
\frac{(q, Cq^{1-n}/A)_\infty}{(Cq^{1-n}, q/A)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)c_j, (x_i q/x_j a_i))_\infty}{((x_i/x_j)q, (x_i c_j/x_j a_i))_\infty} \tag{5.11}
$$

and simplify to obtain (5.9). □

Theorem 5.4 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$, $c, d_1, \ldots, d_n, x_1, \ldots, x_n, y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in (5.12) vanishes. Write $A \equiv a_1 \ldots a_n$, $B \equiv b_1 \ldots b_n$, $D \equiv d_1 \ldots d_n$ and $Z \equiv z_1 \ldots z_n$ for* *short. Then*

(5.12)

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j)_{k_i}}{((x_i/x_j)d_j)_{k_i}} \frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} Z^{|\mathbf{k}|}$$

$$= \frac{(D/A, c/B)_\infty}{(Z, cD/ABZ)_\infty}$$

$$\times \prod_{i,j=1}^{n} \frac{((y_i/y_j)a_j z_j, (y_i d_i q/y_j a_i b_i z_i), (x_i/x_j)q, (x_i d_j/x_j b_i))_\infty}{((x_i/x_j)d_j, (x_i q/x_j b_i), (y_i/y_j)q, (y_i d_i a_j z_j/y_j a_i b_i z_i))_\infty}$$

$$\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_i q^{k_j}}{y_i - y_j}\right)$$

$$\times \prod_{i,j=1}^{n} \frac{(y_i a_j b_j z_j/y_j d_j)_{k_i}}{((y_i/y_j)a_j z_j)_{k_i}} \frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} \left(\frac{D}{A}\right)^{|\mathbf{k}|},$$

provided $|cD/AB| < |Z| < 1$ *and* $|cD/AB| < |D/A| < 1$.

Proof. We have, for $\max(|Z|, |cD/ABZ|, |D/A|, |c/B|) < 1$,

(5.13)

$$ {}_2\psi_2 \left[\begin{matrix} Aq^{1-n}, B \\ c, Dq^{1-n} \end{matrix} ; q, Z\right] = \frac{(AZq^{1-n}, D/A, c/B, Dq/ABZ)_\infty}{(Z, Dq^{1-n}, q/B, cD/ABZ)_\infty} \times {}_2\psi_2 \left[\begin{matrix} Aq^{1-n}, ABZ/D \\ AZq^{1-n}, c \end{matrix} ; q, \tfrac{D}{A}\right],$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.9 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.13) by the $a_i \mapsto b_i$, $b_i \mapsto d_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(Aq^{1-n})_m}{(c)_m} Z^m$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.13) is rewritten by the $a_i \mapsto a_i b_i z_i/d_i$, $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$ and

$$g(m) = \frac{(Aq^{1-n})_m}{(c)_m} \left(\frac{D}{A}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q, Dq^{1-n}/B)_\infty}{(Dq^{1-n}, q/B)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)d_j, (x_i q/x_j b_i))_\infty}{((x_i/x_j)q, (x_i d_j/x_j b_i))_\infty} \tag{5.14}$$

and simplify to obtain (5.12). □

Finally, we provide two multivariable extensions of (5.2) that arise from Lemmas 4.3 and 4.9, respectively.

Theorem 5.5 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, a_2, \dots, a_n, b, c_1, \dots, c_n, d, x_1, \dots, x_n, y_1, \dots, y_n$ and $z_1, \dots, z_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.15) *vanishes. Write $A \equiv a_1 \dots a_n$, $C \equiv c_1 \dots c_n$, and $Z \equiv z_1 \dots z_n$ for short. Then*

(5.15)

$$\begin{aligned}
\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} \\
\times \frac{(Aq^{1-n}, b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n} \binom{k_i}{2}} Z^{|\mathbf{k}|} \Bigg) \\
= \frac{(bZ, Cq/AbZ, dq^n/AbZ)_\infty}{(q^n/A, q/b, d)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, (y_i/y_j)a_j z_j)_\infty}{((y_i/y_j)q, (x_i/x_j)c_j)_\infty} \\
\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \\
\times \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} a_j z_j \right)_{k_i}^{-1} \prod_{i=1}^{n} y_i^{nk_i - |\mathbf{k}|} \\
\times \frac{(AbZ/C, AbZq^{1-n}/d)_{|\mathbf{k}|}}{(bZ)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} \\
\times q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n} \binom{k_i}{2}} \left(\frac{Cd}{AbZ} \right)^{|\mathbf{k}|} \Bigg),
\end{aligned}$$

provided that $|Cd/Ab| < |Z| < |q^{(n-1)/2} x_j^{-n} \prod_{i=1}^{n} x_i|$ and $|Cd/Ab| < |Cd/AbZ| < |q^{(n-1)/2} y_j^{-n} \prod_{i=1}^{n} y_i|$ for $j = 1, \dots, n$.

Proof. We have for $\max(|Z|, |Cd/AbZ|) < 1$,

(5.16)
$$
{}_2\psi_2\left[\begin{matrix} Aq^{1-n}, b \\ Cq^{1-n}, d \end{matrix}; q, Z\right] = \frac{(AZq^{1-n}, bZ, Cq/AbZ, dq^n/AbZ)_\infty}{(q^n/A, q/b, Cq^{1-n}, d)_\infty} \times {}_2\psi_2\left[\begin{matrix} AbZ/C, AbZq^{1-n}/d \\ AZq^{1-n}, bZ \end{matrix}; q, \tfrac{Cd}{AbZ}\right],
$$

by Bailey's ${}_2\psi_2$ transformation in (5.2). Now we apply Lemma 4.3 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.16) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$
f(m) = \frac{(Aq^{1-n}, b)_m}{(d)_m} Z^m
$$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.16) is rewritten by the $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$, and

$$
f(m) = \frac{(AbZ/C, AbZq^{1-n}/d)_m}{(bZ)_m} \left(\frac{Cd}{AbZ}\right)^m
$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.15). □

Theorem 5.6 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$, $c, d_1, \dots, d_n$, $x_1, \dots, x_n$, $y_1, \dots, y_n$ and $z_1, \dots, z_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.17) *vanishes. Write $A \equiv a_1 \dots a_n$, $B \equiv b_1 \dots b_n$, $D \equiv d_1 \dots d_n$ and $Z \equiv z_1 \dots z_n$ for*

short. Then
(5.17)

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \\
&\qquad\qquad \times \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j)_{k_i}}{((x_i/x_j)d_j)_{k_i}} \frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} Z^{|\mathbf{k}|} \\
&= \frac{(BZ, cq^n/ABZ)_\infty}{(q^n/A, c)_\infty} \\
&\quad \times \prod_{i,j=1}^{n} \frac{((y_i/y_j)a_j z_j, (y_i d_i q/y_j a_i b_i z_i), (x_i/x_j)q, (x_i d_j/x_j b_i))_\infty}{((x_i/x_j)d_j, (x_i q/x_j b_i), (y_i/y_j)q, (y_i d_i a_j z_j/y_j a_i b_i z_i))_\infty} \\
&\quad \times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \\
&\quad \times \prod_{i,j=1}^{n} \frac{(y_i a_j b_j z_j/y_j d_j)_{k_i}}{((y_i/y_j)a_j z_j)_{k_i}} \frac{(ABZq^{1-n}/c)_{|\mathbf{k}|}}{(BZ)_{|\mathbf{k}|}} \left(\frac{cD}{ABZ}\right)^{|\mathbf{k}|},
\end{aligned}$$

provided $|cD/AB| < |Z| < 1$.

Proof. We have for $\max(|Z|, |cD/ABZ|) < 1$,
(5.18)

$$\begin{aligned}
{}_2\psi_2\left[\begin{matrix} Aq^{1-n}, B \\ c, Dq^{1-n} \end{matrix}; q, Z\right] &= \frac{(AZq^{1-n}, BZ, cq^n/ABZ, Dq/ABZ)_\infty}{(q^n/A, q/B, c, Dq^{1-n})_\infty} \\
&\quad \times {}_2\psi_2\left[\begin{matrix} ABZq^{1-n}/c, ABZ/D \\ AZq^{1-n}, BZ \end{matrix}; q, \tfrac{cD}{ABZ}\right],
\end{aligned}$$

by Bailey's ${}_2\psi_2$ transformation in (5.2). Now we apply Lemma 4.9 to the ${}_2\psi_2$s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on the left side of (5.18) by $a_i \mapsto b_i$, $b_i \mapsto d_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(Aq^{1-n})_m}{(c)_m} Z^m$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.18) is rewritten by the $a_i \mapsto a_i b_i z_i/d_i$, $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(ABZq^{1-n}/c)_m}{(BZ)_m} \left(\frac{cD}{ABZ} \right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by (5.14) and simplify to obtain (5.17). □

5.2 *Some A_n ${}_2\psi_2$ summations.* Here we work out (all) the A_n extensions of the ${}_2\psi_2$ summation in (5.3) that arise from Lemmas 4.3 and 4.9, respectively.

First we give two multivariable extensions of (5.3) which arise from Lemma 4.3.

Theorem 5.7 (An A_n ${}_2\psi_2$ summation). *Let $a, b, c_1, \dots, c_n$ and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.19) *vanishes. Then*

(5.19)
$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} \\ \times \frac{(a,b)_{|\mathbf{k}|}}{(bq)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} \left(\frac{q}{a} \right)^{|\mathbf{k}|} \Bigg) \\ = \frac{(q, bq/a, c_1 \dots c_n q^{1-n}/b)_\infty}{(q/a, bq, q/b)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q)_\infty}{((x_i/x_j)c_j)_\infty},$$

provided $|c_1 \dots c_n q^{2-n}/a| < |q/a| < |q^{(n-1)/2} x_j^{-n} \prod_{i=1}^{n} x_i|$ *for* $j = 1, \dots, n$.

Proof. We have, for $\max(|q/a|, |c_1 \dots c_n q^{1-n}|) < 1$,

(5.20)
$${}_2\psi_2 \left[\begin{matrix} a, b \\ c_1 \dots c_n q^{1-n}, bq \end{matrix} ; q, \frac{q}{a} \right] = \frac{(q, q, bq/a, c_1 \dots c_n q^{1-n}/b)_\infty}{(q/a, bq, q/b, c_1 \dots c_n q^{1-n})_\infty},$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.3 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.20) by the

$b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$f(m) = \frac{(a,b)_m}{(bq)_m}\left(\frac{q}{a}\right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.19).

For an alternative proof, set $z = q/a$ and $d = bq$ in Theorem 5.1. In this case the multilateral series on the right side of (5.4) reduces to

$$\sum_{\substack{-\infty \le k_1,\dots,k_n \le \infty \\ |\mathbf{k}|=0}} \left(\prod_{1\le i<j\le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \times \prod_{i,j=1}^n \left(\frac{y_i}{y_j} c_j\right)_{k_i}^{-1} \prod_{i=1}^n y_i^{nk_i - |\mathbf{k}|} \times (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^n \binom{k_i}{2}} \right) = \frac{(c_1 \dots c_n q^{1-n})_\infty}{(q)_\infty} \prod_{i,j=1}^n \frac{((y_i/y_j)q)_\infty}{((y_i/y_j)c_j)_\infty},$$

the last evaluation by the $m = 0$ case of Proposition 4.2. □

Theorem 5.8 (An A_n ${}_2\psi_2$ summation). *Let $a, b_1, \dots, b_n, c$ and $x_1, \dots, x_n$ be indeterminate, let $n \ge 1$, and suppose that none of the denominators in* (5.21) *vanishes. Then*

(5.21)
$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \left(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \times \prod_{i,j=1}^n \left(\frac{x_i}{x_j} b_j q\right)_{k_i}^{-1} \prod_{i=1}^n x_i^{nk_i - |\mathbf{k}|} \times \frac{(a, b_1 \dots b_n)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^n \binom{k_i}{2}} \left(\frac{q}{a}\right)^{|\mathbf{k}|} \right) = \frac{(q, b_1 \dots b_n q/a, c/b_1 \dots b_n)_\infty}{(q/a, q/b_1 \dots b_n, c)_\infty} \prod_{i,j=1}^n \frac{((x_i/x_j)q)_\infty}{((x_i/x_j)b_j q)_\infty},$$

provided $|cq/a| < |q/a| < |q^{(n-1)/2} x_j^{-n} \prod_{i=1}^n x_i|$ *for* $j = 1, \dots, n$.

Proof. We have for $\max(|q/a|, |c|) < 1$,

$$(5.22) \quad {}_2\psi_2 \left[\begin{matrix} a, b_1 \dots b_n \\ c, b_1 \dots b_n q \end{matrix} ; q, \frac{q}{a} \right] = \frac{(q, q, b_1 \dots b_n q/a, c/b_1 \dots b_n)_\infty}{(q/a, b_1 \dots b_n q, q/b_1 \dots b_n, c)_\infty},$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.3 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.22) by the $b_i \mapsto b_i q$, $i = 1, \dots, n$, and

$$f(m) = \frac{(a, b_1 \dots b_n)_m}{(c)_m} \left(\frac{q}{a} \right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by

$$\frac{(q)_\infty}{(b_1 \dots b_n q^{1-n})_\infty} \prod_{i,j=1}^n \frac{((x_i/x_j) b_j)_\infty}{((x_i/x_j) q)_\infty}$$

and simplify to obtain (5.21).

For an alternative proof, set $c_i = b_i q$, $z_i = q/a_i$, $i = 1, \dots, n$ and $b \mapsto b_1 \dots b_n$ in Theorem 5.5. In this case, the multilateral series on the right side of (5.15) is terminated from below and from above and reduces just to one term, 1. In the resulting equation, we replace A by aq^{n-1} and d by c. □

Finally, we give four multivariable extensions of (5.3) that arise from Lemma 4.9.

Theorem 5.9 (An A_n ${}_2\psi_2$ summation). *Let* $a_1, \dots, a_n, b, c_1, \dots, c_n$ *and* $x_1, \dots, x_n$ *be indeterminate, let* $n \geq 1$ *and suppose that none of the denominators in* (5.23) *vanishes. Then*

$$(5.23) \quad \sum_{k_1, \dots, k_n = -\infty}^{\infty} \prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^n \frac{((x_i/x_j) a_j)_{k_i}}{((x_i/x_j) c_j)_{k_i}} \times \frac{(b)_{|\mathbf{k}|}}{(bq)_{|\mathbf{k}|}} \left(\frac{q}{a_1 \dots a_n} \right)^{|\mathbf{k}|}$$

$$= \frac{(q, bq/a_1 \dots a_n, c_1 \dots c_n q^{1-n}/b)_\infty}{(bq, q/b, c_1 \dots c_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^n \frac{((x_i/x_j)q, (x_i c_j/x_j a_i))_\infty}{((x_i/x_j)c_j, (x_i q/x_j a_i))_\infty},$$

provided $\max(|c_1 \dots c_n q^{1-n}|, |q/a_1 \dots a_n|) < 1$.

Proof. We have for $\max(|q/a_1 \dots a_n|, |c_1 \dots c_n q^{1-n}|) < 1$,

$$(5.24) \quad {}_2\psi_2 \left[\begin{matrix} a_1 \dots a_n, b \\ c_1 \dots c_n q^{1-n}, bq \end{matrix} ; q, \frac{q}{a_1 \dots a_n} \right] = \frac{(q, q, bq/a_1 \dots a_n, c_1 \dots c_n q^{1-n}/b)_\infty}{(q/a_1 \dots a_n, bq, q/b, c_1 \dots c_n q^{1-n})_\infty},$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.24) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(b)_m}{(bq)_m} \left(\frac{q}{a_1 \dots a_n} \right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by (5.11) and simplify to obtain (5.23).

For an alternative proof, set $z = q/a_1 \dots a_n$ and $d = bq$ in Theorem 5.3. In this case, the multilateral series on the right side of (5.9) reduces to

$$\sum_{\substack{-\infty \le k_1, \dots, k_n \le \infty \\ |\mathbf{k}|=0}} \prod_{1 \le i < j \le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^n \frac{((y_i/y_j)a_j)_{k_i}}{((y_i/y_j)c_j)_{k_i}}$$

$$= \frac{(c_1 \dots c_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, c_1 \dots c_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^n \frac{((y_i/y_j)q, (y_i c_j/y_j a_i))_\infty}{((y_i/y_j)c_j, (y_i q/y_j a_i))_\infty},$$

the last evaluation by Theorem 4.7. □

Theorem 5.10 (An A_n ${}_2\psi_2$ summation). *Let $a, b_1, \dots, b_n, c$ and $x_1, \dots, x_n$ be indeterminate, let $n \ge 1$ and suppose that none of the* *denominators in* (5.25) *vanishes. Then*

(5.25)
$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j)_{k_i}}{((x_i/x_j)b_j q)_{k_i}} \frac{(a)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} \left(\frac{q}{a}\right)^{|\mathbf{k}|}$$
$$= \frac{(b_1\dots b_n q/a, c/b_1\dots b_n)_\infty}{(q/a, c)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, (x_i b_j/x_j b_i)q)_\infty}{((x_i/x_j)b_j q, (x_i q/x_j b_i))_\infty},$$

provided $\max(|c|, |q/a|) < 1$.

Proof. We utilize the ${}_2\psi_2$ summation in (5.20) and apply Lemma 4.9 to the ${}_2\psi_2$ in that summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.20) by the $a_i \mapsto b_i$, $b_i \mapsto b_i q$, $i = 1, \dots, n$, and

$$g(m) = \frac{(a)_m}{(c)_m}\left(\frac{q}{a}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q,q)_\infty}{(b_1\dots b_n q, q/b_1\dots b_n)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j q, (x_i q/x_j b_i))_\infty}{((x_i/x_j)q, (x_i b_j/x_j b_i)q)_\infty}$$

and simplify to obtain (5.25).

For an alternative proof, set $z_i = q/a_i$ and $d_i = b_i q$, $i = 1, \dots, n$, in Theorem 5.4. In this case, the multilateral series on the right side of (5.12) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace A by aq^{n-1}. □

Theorem 5.11 (An A_n ${}_2\psi_2$ summation). *Let* $a, b_1, \dots, b_n, c_1, \dots, c_n$ *and* $x_1, \dots, x_n$ *be indeterminate, let* $n \ge 1$*, and suppose that none of* *the denominators in* (5.26) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \times \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j)_{k_i}}{((x_i/x_j)c_j)_{k_i}} \frac{(a)_{|\mathbf{k}|}}{(b_1\dots b_n q)_{|\mathbf{k}|}} \left(\frac{q}{a}\right)^{|\mathbf{k}|} = \frac{(q, b_1\dots b_n q/a)_\infty}{(q/a, b_1\dots b_n q)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, (x_i c_j/x_j b_i))_\infty}{((x_i/x_j)c_j, (x_i q/x_j b_i))_\infty}, \tag{5.26}$$

provided $\max(|c_1\dots c_n q^{1-n}|, |q/a|) < 1$.

Proof. Write $B \equiv b_1\dots b_n$ and $C \equiv c_1\dots c_n$. We have for $\max(|q/a|, |Cq^{1-n}|) < 1$,

$$ {}_2\psi_2 \left[\begin{matrix} a, B \\ Cq^{1-n}, Bq \end{matrix} ; q, \tfrac{q}{a}\right] = \frac{(q, q, Bq/a, Cq^{1-n}/B)_\infty}{(q/a, Bq, q/B, Cq^{1-n})_\infty}, \tag{5.27}$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.27) by the $a_i \mapsto b_i$, $b_i \mapsto c_i$, $i = 1,\dots,n$, and

$$g(m) = \frac{(a)_m}{(Bq)_m}\left(\frac{q}{a}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q, Cq^{1-n}/B)_\infty}{(Cq^{1-n}, q/B)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)c_j, (x_i q/x_j b_i))_\infty}{((x_i/x_j)q, (x_i c_j/x_j b_i))_\infty}$$

and simplify to obtain (5.26).

For an alternative proof, set $z_i = q/a_i$, $i = 1,\dots,n$, and $c = b_1\dots b_n q$ in Theorem 5.6. In this case, the multilateral series on the right side of (5.17) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace d_i by c_i, $i = 1,\dots,n$, and A by aq^{n-1}. □

Theorem 5.12 (An A_n ${}_2\psi_2$ summation). *Let $a_1, \dots, a_n, b_1, \dots, b_n, c$, and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.28) *vanishes. Then*

(5.28)
$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \times \prod_{i,j=1}^{n} \frac{((x_i/x_j)a_j)_{k_i}}{((x_i/x_j)b_j q)_{k_i}} \frac{(b_1 \dots b_n)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} \left(\frac{q}{a_1 \dots a_n} \right)^{|\mathbf{k}|} = \frac{(q, c/b_1 \dots b_n)_\infty}{(q/b_1 \dots b_n, c)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)q, (x_i b_j q/x_j a_i))_\infty}{((x_i/x_j)b_j q, (x_i q/x_j a_i))_\infty},$$

provided $\max(|c|, |q/a_1 \dots a_n|) < 1$.

Proof. Write $A \equiv a_1 \dots a_n$, $B \equiv b_1 \dots b_n$. We have for $\max(|q/A|, |c|) < 1$,

$$ {}_2\psi_2 \left[\begin{matrix} A, B \\ c, Bq \end{matrix} ; q, \tfrac{q}{A} \right] = \frac{(q, q, Bq/A, c/B)_\infty}{(q/A, Bq, q/B, c)_\infty}, \tag{5.29}$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.29) by the $b_i \mapsto b_i q$, $i = 1, \dots, n$, and

$$g(m) = \frac{(B)_m}{(c)_m} \left(\frac{q}{A} \right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q, Bq/A)_\infty}{(Bq, q/A)_\infty} \prod_{i,j=1}^{n} \frac{((x_i/x_j)b_j q, (x_i q/x_j a_i))_\infty}{((x_i/x_j)q, (x_i b_j q/x_j a_i))_\infty}$$

and simplify to obtain (5.28).

For an alternative proof, set $z_i = q^{1/n}/b_i$ and $d_i = a_i q^{1/n}$, $i = 1, \dots, n$, in Theorem 5.6. In this case, the multilateral series on the right side of (5.17) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace a_i by $b_i q^{1-(1/n)}$ and b_i by a_i for $i = 1, \dots, n$. □

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